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SOME INVARIANT SOLUTIONS OF CERTAIN SOIL WATER EQUATIONS

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Abstract. A mathematical model described by a class of nonlinear partial differential equations was developed to simulate soil water infiltration, redistribution and extraction in a bedded soil profile overlaying a shallow water table and irrigated by a drip irrigation system. We shall obtain exact (invariant) solutions for these equations. The solutions obtained are invariant under two-parameter symmetry groups obtained by the group classification of the governing equation.

1. **Introduction.** A mathematical model was developed to simulate soil water infiltration, redistribution and extraction in a bedded soil profile overlaying a shallow water table and irrigated by a drip irrigation system. This model is described by the class of equations

$$C(\psi)\psi_t = (K(\psi)\psi_x)_x + (K(\psi)(\psi_z - 1))_z - S(\psi), \tag{1}$$

where ψ is soil moisture pressure head, $C(\psi)$ is specific water capacity, $K(\psi)$ is unsaturated hydraulic conductivity, $S(\psi)$ is a sink or source term, t is time, x is the horizontal and z is the vertical axis which is considered positive downward. See [8] and [9].

Equation (1), which is the two-dimensional form of the Richards equation for which the soil is isotropic, homogeneous, nonhysteretic and nondeformable, provides the basis for prediciting soil water movement in both saturated and unsaturated soils. It is highly nonlinear equation because the parameters $C(\psi)$, $K(\psi)$ and $S(\psi)$ depend on the solution $\psi(x, z, t)$.

Many researchers (see, e.g. [9] and references there in) have given analytical and numerical solutions of equation (1) for special cases when the functions $C(\psi)$ and $K(\psi)$ are constants and $S(\psi)$ are linear functions.

Baikov et al [1] used Lie group analysis approach (symmetries, classification and invariant solutions; see, e.g. [3] , [6] or [2] vol.2, chapter 2) to study equation (1). All symmetries of equation (1) were obtained and were used to construct invariant solutions for two particular equations of form (1).

In this paper we shall obtain exact/asymptotic invariant solutions of equation (1) for some particular type of the coefficients $C(\psi)$, $K(\psi)$ and $S(\psi)$ which are not constants nor linear, and when an extension of the principal Lie algebra L_p occurs.

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2. **Invariant solutions.** In this section we shall use Lie group analysis to construct exact/asymptotic (invariant) solutions of equation (1) for some special forms of the functions $C(\psi), K(\psi)$ and $S(\psi)$. We shall consider those particular equations of form (1) when the principal Lie algebra L_p is extended by one or more operators. For each case we shall look for solutions invariant under two-dimensional subalgebras of the symmetry Lie algebra. Then equation (1), in general, reduces to second-order ordinary differential equations which are then solved to obtain solutions. We shall follow the general algorithm for constructing invariant solutions (see, e.g. Lie [4] and Ovsyannikov [7]).

The principal Lie algebra L_p (i.e., the Lie algebra of the Lie transformation group admitted by equation (1) for arbitrary functions $C(\psi)$, $K(\psi)$ and $S(\psi)$, see e.g. [2]) is the three-dimensional Lie algebra spanned by the operators which generate translations along the t, x and z-axis, namely

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial z}.$$

2.1. We first consider the case when $K(\psi) = 1$, $C(\psi) = \psi^\sigma$, where σ is an arbitrary constant and $S(\psi) = 0$.
In this case equation (1) has the form

$$\psi_t = \psi^{-\sigma}\left\{\psi_{xx} + \psi_{zz}\right\}. \tag{2}$$

According to the classification result in [1], equation (2) admits a six-dimensional Lie algebra L_6 obtained by an extension of the principal Lie algebra L_p by the following three operators:

$$X_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},$$
$$X_5 = \sigma t\frac{\partial}{\partial t} + \psi\frac{\partial}{\partial \psi}$$

and

$$X_6 = \sigma x\frac{\partial}{\partial x} + \sigma z\frac{\partial}{\partial z} - 2\psi\frac{\partial}{\partial \psi}.$$

Case 1.
Let us first construct invariant solutions under the operators $X_1 + X_5$ and X_6 where

$$X_1 + X_5 = (1 + \sigma t)\frac{\partial}{\partial t} + \psi\frac{\partial}{\partial \psi}.$$

These operators span a two-dimensional subalgebra L_2 of the algebra L_6 and have two functionally independent invariants. To find these, we start by calculating a basis of invariants $I(t, x, z, \psi)$. This is done by solving the system of linear homogeneous first-order partial differential equations:

$$(X_1 + X_5)I = 0, \quad X_6 I = 0. \tag{3}$$

Since we have $[X_1 + X_5, X_6] = 0$, the subalgebra L_2 is Abelian. Therefore we can solve the equations (3) successively in any order. The second equation provides us with three functionally independent solutions

$$J_1 = \frac{x}{z}, \quad J_2 = z^{\frac{2}{\sigma}}\psi \quad \text{and} \quad J_3 = t.$$

Hence the common solution $I(t, x, z, \psi)$ of our system is defined as a function of J_1, J_2 and J_3 only. Therefore we rewrite the action of $X_1 + X_5$ on the space of J_1, J_2 and J_3 by the formula

$$X_1 + X_5 = (X_1 + X_5)(J_1)\frac{\partial}{\partial J_1} + (X_1 + X_5)(J_2)\frac{\partial}{\partial J_2} + (X_1 + X_5)(J_3)\frac{\partial}{\partial J_3}$$

to obtain

$$X_1 + X_5 = J_2\frac{\partial}{\partial J_2} + (1 + \sigma J_3)\frac{\partial}{\partial J_3}.$$

Thus from the second equation $(X_1 + X_5)I = 0$ we obtain the following two functionally independent solutions (invariants):

$$I_1 = J_2(1 + \sigma J_3)^{\frac{-1}{\sigma}} \equiv z^{\frac{2}{\sigma}}\psi(1 + \sigma t)^{\frac{-1}{\sigma}}, \quad I_2 = J_1 \equiv \frac{x}{z}.$$

Consequently, the invariant solution is given by $I_1 = \Phi(I_2)$, that is

$$z^{\frac{2}{\sigma}}\psi(1 + \sigma t)^{\frac{-1}{\sigma}} = \Phi(\frac{x}{z})$$

or

$$\psi = z^{\frac{-2}{\sigma}}(1 + \sigma t)^{\frac{1}{\sigma}}\ \Phi(\frac{x}{z}).$$

Substituting this value of ψ into equation (2), we obtain the following second-order nonlinear ordinary differential equation:

$$(1 + \xi^2)\Phi'' + 2\xi\Phi' + \frac{2}{\sigma}\left(\frac{2}{\sigma} + 1\right)\Phi - \Phi^{1+\sigma} = 0, \quad \text{where} \quad \xi = \frac{x}{z}. \tag{4}$$

It can be seen that

$$\Phi_0 = \left[\frac{2}{\sigma}\left(\frac{2}{\sigma} + 1\right)\right]^{\frac{1}{\sigma}}$$

is a constant solution of equation (4). We now obtain an approximate solution of equation (4) near Φ_0. By letting $\Phi = \Phi_0 + \Phi_1$ we linearize equation (4) near the constant solution Φ_0. We obtain

$$(1 + \xi^2)\Phi_1'' + \left(2 + \frac{4}{\sigma}\right)\xi\Phi_1' - 2\left(\frac{2}{\sigma} + 1\right)\Phi_1 = 0$$

whose general solution is given by (see for example [5])

$$\Phi_1 = C_1\phi_1 + C_2\phi_2$$

where C_1 and C_2 are arbitrary constants,

$$\phi_1 = F\left(-\frac{a}{2} + \frac{2}{\sigma}, \frac{a}{2} + \frac{2}{\sigma} + \frac{1}{2}; \frac{1}{2}; -\xi^2\right)$$

and

$$\phi_2 = \xi F\left(-\frac{a}{2} + \frac{2}{\sigma} + \frac{1}{2}, \frac{a}{2} + \frac{2}{\sigma} + 1; \frac{3}{2}; -\xi^2\right).$$

Here F is a hypergeometric function and a satisfies the quadratic equation

$$\alpha^2 + \alpha - \frac{4}{\sigma^2} - \frac{6}{\sigma} - 2 = 0.$$

Hence the approximate invariant solution of equation (2) is given by

$$\psi = z^{\frac{-2}{\sigma}}(1 + \sigma t)^{\frac{1}{\sigma}}\left[\left(\frac{2}{\sigma}\left(\frac{2}{\sigma} + 1\right)\right)^{\frac{1}{\sigma}} + C_1\phi_1 + C_2\phi_2\right].$$

As a special case when $\sigma = 1$, the invariant solution of equation (2) is

$$\psi = z^{-2}t\left\{C_1\Phi_1 + C_2\Phi_2 + 6\right\}$$

where C_1 and C_2 are arbitrary constants and

$$\Phi_1 = F\left(\frac{1}{2}, 4; \frac{1}{2}; -\xi^2\right)$$

and

$$\Phi_2 = \xi F\left(1, \frac{9}{2}, \frac{3}{2}; -\xi^2\right), \quad \xi = \frac{x}{z}.$$

Case 2.
Let us now construct invariant solutions under the operators X_4 and $X_1 + X_5$ which span a two-dimensional subalgebra L_2 of algebra L_6. Repeating the calculations described above, we obtain the invariant solution of equation (2) as

$$\psi = (1+\sigma t)^{\frac{1}{\sigma}} \phi(x^2 + z^2)$$

where ϕ satisfies the second-order nonlinear ordinary differential equation

$$4(x^2 + z^2)\phi'' + 4\phi' - \phi^{\sigma+1} = 0.$$

This equation has a special solution

$$\phi = \left(\frac{4}{\sigma^2}\right)^{\frac{1}{\sigma}} (x^2 + z^2)^{\frac{-1}{\sigma}}$$

and consequently the invariant solution of equation (2) is

$$\psi = \left(\frac{4(1+\sigma t)}{\sigma^2}\right)^{\frac{1}{\sigma}} (x^2 + z^2)^{\frac{-1}{\sigma}}.$$

Likewise we can construct more invariant solutions with respect to different two-dimensional subalgebras of the six-dimensional algebra L_6.

2.2. We now consider the case when $K(\psi) = 1$, $C(\psi) = \psi^\sigma$, and $S(\psi) = B\psi^{\sigma+1}$, where $B \neq 0$ and σ are arbitrary constants.
In this case equation (1) has the form

$$\psi_t = \psi^{-\sigma}(\psi_{xx} + \psi_{zz}) - B\psi \tag{5}$$

and the principal Lie algebra is extended by the following three operators, namely

$$X_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},$$

$$X_5 = e^{B\sigma t}\frac{\partial}{\partial t} + Be^{B\sigma t}\psi\frac{\partial}{\partial \psi}$$

and

$$X_6 = \sigma x\frac{\partial}{\partial x} + \sigma z\frac{\partial}{\partial z} - 2\psi\frac{\partial}{\partial \psi}.$$

Let us construct invariant solutions based on the subalgebra L_2 of the algebra L_6 spanned by the operators $X_2 + X_6$ and L_5, where

$$X_2 + X_6 = (1+\sigma x)\frac{\partial}{\partial x} + \sigma z\frac{\partial}{\partial z} - 2\psi\frac{\partial}{\partial \psi}.$$

Following the above procedure, the invariant solution of equation (5) is given by

$$\psi = e^{Bt} z^{\frac{-2}{\sigma}} \Phi(\xi), \quad \xi = \frac{1 + x\sigma}{z}$$

where Φ satisfies

$$(\sigma^2 + \xi^2)\Phi'' + \left(\frac{4}{\sigma} + 2\right)\xi\Phi' + \frac{2}{\sigma}\left(\frac{2}{\sigma} + 1\right)\Phi - 2Be^{B\sigma t}\Phi^{\sigma+1} = 0.$$

This is a second-order nonlinear ordinary differential equation. Its approximate solution can be obtained in a similar manner described in case 1 above. Consequently we can then write the approximate invariant solution of equation (5).

2.3. Finally we consider equation (1) when $K(\psi)$=1, $C(\psi) = 1$ and $S(\psi) = -\psi^2$, that is

$$\psi_t = \psi_{xx} + \psi_{zz} + \psi^2. \tag{6}$$

According to the classification result in [1], equation (6) admits a five-dimensional Lie algebra L_5 obtained by an extension of the principal Lie algebra L_p by the following two operators:

$$X_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

and

$$X_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - 2\psi\frac{\partial}{\partial \psi}.$$

Let us construct invariant solutions under the operators $X_1 + X_5$ and X_4, where

$$X_1 + X_5 = (1 + 2t)\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - 2\psi\frac{\partial}{\partial \psi}.$$

These two operators span a two-dimensional subalgebra L_2 of the algebra L_5. Proceeding as above, the invariant solution of equation (6) is given by

$$\psi = \frac{1}{x^2 + z^2}\Phi(\xi), \quad \text{where} \quad \xi = \frac{x^2 + z^2}{1 + 2t}$$

and Φ satisfies the following nonlinear differential equation:

$$4\xi^2\Phi'' + (2\xi^2 - 4\xi)\Phi' + 4\Phi + \Phi^2 = 0. \tag{7}$$

We observe that $\Phi_0 = -4$ is a constant solution of equation (7). By letting $\Phi = \Phi_0 + \Phi_1$ we linearize the equation and obtain

$$\Phi_1'' + \left(\frac{1}{2} - \frac{1}{\xi}\right)\Phi_1' - \frac{1}{\xi^2}\Phi_1 = 0. \tag{8}$$

If we let

$$\Phi_1 = ye^{-\frac{1}{2}\int\left(\frac{1}{2} - \frac{1}{\xi}\right)d\xi}$$

and substitute in equation (8), we see that y satisfies the second-order differential equation

$$y'' = q(\xi)y \tag{9}$$

where

$$q(\xi) = \frac{1}{4}\left(\frac{1}{4} - \frac{1}{\xi} + \frac{7}{\xi^2}\right).$$

The Liouville-Green approximation for the general solution of equation (9) is given by (see for example [5])

$$y = Aq^{\frac{-1}{4}}e^{\int q^{\frac{1}{2}}d\xi} + Bq^{\frac{-1}{4}}e^{-\int q^{\frac{1}{2}}d\xi},$$

where A and B are arbitrary constants.

Hence an approximate invariant solution of equation (6) is given by

$$\psi = \frac{1}{x^2+z^2}\left[-4 + e^{-\frac{1}{2}\int\left(\frac{1}{2}-\frac{1}{\xi}\right)d\xi}\left\{Aq^{\frac{-1}{4}}e^{\int q^{\frac{1}{2}}d\xi} + Bq^{\frac{-1}{4}}e^{-\int q^{\frac{1}{2}}d\xi}\right\}\right].$$

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