PROCEEDINGS OF THE INTERNATIONAL CONFERENCE ON
DYNAMICAL SYSTEMS AND DIFFERENTIAL EQUATIONS
May 18 – 21, 2000, Atlanta, USA pp. 241–247

EIGENVALUES OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH SINGULARITIES

KUNQUAN LAN

Department of Mathematics and Statistics
York University, Toronto, Canada, M3J 1P3

Abstract. We study the existence of eigenvalues of a Hammerstein integral equation of the form

$$\lambda z(t) = \int_G k(t,s)f(s,z(s))\,ds, \quad t \in G,$$

where k and f need not be continuous. As applications of our results, we consider the existence of eigenvalues of the equation

$$\lambda z''(t) + f(t,z(t)) = 0, \quad \text{a.e. on } [0,1],$$

subject to the well-known general separated boundary conditions. Our results generalize many known results.

1. **Introduction.** We consider the existence of eigenvalues for a Hammerstein integral equation of the form

$$\lambda z(t) = \int_G k(t,s)f(s,z(s))\,ds, \quad t \in G. \tag{1}$$

We allow k and f to have singularities. We shall apply our results to the existence of eigenvalues for a second order differential equation of the form

$$\lambda z''(t) + f(t,z(t)) = 0, \quad \text{a.e. on } [0,1], \tag{2}$$

with the well-known general separated boundary conditions.

Eqs. (1) and (2) were studied in [4], where $f(t,u) \equiv h(u)$ and k and h are continuous. A well-known result on the existence of eigenvalues for compact maps was used. Eq. (2) was studied by using iterative techniques in [11]-[14], where $f(t,u) = g(t)u^{\mu}$. Existence of positive solutions for Eqs. (1) and (2) with $\lambda = 1$ has been widely studied in [1]-[9] and [15].

In this paper we shall prove new results on the existence of eigenvalues for Eqs. (1) and (2) under weak assumptions on k and f. We also use the well-known eigenvalue result for compact maps and employ a particular cone in $C(G)$, which was used in, for example, [1]-[2], [4] and [7]-[9]. Our hypotheses imposed on k and f are very general. Moreover, our method is much simpler than and different from that used in [4]. Our results generalize many well-known results.

1991 *Mathematics Subject Classification.* Primary: 47H30; Secondary: 34B16, 47H10.

2. **Eigenvalues of Hammerstein integral equations with singularities.** In this section we shall consider the existence of eigenvalues of a Hammerstein equation of the form

$$\lambda z(t) = \int_G k(t,s)f(s,z(s))\,ds \equiv Az(t), \quad t \in G, \tag{3}$$

where G is a bounded closed set in $\mathbb{R}^N$ with meas$(G) > 0$.

Let $r > 0$. We always assume the following conditions.

(C_1) $f : G \times [0,r] \to \mathbb{R}_+$ satisfies Carathéodory conditions on $G \times [0,r]$ and there exists a measurable function $g_r : G \to \mathbb{R}_+$ such that

$$|f(t,u)| \leq g_r(t) \quad \text{for almost all } t \in G \quad \text{and all } u \in [0,r].$$

(C_2) $k : G \times G \to \mathbb{R}_+$ satisfies the following conditions.

(*i*) For each $t \in G$, $k(t,.) : G \to \mathbb{R}_+$ is measurable.

(*ii*) For each $t \in G$, $\int_G k(t,s)g_r(s)\,ds < \infty$.

(*iii*) For each $\tau \in G$, $\lim_{t\to\tau} \int_G |k(t,s) - k(\tau,s)|g_r(s)\,ds = 0$.

(C_3) There exist a closed subset $G_0 \subset G$ with meas$(G_0) > 0$, $c \in (0,1)$ and a measurable function $\Phi : G \to \mathbb{R}_+$ such that $k(t,s) \leq \Phi(s)$ for all $t \in G$ and almost all $s \in G$ and

$$c\Phi(s) \leq k(t,s) \quad \text{for all } t \in G_0 \quad \text{and almost all } s \in G.$$

Conditions (C_1)-(C_3) can be found in [8]. As remarked in [8], f is allowed to have singularities in its first variable and k is allowed to have singularities in its second variable.

We denote by $C(G)$ the Banach space of all continuous functions from G into $\mathbb{R}$ with the usual maximal norm. Let $P = \{x \in C(G) : x(t) \geq 0 \text{ for all } t \in G\}$. We define $K = \{x \in P : \min\{x(t) : t \in G_0\} \geq c\|x\|\}$. Then P and K are cones in $C(G)$.

Let $K_r = \{x \in K : \|x\| < r\}$, $\overline{K}_r = \{x \in K : \|x\| \leq r\}$ and $\partial K_r = \{x \in K : \|x\| = r\}$.

We need the following well-known compactness result (see Lemma 2.2 in [8]).

Lemma 1. *Under the hypotheses (C_1)-(C_3) the map A defined in (3) maps $\overline{K}_r$ into K and is compact.*

The following result is a special of Theorem 2.3.6 in [4].

Lemma 2. *Assume that $A : \overline{K}_r \to K$ is a compact map and satisfies*

$$\inf\{\|Ax\| : x \in \partial K_r\} > 0.$$

Then there exist $\lambda_0 > 0$ and $z_0 \in \partial K_r$ such that $\lambda_0 z_0 = Az_0$.

Now, we are in a position to give our main result.

Theorem 1. *Assume that there exist $\rho \in (0,r]$ and a measurable function $\psi_\rho : G_0 \to \mathbb{R}_+$ such that*

(i) $\tau := \sup\{\int_{G_0} k(t,s)\psi_\rho(s)\,ds : t \in G\} > 0$ and

(ii) $f(s,u) \geq \psi_\rho(s)$ for all $u \in [c\rho, \rho]$ and almost all $s \in G_0$.

Then there exist $\lambda_0 > 0$ and $z_0 \in \partial K_\rho$ such that Eq. (3) holds.

Proof. By Lemma 1 $A : \overline{K}_r \to K$ is compact. Let $x \in \partial K_\rho$. Then $c\rho \leq x(t) \leq \rho$ for $t \in G_0$. By hypotheses (*i*) and (*ii*), we have for $t \in G$,

$$\|Ax\| \geq \int_{G_0} k(t,s)f(s,x(s))\,ds \geq \int_{G_0} k(t,s)\psi_\rho(s)\,ds.$$

This implies $\|Ax\| \geq \tau$ and $\inf\{\|Ax\| : x \in \partial K_\rho\} \geq \tau > 0$. The result follows from Lemma 2. □

As special cases of Theorem 1, we obtain the following results.

Corollary 1. *Assume that $f : G \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (C_1) for each $r > 0$ and (ii) and (iii) of (C_2) hold for each $r > 0$. Assume that k satisfies (C_3) and $k(t,.) : G \to \mathbb{R}_+$ is integrable for each $t \in G$. Assume that the following conditions hold.*

(i') $\sup\{\int_{G_0} k(t,s)\,ds : t \in G\} > 0$.

(ii') There exist $\sigma > 0$ and $r_0 > 0$ such that

$$f(t,u) \geq \sigma \quad \text{for all } u \in (0, r_0) \quad \text{and almost all } s \in G_0.$$

Then for each $\rho \in (0, r_0)$ there exist $\lambda_\rho > 0$ and $z_\rho \in \partial K_\rho$ such that Eq. (3) holds.

Proof. Let $\rho \in (0, r_0)$. We define $\phi_\rho : G_0 \to \mathbb{R}_+$ by $\phi_\rho(t) = \sigma$. Then (i') and (ii') imply (i) and (ii) of Theorem 1 hold. The result follows from Theorem 1. □

Corollary 2. *In corollary 1, if (ii') is replaced by the following condition:*

(ii'') There exist $\sigma > 0$ and $r_0 > 0$ such that

$$f(t,u) \geq \sigma \quad \text{for all } u \in (r_0, \infty) \quad \text{and almost all } s \in G_0.$$

Then for each $\rho \in (r_0/c, \infty)$ there exist $\lambda_\rho > 0$ and $z_\rho \in \partial K_\rho$ such that Eq. (3) holds.

Proof. Let $\rho \in (r_0/c, \infty)$. We define $\phi_\rho : G_0 \to \mathbb{R}_+$ by $\phi_\rho(t) = \sigma$. Then ϕ_ρ satisfies (i) and (ii) of Theorem 1 hold. The result follows from Theorem 1. □

Now, we consider eigenvalue problems of the following equation

$$\lambda z(t) = \int_G k(t,s) g(s) h(z(s))\,ds, \quad t \in G. \tag{4}$$

We assume that the following conditions hold.

(1) $h : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous.

(2) (C_2) with $g_r = g$ and (C_3) hold.

Theorem 2. *Assume that $\sup\{\int_{G_0} k(t,s)g(s)\,ds : t \in G\} > 0$ and the following condition holds.*

(S) There exist $\rho > 0$ and $\sigma > 0$ such that

$$h(u) \geq \sigma \quad \text{for all } u \in [c\rho, \rho].$$

Then there exist $\lambda_0 > 0$ and $z_0 \in \partial K_\rho$ such that Eq. (4) holds.

Proof. We define $f : G \times [0, \rho] \to \mathbb{R}_+$ by $f(t,u) = g(t)h(u)$. It is easy to verify that all the conditions of Theorem 1 hold. The result follows from Theorem 1. □

As special cases of Theorem 2 we obtain the following results which generalize known results.

Corollary 3. *Assume that $\sup\{\int_{G_0} k(t,s)g(s)\,ds : t \in G\} > 0$ and there exists $r_0 > 0$ such that*

$$h(u) > 0 \quad \text{for all } u \in (0, r_0).$$

Then for each $\rho \in (0, r_0)$ there exist $\lambda_\rho > 0$ and $z_\rho \in \partial K_\rho$ such that Eq. (4) holds.

Proof. Let $\rho \in (0, r_0)$. Since h is continuous on $[c\rho, \rho]$ and $h(u) > 0$ for $u \in [c\rho, \rho]$, there exists $\sigma > 0$ such that $f(u) \geq \sigma$ for all $u \in [c\rho, \rho]$. The result follows from Theorem 2. □

Remark 1. Corollary 3 generalizes Theorem 3.2.7 in [4], where $g \equiv 1$. Moreover, our proof is different from that in [4].

By a similar argument, we obtain the following result which generalizes Theorem 2.3.8 in [4].

Corollary 4. *Assume that* $\sup\{\int_{G_0} k(t,s)g(s)\,ds : t \in G\} > 0$ *and there exists* $r_0 > 0$ *such that*

$$f(u) > 0 \quad \text{for all } u \in (r_0, \infty).$$

Then for each $\rho \in (r_0/c, \infty)$ *there exist* $\lambda_\rho > 0$ *and* $z_\rho \in \partial K_\rho$ *such that Eq.* (4) *holds.*

3. **Eigenvalues of $\lambda z'' + f(t, z(t)) = 0$.** We consider the existence of eigenvalues for a second order differential equation of the form

$$\lambda z''(t) + f(t, z(t)) = 0, \quad \text{a.e. on } [0,1], \tag{5}$$

subject to the following general separated boundary conditions.

$$\begin{cases} \alpha z(0) - \beta z'(0) = 0 \\ \gamma z(1) + \delta z'(1) = 0, \end{cases} \tag{6}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ and $\Gamma := \gamma\beta + \alpha\gamma + \alpha\delta > 0$.

It is known that (6) contains eight sorts of boundary conditions.

(B_1) $z(0) = z(1) = 0$.
(B_2) $z(0) = z'(1) = 0$.
(B_3) $\alpha z(0) = \beta z'(0)$ and $z(1) = 0$ with $\alpha, \beta > 0$.
(B_4) $\alpha z(0) = \beta z'(0)$ and $z'(1) = 0$ with $\alpha, \beta > 0$.
(B_5) $\alpha z(0) = \beta z'(0)$ and $\gamma z(1) = -\delta z'(1)$ with $\alpha, \beta > 0, \gamma, \delta > 0$.
(B_2') $z'(0) = z(1) = 0$.
(B_3') $\gamma z(1) = -\delta z'(1)$ with $\gamma, \delta > 0$ and $z(0) = 0$.
(B_4') $\gamma z(1) = -\delta z'(1)$ with $\gamma, \delta > 0$ and $z'(0) = 0$.

Let k be the Green's function to $-z'' = 0$ subject to (6). We define $k : [0,1] \times [0,1] \to \mathbb{R}_+$ by

$$k(t,s) = \frac{1}{\Gamma}\begin{cases} (\gamma + \delta - \gamma t)(\beta + \alpha s) & \text{if } 0 \leq s \leq t \leq 1, \\ (\beta + \alpha t)(\gamma + \delta - \gamma s) & \text{if } 0 \leq t < s \leq 1. \end{cases} \tag{7}$$

We always choose a, b, c which satisfy the following conditions.

(i) $a, b \in [0,1]$ satisfy $-\beta/\alpha < a < b < 1 + \delta/\gamma$, where $\beta/\alpha = \infty$ if $\alpha = 0$ and $\delta/\gamma = \infty$ if $\gamma = 0$.

(ii) $c = \min\{(\gamma + \delta - \gamma b)/(\gamma + \delta), (\beta + \alpha a)/(\alpha + \beta)\}$.

It is well-known that k satisfies $k(t,s) \leq k(s,s)$ for $t, s \in [0,1]$ and $ck(s,s) \leq k(t,s)$ for $t \in [a,b]$ and $s \in [0,1]$ (see [7]). Hence, k satisfies (C_3) with $\Phi(s) = k(s,s)$, $G = [0,1]$ and $G_0 = [a,b]$.

We assume the following conditions.

(C) $f : [0,1] \times [0,r] \to \mathbb{R}_+$ satisfies Carathéodory conditions on $[0,1] \times [0,r]$ and there exists a measurable function $g_r : [0,1] \to \mathbb{R}_+$ such that

(i) $\int_0^1 k(s,s)g_r(s)\,ds < \infty$ and

(ii) $f(t,u) \leq g_r(t)$ for almost all $t \in [0,1]$ and all $u \in [0,r]$.

As mentioned in [8], the condition (i) of (C) implies that g_r is integrable under (B_4), (B_4') and (B_5) since $k(s,s) \geq k(1,0) = \beta\delta/\Gamma > 0$ for $t, s \in [0,1]$. However, under other boundary conditions, g_r need not be integrable.

Eq. (5)-(6) can be transformed into the following Hammerstein integral equation

$$\lambda z(t) = \int_0^1 k(t,s) f(s, z(s))\, ds, \quad t \in [0,1], \tag{8}$$

where k is defined by (7). It is easy to see that if λ and $z \in \overline{K}_r$ satisfy Eq. (8), then λ and z also satisfy Eq. (5)-(6). Moreover, we remark that, under condition (C), f and k satisfy conditions (C_1)-(C_3) with $G = [0,1]$, $G_0 = [a,b]$ and $\Phi(s) = k(s,s)$. Hence, the results obtained in the above section can be applied to treat Eq. (5) with (6).

We are now in a position to give our new results on the existence of eigenvalues of Eq. (5)-(6).

By Theorem 1, we obtain

Theorem 3. *Assume that there exist $\rho \in (0, r]$ and a measurable function $\psi_\rho : [a,b] \to \mathbb{R}_+$ such that*
(i) $\sup\{\int_a^b k(t,s)\psi_\rho(s)\, ds : t \in [0,1]\} > 0$ *and*
(ii) $f(s,u) \geq \psi_\rho(s)$ *for all $u \in [c\rho, \rho]$ and almost all $s \in [a,b]$.*
Then there exist $\lambda > 0$ and $z \in \partial K_\rho$ such that Eq. (5)-(6) holds.

As a special case of Theorem 3, we obtain

Corollary 5. *Assume that there exists $\alpha > 0$ such that*

$$f(s,u) \geq \alpha \quad \text{for } u \in [c\rho, \rho] \quad \text{and almost all } s \in [a,b].$$

Then the result of Theorem 3 holds.

Proof. By (3.5) in [8] we have $k(t,s) > ck(s,s)$ for $t \in [a,b]$ and $s \in (0,1)$. Hence, $\int_a^b k(t,s)\, ds > 0$ for each $t \in [a,b]$. The result follows from Theorem 3. □

Now, we consider the existence of an equation of the form

$$\lambda z''(t) + g(t)h(u) = 0, \quad \text{a.e. on } [0,1]. \tag{9}$$

with the boundary condition (6).

We always assume that the following conditions.
(i) $g : [0,1] \to \mathbb{R}_+$ is measurable.
(ii) $\int_0^1 k(s,s)g(s)\, ds < \infty$.
(iii) $h : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous.
By Theorem 2, we have

Theorem 4. *Assume that* $\sup\{\int_a^b k(t,s)g(s)\, ds : t \in [0,1]\} > 0$ *and the following condition holds.*
(S') There exist $\rho > 0$ and $\sigma > 0$ such that

$$h(u) \geq \sigma \quad \text{for all } u \in [c\rho, \rho].$$

Then there exist $\lambda > 0$ and $z \in \partial K_\rho$ such that Eq. (9)-(6) holds.

As special cases of Theorem 4 we obtain the following results.

Corollary 6. *Assume that $\sup\{\int_a^b k(t,s)g(s)\, ds : t \in [0,1]\} > 0$ and there exists $\sigma > 0$ such that*

$$h(u) > 0 \quad \text{for all } u \in [c\rho, \rho].$$

Then there exist $\lambda > 0$ and $z \in \partial K_\rho$ such that Eq. (9)-(6) holds.

Corollary 7. *Assume that* $\sup\{\int_a^b k(t,s)g(s)\,ds : t \in [0,1]\} > 0$ *and there exists* $r_0 > 0$ *such that*

$$h(u) > 0 \quad \textit{for all } u \in (0, r_0).$$

Then for each $\rho \in (0, r_0)$ *there exist* $\lambda_\rho > 0$ *and* $z_\rho \in \partial K_\rho$ *such that Eq.* (9)-(6) *holds.*

Remark 2. Corollary 7 generalizes Theorem 3.2.9 in [4], where $g \equiv 1$. Moreover, our proof is different from that in [4].

Corollary 8. *Assume that* $h : [0, r] \to \mathbb{R}$ *is continuous and* $h(0) = 0$. *Assume that the following condition holds.*

$$0 < |h'_+(0)| \leq \infty.$$

Then there exists $r_0 \in (0, r]$ *such that for each* $\rho \in (0, r_0)$ *there exist* $\lambda_\rho \neq 0$ *and* $z_\rho \in \partial K_\rho$ *such that the equation* $\lambda z''(t) + h(z) = 0$ *with the boundary condition* (6) *holds.*

Proof. We consider two cases.

(1) If $0 < h'_+(0) \leq \infty$, then there exist $r_0 \in (0, r]$ and $\sigma > 0$ such that $h(u) \geq \sigma u$ for $u \in (0, r_0)$. We have shown in the proof of Corollary 5 that $\int_a^b k(t,s)\,ds > 0$ for $t \in [a, b]$. The result follows from Theorem 4.

(2) If $-\infty \leq h'_+(0) < 0$, then there exist $r_0 \in (0, r]$ and $\sigma > 0$ such that $-h(u) \geq \sigma u$ for $u \in (0, r_0)$. The result follows from (1). □

Remark 3. Corollary 8 generalizes Theorem 3.2.5 (a), (b) in [4], where the boundary conditions are (B_1) and (B_2). Moreover, our method is different from that used in [4].

Corollary 9. *Assume that* $\sup\{\int_a^b k(t,s)g(s)\,ds : t \in [0,1]\} > 0$ *and there exists* $r_0 > 0$ *such that*

$$h(u) > 0 \quad \textit{for all } u \in (r_0, \infty).$$

Then for each $\rho \in (r_0/c, \infty)$ *there exist* $\lambda_\rho > 0$ *and* $z_\rho \in K$ *with* $\|z_\rho\| = \rho$ *such that the equation* $\lambda z''(t) + h(z) = 0$ *with the boundary condition* (6) *holds.*

Remark 4. Corollary 9 generalizes Theorem 3.2.10 in [4]. Corollary 9 also generalizes Example 2.3.2 in [4], where h satisfies $0 < \liminf_{u \to +\infty} h(u)/u \leq \infty$.

REFERENCES

[1] L. H. Erbe, S. Hu and H. Wang, *Multiple positive solutions of some boundary valued problems*, J. Math. Anal. Appl. 184 (1994), 640-648.

[2] L. H. Erbe and H. Wang, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc., 120 (1994), 743-748.

[3] A. M. Fink, J. A. Gatica and G. E. Hernandez, *Eigenvalues of generalized Gelfand models*, Nonlinear Analysis, 20 (1993), 1453-1468.

[4] D. Guo and V. Lakshmikantham, "Nonlinear Problems in Abstract Cones," Academic Press, 1988.

[5] J. A. Gatica and H. L. Smith, *Fixed point techniques in a cone with applications*, J. Math. Anal. Appl. 61 (1977), 58-71.

[6] G. B. Gustafson and K. Schmitt, *Nonzero solutions of boundary value problems for second order ordinary and delay-differential equations*, J. Differential Equations 12 (1972), 129-147.

[7] K. Q. Lan, *Multiple positive solutions of semilinear differential equations with singularities*, J. London Math. Soc., to appear.

[8] K. Q. Lan, *Multiple positive solutions of Hammerstein integral equations with singularities*, Differential Equations and Dynamical Systems, 8 (2000), 175-192.

[9] K. Q. Lan and J. R. L. Webb, *Positive solutions of semilinear differential equations with singularities*, J. Differential Equations 148 (1998), 407-421.
[10] Z. Liu and F. Li, *Multiple positive solutions of nonlinear two-point boundary value problems*, J. Math. Anal. Appl. 203 (1996), 610-625.
[11] C. D. Luning, *An iterative technique for obtaining solutions of a Thomas-Fermi equation*, SIAM J. Math. Anal. 9 (1978), 515-522.
[12] C. D. Luning and W. L. Perry, *An iterative technique for solution of the Thomas-Fermi equation utilizing a nonlinear eigenvalue problem*, Quart. Appl. Math. 35 (1977), 257-268.
[13] C. D. Luning and W. L. Perry, *Positive solutions of superlinear eigenvalue problems via a monotone iterative technique*, J. Differential Equations 33 (1979), 359-367.
[14] C. D. Luning and W. L. Perry, *Positive solutions of negative exponent generalized Emden-Fowler boundary value problems*, SIAM J. Math. Anal. 12 (1981), 874-879.
[15] J. S. W. Wong, *On the generalized Emden-Fowler equation*, SIAM Rev. 17 (1975), 339-360.

E-mail address: kunquan@yorku.ca