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EXISTENCE OF SOLUTIONS TO SOME PHI-LAPLACIAN BOUNDARY VALUE PROBLEMS

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Abstract. We shall derive existence results for a periodic boundary value problem at resonance involving a phi-Laplacian operator, special cases of which are the p-Laplacian operator $\phi(x) = |x|^{p-2}x$ and the (radial) capillary surface operator $\phi(x) = \frac{x}{\sqrt{1+x^2}}$.

1. Introduction. We shall consider the periodic boundary value problem

$$\begin{cases} \frac{d}{dt}[\phi(y'(t))] = f(t, y(t), y'(t)), \quad 0 < t < T, \\ y(0) = y(T),\ y'(0) = y'(T), \end{cases} \tag{1}$$

where $T > 0$, $\phi : (c,d) \to (a,b)$ is an increasing homeomorphism, the interval (a,b) contains zero, $\phi(0) = 0$ and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

The differential operator in (1) is a so called phi-Laplacian operator, a special case of which is the p-Laplacian operator with $\phi(x) = |x|^{p-2}x$, $p > 1$ and $(c,d) = (a,b) = \mathbb{R}$. Other examples of ϕ are the (radial) capillary surface operator

$$\phi(x) = \frac{x}{\sqrt{1+x^2}}, \quad -\infty < x < \infty, \tag{2}$$

for which $(c,d) = \mathbb{R}$ and $(a,b) = (-1,1)$ and

$$\phi(x) = \frac{x}{\sqrt{1-x^2}}, \quad -1 < x < 1, \tag{3}$$

arising in relativistic dynamics, with $(c,d) = (-1,1)$ and $(a,b) = \mathbb{R}$.

We shall derive existence results for problem (1) by using upper and lower solutions, by a shooting method and by the alternative method. The results generalize those of [6]. A different kind of approach can be found in [3], for instance.

Obviously, in the case $f(t,y,v) = f(t)$,

$$\int_0^T f(s)ds = 0 \tag{4}$$

is a necessary condition for the existence of a solution. It can be shown that another necessary condition is

$$m_1 - m_0 < b - a, \tag{5}$$

where $m_0 = \min_{0 \le t \le T} g(t)$, $m_1 = \max_{0 \le t \le T} g(t)$, with

$$g(t) = \int_0^t f(s)ds,\ 0 \le t \le T,$$

and that this condition cannot be improved, in general. An existence result is given by

Lemma 1. *If $m_1 - m_0 < min(-a,b)$, then the BVP (1), with $f(t,y,v) = f(t)$, has a solution.*

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2. The Results

In the following we will use the assumption

($f0$) there exist $m_0, m_1 \in \mathbb{R}$ such that $m_1 - m_0 < min(-a, b)$ and

$$m_0 < \int_0^t f(s, y(s), v(s))ds < m_1 \tag{7}$$

for all $t \in [0, T]$, $y \in C^2[0, T]$ and $v \in C^1[0, T]$.

Theorem 1. *Let f satisfy ($f0$) and be bounded and assume that there exists an $N > 0$ such that*

$$f(t, N, 0) \geq 0,\ f(t, -N, 0) \leq 0,\ 0 \leq t \leq T, \tag{8}$$

and that ϕ' is continuous and positive valued. Then the boundary value problem (1) has at least one solution.

Proof. Define $g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$g(t, u, v) = \begin{cases} [\phi'(v)]^{-1} f(t, u, v), & |v| < M = max\{\phi^{-1}(m_1 - m_0), |\phi^{-1}(m_0 - m_1|\} \\ [\phi'(M)]^{-1} f(t, u, v), & v \geq M \\ [\phi'(-M)]^{-1} f(t, u, v), & v \leq -M \end{cases}$$

for $t \in [0, T]$, $u, v \in \mathbb{R}$. Obviously g is continuous and bounded and $u = N$ and $w = -N$ are lower and upper solutions, respectively, of the boundary value problem

$$\begin{cases} y''(t) = g(t, y(t), y'(t)), & 0 < t < T, \\ y(0) = y(T),\ y'(0) = y'(T). \end{cases} \tag{9}$$

Then, by Theorem 3.1 of [1](see also Remark 2.2 of [2]), problem (9) has a solution y. We will show that y is a solution of (1). Because $y(0) = y(T)$, there exists a $t_0 \in [0, T]$ for which $y'(t_0) = 0$. Hence there exists a subinterval I of $[0, T]$ containing t_0 such that $|y'(t)| < M$, $t \in I$ and hence

$$[\phi(y')]' = f(t, y(t), y'(t)), \quad t \in I.$$

Then for $t \in I$,

$$\phi(y'(t)) = \phi(y'(t_0)) + \int_{t_0}^t f(s, y(s), y'(s))ds = \int_{t_0}^t f(s, y(s), y'(s))ds,$$

which implies that

$$y'(t) = \phi^{-1}\left(\int_{t_0}^t f(s, y(s), y'(s))ds\right) = \phi^{-1}\left[\int_0^t f(s, y(s), y'(s))ds - \int_0^{t_0} f(s, y(s), y'(s))ds\right].$$

Here we have used the assumption that $m_1 - m_0 < min(-a, b)$, which implies that

$$\int_{t_0}^t f(s, y(s), y'(s))ds \in (a, b)$$

for all $t_0, t \in [0, T]$. From the last equality and from inequality (7) it follows that the interval I can be extended to $[0, T]$. Hence y is a solution of (1). □

Example 1. Let us consider the boundary value problem

$$\begin{cases} \frac{y''(t)}{(1+y'(t)^2)^{3/2}} = \frac{1}{2\pi}[e^{-\frac{|y(t)|}{2\pi}} \sin(\frac{y(t)}{2\pi}) \cos(y'(t)) + \frac{t}{2\pi}(1 - \frac{t}{2\pi})], \quad 0 < t < 2\pi, \\ y(0) = y(2\pi), \ y'(0) = y'(2\pi). \end{cases} \tag{10}$$

Now ϕ is defined by (2), $(a, b) = (-1, 1)$, ϕ satisfies the assumptions of Theorem 1, and for $N = \pi^2/2$ we have

$$f(t, N, 0) = \frac{1}{T}[\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}} + \frac{t}{T}(1 - \frac{t}{T})] > 0,$$

and

$$f(t, -N, 0) = \frac{1}{T}[-\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}} + \frac{t}{2\pi}(1 - \frac{t}{2\pi})] < 0.$$

Moreover, $m_1 - m_0 < \sqrt{2}e^{-\frac{\pi}{4}} = 0.6448 < 1$. Hence, by Theorem 1, problem (10) has at least one solution. Numerical calculations indicate that problem (10) has two solutions as we can see from the curve $\delta(\lambda)$ of Figure 1. The curve $\delta(\lambda)$ is defined by

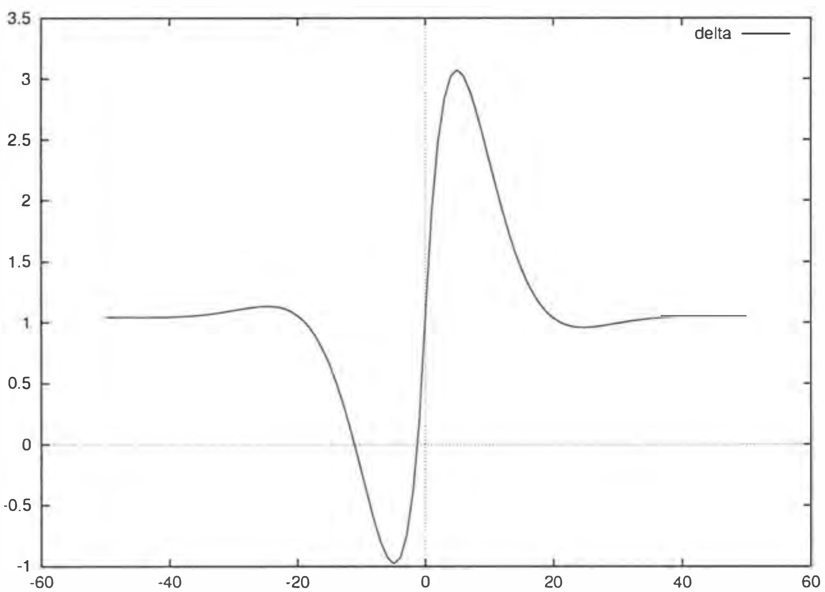
$$\delta(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} (1 + y_\lambda'(t)^2)^{3/2} f(t, y_\lambda(t), y_\lambda'(t))dt,$$

where y_λ is the solution of the parametrized problem (see [5])

$$\begin{cases} y''(t) = (1 + y'(t)^2)^{3/2} f(t, y(t), y'(t)) - \delta, \quad 0 < t < 2\pi, \\ y(0) = y(2\pi), \ y'(0) = y'(2\pi), \ \int_0^{2\pi} y(t)dt = \lambda. \end{cases} \tag{11}$$

A zero λ_0 of the $\delta(\lambda)$ curve indicates the existence of a solution with mean value $\frac{\lambda_0}{2\pi}$.

Figure 1.



Next we will use the following hypotheses

(ϕ1) to each choice of s_0, $s_1 \in (c,d)$, $s_0 < s_1$, there corresponds such an $M > 0$ that $\phi(y) - \phi(z) \geq M(y-z)$ whenever $s_0 \leq z < y \leq s_1$;

(f1) $f(t,x,z) \leq f(t,y,z)$ for $t \in [0,T]$ and for x, y, $z \in \mathbb{R}$, $x \leq y$;

(f2) $|f(t,x,y) - f(t,x,z)| \leq p(t)\varphi(|y-z|)$ for $t \in [0,T]$ and for x, y, $z \in \mathbb{R}$, $0 < |y-z| \leq r$, where $r > 0$, $p \in L^1_+(J)$, the function φ: $(0,r] \to (0,\infty)$ is increasing and $\int_{0+}^{r} \frac{dz}{\varphi(z)} = \infty$;

Theorem 2. *Assume that conditions (ϕ1), (f0), (f1) and (f2) are satisfied and that there exists a $K > 0$ such that*

$$\int_0^T f(t,u(t),v(t))dt > 0, \quad \int_0^T f(t,-u(t),v(t))dt < 0, \tag{12}$$

if $u(t) \geq K, 0 \leq t \leq T$ and if $v \in C[0,T]$. Then the boundary value problem (1) has at least one solution.

Proof. Let $d \in \mathbb{R}$ be given. By Theorem 2.1 of [4] the boundary value problem

$$\frac{d}{dt}[\phi(y'(t))] = f(t,y(t),y'(t)), \; y'(0) = c, \; y(T) = d \tag{13}$$

has at most one solution y on $[0,T]$ for all $c, d \in \mathbb{R}$. Next we will show, by using successive approximations, that this solution $y = y_\alpha$,

$$y_\alpha(t) = d + \int_T^t \phi^{-1}[\alpha + \int_0^s f(x,y_\alpha(x),y'_\alpha(x))dx]ds, \tag{14}$$

where $\alpha = \phi(c) \in (a - m_0, b - m_1)$, also exists and is continuous in α.
Let

$$\begin{cases} y_n(t) = d + \int_T^t \phi^{-1}[\alpha + \int_0^s f(x,y_{n-1}(x),y'_{n-1}(x))dx]ds, \; n = 1,2..., \; 0 \leq t \leq T, \\ y_0(t) = d + \int_T^t \phi^{-1}[\alpha + m_0]ds, \; 0 \leq t \leq T. \end{cases}$$

Then, by (7) and by the monotonicity of ϕ^{-1},

$$d + \int_T^t \phi^{-1}(\alpha + m_1)ds \leq y_n(t) \leq d + \int_T^t \phi^{-1}(\alpha + m_0)ds, \; n = 0,1,..., \; 0 \leq t \leq T$$

and

$$\phi^{-1}(\alpha + m_0) \leq y'_n(t) \leq \phi^{-1}(\alpha + m_1), \; n = 0,1,...,0 \leq t \leq T,$$

i.e. both sequences y_n and y'_n are equi-bounded. It is easy to show that they are also equi-continuous. Hence, there exists such a subsequence n_k that both (y_{n_k}) and (y'_{n_k}) converge uniformly on $[0,T]$. Then the limit of (y'_{n_k}) is y'_α where y_α is the limit of (y_{n_k}) and by the dominated converge theorem y_α satisfies (14), i.e. y_α is a solution of (13). But that solution is unique and hence the whole sequence (y_n) converges. The continuity of y_α in α can be shown in a similar way using the equi-continuity and equi-boundedness argument and the uniqueness of the solution of (13).

Then, from (7) we obtain $T(-m_1) \geq 0$ and $T(-m_0) \leq 0$, for T defined by

$$T(\alpha) = \int_T^0 \phi^{-1}[\alpha + \int_0^t f(s, y_\alpha(s), y'_\alpha(s))ds]dt.$$

Since $m_1 - m_0 < min(-a, b)$, $T(\alpha)$ is well defined on $[-m_1, -m_0]$. By the continuity of T there exists an α_0 for which $T(\alpha_0) = 0$ and hence $y_{\alpha_0}(0) = d = y_{\alpha_0}(T)$. By Theorem 2.1 of [4] this α_0 is unique.

Let then $\alpha \in [-m_1, -m_0]$ be given and for a $d \in \mathbb{R}$ denote by y_d the solution of (13),

$$y_d(t) = d + \int_T^t \phi^{-1}[\alpha + \int_0^s f(x, y_d(x), y'_d(x))dx]ds.$$

The condition $y_d{}'(0) = y'_d(T)$ is satisfied if

$$S(d) = \int_0^T f(t, y_d(t), y'_d(t))dt = 0,$$

i.e.

$$S(d) = \int_0^T f(t, d + \int_T^t \phi^{-1}[\alpha + \int_0^s f(x, y_d(x), y'_d(x))dx]ds, y'_d(t))dt = 0.$$

From (12) it follows that there exists a d_0 such that $S(d_0) = 0$, because y_d is continuous in d. This d_0 is unique, again by Theorem 2.1 of [4]. Define the functions $\psi_1 : \mathbb{R} \to [-m_1, -m_0]$ and $\psi_2 : [-m_1, -m_0] \to \mathbb{R}$ by

$$d \to \psi_1(d) = \alpha_0, \quad \alpha \to \psi_2(\alpha) = d_0,$$

where $T(\alpha_0) = 0,\ S(d_0) = 0$. Then the composite map

$$\psi_1 \circ \psi_2 : [-m_1, -m_0] \to [-m_1, -m_0]$$

has, as a continuous function, a fixed point α,

$$\psi_1(\psi_2(\alpha)) = \alpha.$$

For this α the solution y of (13) satisfies both boundary conditions, i.e. y is a solution of (1). □

Theorem 3. *Assume that the hypotheses of Theorem 2 are satisfied and that*

(f1') *There is a nonnegligible subset J of $[0,T]$ such that $f(t,x,z) < f(t,y,z)$ for all $t \in J$ and x, y, $z \in \mathbb{R}$, $x < y$.*

Then the boundary value problem (1) has a unique solution.

Proof. By Theorem 2 we have at least one solution and by Theorem 3.3 of [4] at most one solution. □

Example 2. The boundary value problem

$$\begin{cases} \frac{y''(t)}{(1+y'(t)^2)^{3/2}} = \frac{1}{4\pi}[\tan^{-1} y(t) + \frac{y'(t)}{\sqrt{(1+y'(t)^2)}}], \quad 0 < t < 2\pi \\ y(0) = y(2\pi), \ y'(0) = y'(2\pi). \end{cases}$$

has, by Theorem 3, exactly one solution. This can also be seen from the $\delta(\lambda)$ curve (see Ex.1) given in Figure 2 below.

Figure 2.

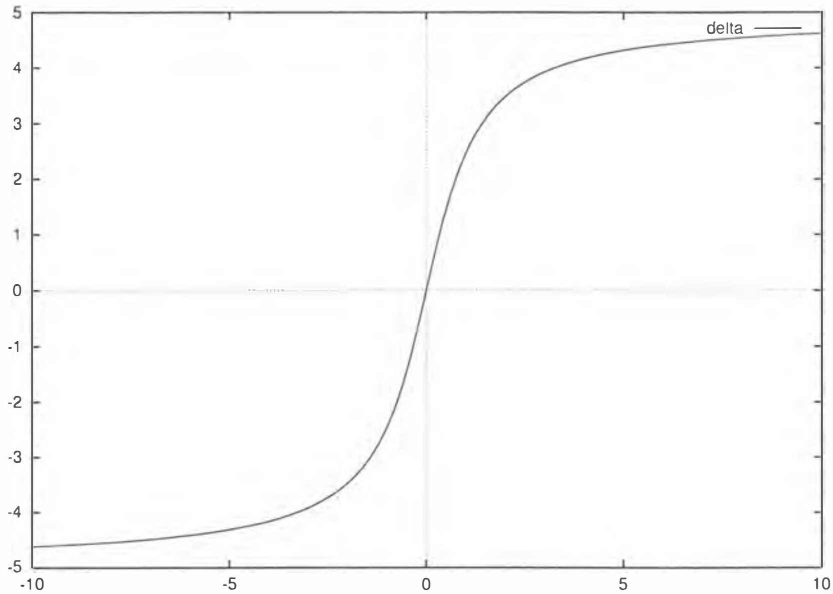
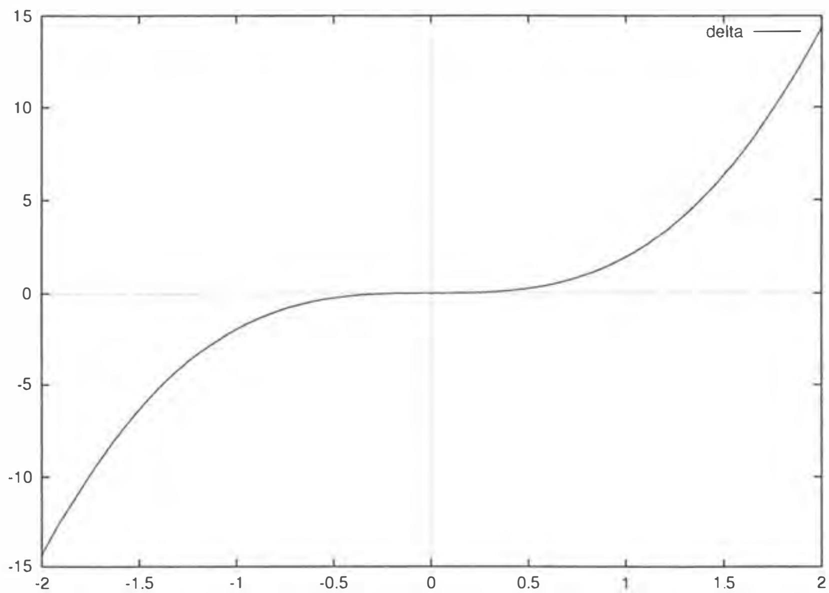


Figure 3.



In the following theorem we do not assume the integral condition $(f0)$.

Theorem 4. *Assume that the hypotheses of Theorem 2 are satisfied except that instead of $(f0)$ we assume that ϕ is a mapping from a bounded interval (c,d) into $\mathbb{R}$. Then the boundary value problem (1) has at least one solution.*

Proof. The proof proceeds as the proof of Theorem 2 except that now in proving that Eq. (14) has a solution we confine to the subsets Y_d of $C^1[0,1]$ defined by $Y_d = \{y \in C^1[0,1] : |y(t)| \le |d| + MT, |y'(t)| \le M\}$, where M is the bound for $|\phi^{-1}|$, and use $m_1 = KT, m_0 = -KT$, where K is the bound for $\{|f(t,y(t),y'(t)| : y \in Y_d\}$. □

Example 3. The boundary value problem

$$\begin{cases} \frac{y''(t)}{(1-y'(t)^2)^{3/2}} = \frac{1}{4\pi}\{[y(t)]^3 \sin\frac{t}{2} + \tan^{-1} y'(t)\}, \quad 0 < t < 2\pi, \\ y(0) = y(2\pi), \ y'(0) = y'(2\pi). \end{cases}$$

has, by Theorem 4, a solution. The $\delta(\lambda)$ curve is given in Figure 3.

For the Neumann boundary value problem

$$\begin{cases} \frac{d}{dt}[\phi(y'(t))] = f(t,y(t),y'(t)), \quad 0 < t < T, \\ y'(0) = 0, \ y'(T) = 0, \end{cases} \tag{15}$$

we have

Theorem 5. *Assume that either ϕ is a mapping from a bounded interval (c,d) into $\mathbb{R}$ or $(f0)$ holds, except that only $a < m_0 \le m_1 < b$ is required. Moreover, assume that $(\phi 1)$, $(f1)$ and $(f2)$hold and that there exists a $K > 0$ such that*

$$\int_0^T f(t,u(t),v(t))dt > 0, \ \int_0^T f(t,-u(t),v(t))dt \ < \ 0, \tag{12}$$

if $u(t) \ge K, 0 \le t \le T$ and if $v \in C[0,T]$. Then the boundary value problem (15) has at least one solution.

Proof. Set $\alpha = 0$ in the proof of Theorem 2. The assumptions guarantee that Eq. (14) has a unique solution y_d for any $d \in \mathbb{R}$. From inequalities (12) and from the continuous dependendence of y_d on d it follows that there exists a d_0 such that $y'_{d_0}(T) = 0$. □

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