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IMPULSIVE EFFECTS ON THE EXISTENCE OF SOLUTIONS FOR A FAST DIFFUSION EQUATION

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Abstract. Let α (< 1) be a positive constant. This article studies the following impulsive problem: for $n = 1, 2, 3, \ldots$,

$$u_t - (u^\alpha)_{xx} = \lambda f(u),\ 0 < x < 1,\ (n-1)T < t \leq nT^-,$$

$$u(x, 0) = 0,\ 0 \leq x \leq 1,$$

$$u(x, nT) = \sigma u(x, nT^-),\ 0 \leq x \leq 1,$$

$$u(0, t) = 0 = u(1, t),\ t > 0.$$

The number λ^* is called the critical value if the problem has a unique global solution u for $\lambda < \lambda^*$, and the solution quenches in a finite time for $\lambda > \lambda^*$. The existence of a unique λ^* is established.

1. **Introduction.** Owing to short-term perturbations, many evolution processes at certain moments of time experience changes of state abruptly. Since the durations of the perturbations are negligible in comparison with the duration of each process, it is natural to assume that these perturbations act instantaneously in the form of impulses (cf. Lakshmikantham, Bainov, and Simeonov [6]). The impulsive effects on quenching were first studied by Chan, Ke and Vatsala [2] for a semilinear heat equation, and Chan and Kong [3] for a degenerate semilinear equation. The main purpose here is to extend these results to the fast diffusion operator L, where

$$Lu = u_t - (u^\alpha)_{xx},$$

for some positive constant α ($\alpha < 1$). This problem arises in the area of plasma physic, where u denotes plasma density, and the diffusion coefficient, $\alpha u^{\alpha-1}$, tends to infinity as u tends to 0.

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Let σ, λ, and T be positive constants. We consider the following impulsive problem: for $n = 1, 2, 3, \cdots$,

$$\left.\begin{aligned} Lu = \lambda f(u),\ 0 < x < 1,\ (n-1)T < t \le nT^-, \\ u(x,0) = 0,\ 0 \le x \le 1, \\ u(x,nT) = \sigma u(x,nT^-),\ 0 \le x \le 1, \\ u(0,t) = 0 = u(1,t),\ t > 0, \end{aligned}\right\} \tag{1.1}$$

where $f(0) > 0$, $f' > 0$, $f'' \ge 0$, and $\lim_{u\to 1^-} f(u) = +\infty$. A solution u of the problem (1.1) is said to quench if there is a $m \in \mathbb{N}$ and a positive number τ such that $(m-1)T < \tau \le mT^-$ and

$$\max\{u(x,t) : 0 \le x \le 1\} \to 1 \text{ as } t \to \tau.$$

The number λ^* is called the critical value if the problem has a unique global solution u for $\lambda < \lambda^*$, and the solution quenches in a finite time for $\lambda > \lambda^*$. If $\sigma = 1$, then there is no impulse; Deng [4] shows that for $f(u) = (1-u)^{-p}$ with $p > 0$, the problem has a critical value λ^*. When $\sigma \ne 1$, the problem (1.1) describes an impulse proportional to u (with σ as the proportionality constant) being given at each time interval T.

By the transformation,

$$u_n(x, t-(n-1)T) = u(x,t),\ (n-1)T < t \le nT^-,\ n = 1, 2, 3, \cdots,$$

the problem (1.1) can be written as

$$\left.\begin{aligned} Lu_n(x,t) = \lambda f(u_n(x,t)) \text{ in } (0,1)\times(0,T^-], \\ u_1(x,0) = 0,\ 0 \le x \le 1, \\ u_{n+1}(x,0) = \sigma u_n(x,T^-),\ 0 \le x \le 1, \\ u_n(0,t) = 0 = u_n(1,t),\ 0 < t \le T^-. \end{aligned}\right\} \tag{1.2}$$

Thus, global existence of a solution of the problem (1.1) is now equivalent to existence of u_n for all positive integers n.

2. **Main Results.** We begin with the definition of solution, subsolution, and supersolution of the problem (1.2). Let $D_{T_0} = (0,1)\times(0,T_0)$ for $T \ge T_0 > 0$.

Definition 1. *A solution u_n of the problem (1.2) on $[0,T_0]$, is a function u_n with the following properties:*

1. $u_n \in C\left([0,T_0]; L^1((0,1))\right) \cap L^\infty(D_{T_0})$*;*
2. *u_n satisfies*

$$\begin{aligned} &\int_0^1 u_n(x,t)\varphi(x,t)dx \\ &\quad - \int_0^t\int_0^1 \left(u_n(x,s)\varphi_t(x,s) + u_n^\alpha(x,s)\varphi_{xx}(x,s)\right)dxds \\ &= \int_0^1 u_n(x,0)\varphi(x,0)dx + \lambda\int_0^t\int_0^1 f(u_n(x,s))\varphi(x,s)dxds, \quad (1.3) \\ u_{n+1}(x,0) &= \sigma u_n(x,T^-)(<1),\ u_0(x) = 0, \end{aligned}$$

for $0 \le t \le T_0$ *and* $\varphi \in C^{2,1}\left(\bar{D}_{T_0}\right)$, *with* $\varphi \ge 0$ *on* D_{T_0}, $\varphi = 0$ *at* $x = 0$ *and* $x = 1$.

A subsolution (supersolution) of problem (1.2) is defined by 1. and 2. with equality replaced by $\le$ $(\ge)$.

By the Theorem 2.4 of Filo [5], since $f(u)$ is locally Lipschitz, we have the following comparison theorem.

Theorem 1. *Let* $u(< 1)$ *be a supersolution and* $v(< 1)$ *be a subsolution of the problem (1.2) with* $u(x, 0) \ge v(x, 0)$ *on* $(0, 1)$, *then* $u \ge v$ *on* D_T.

In Theorem 2.2 of Deng [4], the following local existence of the solution of the problem (1.1) was proved by use of the Theorem 1.

Theorem 2. *There exists a* $T_0 > 0$ *such that problem (1.1) has a unique nonnegative solution on* D_{T_0}.

We now prove that u_n increases as λ increases.

Lemma 3. *Let* $u_n(x, t; \lambda)$ *denotes the solution of the problem (1.2). If* $\lambda_1 > \lambda_2$, *then*

$$u_n(x, t; \lambda_1) \ge u_n(x, t; \lambda_2).$$

Proof: Since $u_1(x, 0; \lambda_1) = u_1(x, 0; \lambda_2) = 0$, and

$$\lambda_1 \int_0^t \int_0^1 f(u_1(x, s; \lambda_1))\varphi(x, s)dxds \ge \lambda_2 \int_0^t \int_0^1 f(u_1(x, s; \lambda_1))\varphi(x, s)dxds$$

for any $\varphi \ge 0$, we have $u_1(x, t; \lambda_1) \ge u_1(x, t; \lambda_2)$. By the principle of mathematical induction, the lemma is then proved.

Let $\varphi(x) = \frac{\pi}{2} \sin \pi x$. Then $\varphi(x)$ is the solution of the eigenvalue problem

$$\begin{aligned} \varphi'' &= -\mu\varphi, \text{ for } 0 < x < 1 \\ \varphi(0) &= 0 = \varphi(1), \end{aligned}$$

with $\mu = \pi^2$ and $\int_0^1 \varphi(x)dx = 1$.

Let $F_n(t) = \int_0^1 u_n(x, t)\varphi(x)dx$. We now prove the existence of λ^*.

Theorem 4. *The problem (1.2) has a unique critical value.*

Proof: Firstly, we claim that if $\lambda > \dfrac{1 + \mu T}{T f(0)}$, then the solution of the problem (1.2) quenches in finite time. Suppose that this is false, then for some $\lambda > \dfrac{1 + \mu T}{T f(0)}$ the problem (1.2) has a nononegative solution u_n (< 1), and hence $F_n(t) < 1$ for $t \in [0, T]$. By (1.3), with $u_1(x, 0) = 0$, we get

$$\begin{aligned} F_1(T) &= \int_0^T \int_0^1 (-\mu)\, u_1^\alpha(x, s)\varphi(x)dxds + \lambda \int_0^T \int_0^1 f(u_1(x, s))\varphi(x)dxds \\ &\ge -\mu T + \lambda f(0)T \\ &= T\left(\lambda f(0) - \mu\right). \end{aligned}$$

But $T\left(\lambda f(0) - \mu\right) > 1$, a contradiction. Therefore the solution of the problem quenches.

Next, we claim that the solution exists globally for λ is small.

For the case $\sigma > 1$, let $w(x,t) = kg(t)\left[x(1-x)\right]^{1/\alpha}$ with $g(t) = \left(\sigma + (1-\sigma)\frac{t}{T}\right)$, $0 \le t \le T$, where k is a positive constant to be determined. Now

$$w_t = -k\left[x(1-x)\right]^{1/\alpha}\frac{(\sigma-1)}{T} < 0,$$

$$\left(w^\alpha\right)_{xx} = -2k^\alpha g^\alpha(t),$$

and $w(x,0) = \sigma w(x,T)$, $w(0,t) = 0 = w(1,t)$. Since $w_t < 0$, we have $\max w(x,t) = \max w(x,0) = \frac{k\sigma}{4^{1/\alpha}}$. Now if we take $k < \min\left[\left(\frac{2T}{4^{1/\alpha}}(\sigma-1)\right)^{1/(1-\alpha)}, \frac{4^{1/\alpha}}{\sigma}\right]$, then when

$$\lambda < \frac{\frac{k(1-\sigma)}{4^{1/\alpha}T} + 2k^\alpha}{f\left(\frac{k\sigma}{4^{1/\alpha}}\right)},$$

we get

$$w_t - \left(w^\alpha\right)_{xx} \ge -\frac{\sigma-1}{T}k\left(\frac{1}{4}\right)^{1/\alpha} + 2k^\alpha > \lambda f\left(\frac{k\sigma}{4^{1/\alpha}}\right) \ge \lambda f(w).$$

Therefore by the Theorem 1, $w(x,t) \ge u_1(x,t)$ on $\bar{D}_T$.

Let us assume that for some $n = i$, $w \ge u_i$ on $\bar{D}_T$, then $u_{i+1}(x,0) = \sigma u_i(x,T) \le \sigma w(x,T) = w(x,0)$. By Theorem 1 again, we get $w \ge u_{i+1}$ on $\bar{D}_T$, and hence $w \ge u_n$ for any positive integer n with $w(x,t) < 1$. Therefore, u_n exists.

For the case $\sigma < 1$, let $w(x) = k\left[x(1-x)\right]^{1/\alpha}$, for $k < 4^{1/\alpha}$. Then $w(x) < 1$ on $[0,1]$.

When $\lambda \le \frac{2k^\alpha}{f\left(\frac{k}{4^{1/\alpha}}\right)}$, we have

$$w_t - \left(w^\alpha\right)_{xx} = 2k^\alpha \ge \lambda f\left(\frac{k}{4^{1/\alpha}}\right) \ge \lambda f(w).$$

Now, $w(x) \ge 0$, and $w(0) = 0 = w(1)$, by Theorem 1, we get $w \ge u_1$ on $\bar{D}_T$. Since $w(x) \ge u_1(x,T) \ge \sigma u_1(x,T) = u_2(x,0)$, by Theorem 1 again, we have $w \ge u_2$. Thus, by the principle of mathematical induction $w \ge u_n$ on $\bar{D}_T$ for any positive integer n, and hence u_n exists.

Thus for λ is sufficiently small, the problem have a unique solution; and when λ is large, the solution quenches in a finite time, and hence the theorem follows.

We remark that in the Theorem 3 of Chan, Ke and Vatsala, they show that if $\sigma < 1$, the problem has a unique critical value when $T > (1-\sigma)/f(0)$ for the semilinear heat equation.

Next, we consider the problem with nontrivial initial condition. Let $u_0(x) \in L^\infty\left((0,1)\right)$ with $u_0 \ge 0$, $0 < \sup u_0 < 1$, and $0 < \int_0^1 u_0(x)\varphi(x)dx < 1$. We show that if $\sigma > 1$, then the solution quenches in finite time if T is small enough.

Recall that $F_n(t) = \int_0^1 u_n(x,t)\varphi(x)dx$, and hence $F_1(0) = \int_0^1 u_0(x)\varphi(x)dx > 0$.

Theorem 5. *For $\sigma > 1$, if $T < \dfrac{(\sigma-1)F_1(0)}{\mu\sigma}$, then the solution of the problem (1.2) with $u_1(x,0) = u_0(x)$ quenches in finite time for any $\lambda > 0$.*

Proof: By using a similar argument as in the proof of Theorem 2.2 of Deng [4] we can show that the problem (1.2) has a local solution.

If the theorem is false, then there exists $\lambda > 0$ such that the problem has a solution, and hence $F_n(t) < 1$ on $[0,T]$ for $n = 1,2,\cdots$. By using (1.3) and the Jensen's inequality, we get

$$\begin{aligned} F_n(t) &\geq \lambda\int_0^t f(F_n(s))\,ds - \mu\int_0^t (F_n(s))^\alpha\,ds + F_n(0) \\ &\geq \lambda f(0)t - \mu t + F_n(0) \end{aligned} \tag{1.4}$$

for $t \in [0,T]$ and $n = 1,2,\cdots$.

Since $\sigma\mu T < (\sigma-1)F_1(0)$, we get $F_1(0) - \mu T > \dfrac{1}{\sigma}F_1(0)$. Thus, for $n = 1$,

$$\begin{aligned} F_1(T) &\geq \lambda f(0)T - \mu T + F_1(0) \\ &> \lambda f(0)T + \frac{1}{\sigma}F_1(0). \end{aligned}$$

By (1.4), when $n = 2$,

$$\begin{aligned} F_2(T) &\geq \lambda f(0)T - \mu T + F_2(0) \\ &= \lambda f(0)T - \mu T + \sigma F_1(T) \\ &> \lambda f(0)T - \mu T + \sigma\left(\lambda f(0)T + \frac{1}{\sigma}F_1(0)\right) \\ &= \lambda(1+\sigma)f(0)T - \mu T + F_1(0) \\ &> \lambda(1+\sigma)f(0)T + \frac{1}{\sigma}F_1(0). \end{aligned}$$

By continuing the process, we get, for any positive integer n,

$$\begin{aligned} F_n(T) &> \lambda\left(1+\sigma+\sigma^2+\cdots+\sigma^{n-1}\right)f(0)T + \frac{1}{\sigma}F_1(0) \\ &= \lambda\frac{\sigma^n-1}{\sigma-1}f(0)T + \frac{1}{\sigma}F_1(0). \end{aligned}$$

Since $\sigma > 1$, it follows that $F_n(T) > 1$ for some n, a contradiction. This prove the theorem.

For the case $\sigma < 1$, we have the following global existence of the solution for the problem.

Theorem 6. *When $\sigma < 1$, for $\lambda > 0$, there exists $T > 0$, such that the solution u_n of the problem (1.2) with $u_1(x,0) = u_0$ exists.*

Proof: Let $b = \sup u_0$, and $g(t)$ be the solution of the problem

$$g'(t) = \lambda f(g(t)),\; g(0) = b. \tag{1.5}$$

We get $\dfrac{1}{\lambda}\displaystyle\int_b^{g(t)}\frac{1}{f(s)}ds = t$. Then $g(t)$ exists and bounded away from 1 for $0 \leq t \leq T_1$ where $T_1 < \dfrac{1}{\lambda}\displaystyle\int_b^1\frac{1}{f(s)}ds$.

If $b \geq \sigma$, we have $g(0) = b \geq \sigma \cdot 1 > \sigma g(t)$ for $0 \leq t \leq T_1$; if $b < \sigma$, since $g(t)$ is increasing and $\dfrac{b}{\sigma} > b$, there exists $T_2 \leq T_1$ such that $g(T_2) \leq \dfrac{b}{\sigma}$.

Therefore, in either case there exists $T > 0$ such that $g(t)\,(< 1)$ satisfies the problem (1.5) on $[0, T]$ with $g(0) \geq \sigma g(T)$. Then $g(t)$ is a supersolution of the problem (1.2) with $g(t) \geq u_n(x,t)$ for $n = 1, 2, \cdots$, and hence the solution of the problem exists.

Here we give an example for the Theorem 6: for $\sigma < 1$ and $f(u) = (1-u)^{-p}$, $p > 0$, we get

$$g(t) = 1 - \left[(1 - \sup u_0)^{1+p} - \lambda\,(p+1)\,t\right]^{1/(p+1)},$$

and hence if

$$T < \begin{cases} \dfrac{(1-\sup u_0)^{p+1}}{\lambda\,(p+1)}, & \text{if } \sup u_0 \geq \sigma \\[2ex] \dfrac{(1-\sup u_0)^{p+1} - \left(1 - \frac{\sup u_0}{\sigma}\right)^{p+1}}{\lambda\,(p+1)} & \text{if } \sup u_0 < \sigma, \end{cases}$$

the problem has a unique nonnegative solution.

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