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A TOPOLOGICAL CHARACTERIZATION OF ω-LIMIT SETS FOR CONTINUOUS FLOWS ON THE PROJECTIVE PLANE

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Abstract. We characterize topologically ω-limit sets for continuous flows on the projective plane $\mathbb{P}^2$. Namely, it is shown that if Ω is the ω-limit set of some orbit of a continuous flow in $\mathbb{P}^2$ then it is the boundary of a region $\emptyset \subsetneq O \subsetneq \mathbb{P}^2$ whose complementary is connected. Conversely, if Ω is the boundary of a region $\emptyset \subsetneq O \subsetneq \mathbb{P}^2$ whose complementary is connected then there is a (C^∞) smooth flow on $\mathbb{P}^2$ having Ω as the ω-limit set of one of its orbits. This answers a question by Anosov.

1. **Introduction.** In what follows, $\mathcal{M}$ will always denote a compact connected surface (here the word "surface" is used in a topological sense, that is, $\mathcal{M}$ is Hausdorff and second countable, and any point of $\mathcal{M}$ has a neighborhood which is homeomorphic to a disk). Recall that if O is an open subset of $\mathcal{M}$, a *continuous flow* on O is a continuous map $\Phi : \mathbb{R} \times O \to O$ satisfying: (i) $\Phi(0,u) = u$ for any $u \in O$; (ii) $\Phi(t+s,u) = \Phi(t,\Phi(s,u))$ for any $t, s \in \mathbb{R}$ and any $u \in O$.

We are interested in the study of the dynamics of continuous flows on $\mathcal{M}$, that is, the asymptotic behavior of the *orbits* $\Phi_u(t) = \Phi(t,u)$ when $t \to \infty$. In particular, the problem of characterizing topologically the subsets of $\mathcal{M}$ which can be ω-limit sets for continuous flows in $\mathcal{M}$ arises in a natural way. Here recall that the *ω-limit set* of an orbit $\Phi_u(t)$ [denoted $\omega_\Phi(u)$] is the set of its accumulation points when $t \to \infty$, that is,

$$\omega_\Phi(u) = \{v \in \mathcal{M} : \exists (t_n)_n \to \infty; (\Phi_u(t_n))_n \to v\}.$$

Curiously enough, up to now this problem has been only solved in the sphere case.

Theorem (Vinograd [8], cf. also Balibrea and Jiménez López [1]). *Let Φ be a continuous flow on $\mathbb{S}^2$ and let $u \in \mathbb{S}^2$. Then $\omega_\Phi(u)$ is the boundary of a simply connected region $\emptyset \subsetneq O \subsetneq \mathbb{S}^2$. Conversely, if Ω is the boundary of a simply connected region $\emptyset \subsetneq O \subsetneq \mathbb{S}^2$ then there are a smooth flow Φ on $\mathbb{S}^2$ and a point $u \in \mathbb{S}^2$ such that $\Omega = \omega_\Phi(u)$.*

Notice that we have spoken about "smooth flows" thus taking in fact into account the smooth structure on $\mathbb{S}^2$ (the word "smooth" will always refer to C^∞ differentiability). It is interesting to emphasize that, as is well known, any compact connected surface $\mathcal{M}$ admits a smooth structure (which is unique up to diffeomorphisms), see

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e.g. [2, p. 207]. Moreover, $\mathcal{M}$ can be embedded in $\mathbb{R}^n$ (with $n = 3$ or $n = 4$ depending on the surface) and thus, if $u \in \mathcal{M}$, its tangent plane $T\mathcal{M}_u$ can be seen as a subset of $\mathbb{R}^n$. Hence a *smooth vector field* on an open subset O of $\mathcal{M}$ can be conveniently defined as a smooth map $F : O \to \mathbb{R}^n$ such that $F(u) \in T\mathcal{M}_u$ for any $u \in O$, thus allowing us to relate flows with autonomous differential equations as usual. Namely, it can be proved that if $\Phi : \mathbb{R} \times O \to O$ is a smooth flow then there is a smooth vector field $F : O \to \mathbb{R}^n$ such that $\frac{\partial\Phi}{\partial t}(t, u) = F(\Phi(t, u))$ for any t and u and that, conversely, for any smooth vector field F on $\mathcal{M}$ there is a smooth flow Φ on $\mathcal{M}$ such that $\frac{\partial\Phi}{\partial t}(t, u) = F(\Phi(t, u))$ for any t and u.

As we said before, the problem of characterizing ω-limit sets for flows on surfaces remains open even in the torus case. Anosov proposed it in the particular case of the projective plane $\mathbb{P}^2$ (cf. [5, p. 39]), emphasizing that Vinograd's result does not further work here. Indeed, the figure below shows a subset of $\mathbb{P}^2$ (the union of the boundaries of the regions A, B and C) which is not the boundary of a simply connected region (because O includes non null homotopic Jordan curves) but it is the ω-limit set of the orbit having γ as its image for some appropriate flow on $\mathbb{P}^2$.

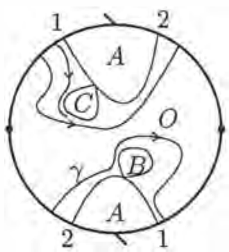

FIGURE 1. A counterexample for Vinograd's theorem in the projective plane.

The aim of this note is to obtain an analogous to Vinograd's result for $\mathbb{P}^2$, thus answering Anosov's question. Let us remark that ω-limit sets with non-empty interior have been characterized in the general case in [4].

Theorem. *Let Φ be a continuous flow on $\mathbb{P}^2$ and let $u \in \mathbb{P}^2$. Then $\omega_\Phi(u)$ is the boundary of a region $\emptyset \subsetneq O \subsetneq \mathbb{P}^2$ whose complementary is connected. Conversely, if Ω is the boundary of a region $\emptyset \subsetneq O \subsetneq \mathbb{P}^2$ with connected complementary, then there are a smooth flow Φ on $\mathbb{P}^2$ and a point $u \in \mathbb{P}^2$ such that $\Omega = \omega_\Phi(u)$.*

Of course simply connected regions in $\mathbb{S}^2$ are those having connected complementary so both ours and Vinograd's theorem admits a similar formulation.

Let us finish this section with some additional terminology. Recall that if γ is a Jordan curve (i.e., a homeomorphic curve to $\mathbb{S}^1$) in $\mathcal{M}$ then it has a neighborhood which is homeomorphic either to an annulus or a Möbius band. In the first case it is called *orientable* and in the second case *nonorientable.* Later it will be convenient to handle $\mathbb{P}^2$ either as the quotient space $\mathbb{S}^2/\sim$, where $u \sim u'$ if $u = -u'$ (when we will denote the equivalence class containing u by $[u]$ and $p : \mathbb{S}^2 \to \mathbb{P}^2$ will denote the quotient map) or a submanifold of $\mathbb{R}^4$ [as for instance the image of $\mathbb{S}^2$ under the map $f(x, y, z) = (yz, xz, xy, x^2 + 2y^2 + 3z^2)$]. Bd$(A)$ and Cl(A) will denote the *boundary* and the *closure* of A.

2. **An embedding lemma.** The aim of the section is to show that if we take off a nonorientable Jordan curve from $\mathbb{P}^2$ then the resultant surface is diffeomorphic to an open disk (see the proposition below). This result will be strongly used in Section 4. Although of course it is well known, we are unable to provide a precise reference; hence we give here a simple proof.

Proposition. *Let $\delta \subset \mathbb{P}^2$ be a nonorientable Jordan curve. Then $\mathbb{P}^2 \backslash \delta$ is diffeomorphic to an open disk.*

Proof. As according to [6] all smooth structures in $\mathbb{R}^2$ are diffeomorphic, it will be sufficient to show that $\mathbb{P}^2 \backslash \delta$ is homeomorphic to the complementary (in $\mathbb{S}^2$) of a closed disk. Here it will be convenient to see $\mathbb{P}^2$ as a subset of $\mathbb{R}^4$.

To begin with, fix open neighborhoods U, V of δ with U homeomorphic to the Möbius band and $\mathrm{Cl}(V) \subset U$, and let D_1 denote the open disk $\{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$. Then there is a homeomorphism $h: U \setminus \delta \to D_1 \setminus \{(0,0)\}$ such that $\|h(u)\| \to 1$ as u approaches δ. Extend h to a continuous map on the whole $\mathbb{P}^2 \setminus \delta$ by writing $h(u) = (0,0)$ for any $u \in \mathbb{P}^2 \setminus U$. Further, construct a continuous map $g: \mathbb{P}^2 \setminus \delta \to \mathbb{R}^4$ satisfying $g(u) = u$ for any $u \in \mathbb{P}^2 \setminus U$ and $g(u) = (0,0,0,0)$ for any $u \in V \setminus \delta$. Finally, let $f: \mathbb{P}^2 \setminus \delta \to \mathbb{R}^7$ be defined by $f(u) = (g(u), h(u), \|h(u)\|)$. Clearly f is injective and, since if $(u_n)_n$ is a sequence of points in $\mathbb{P}^2 \setminus \delta$ tending to δ then $(f(u_n))_n$ has no accumulation points in $\mathcal{N} := f(\mathbb{P}^2 \setminus \delta)$, we conclude that $\mathbb{P}^2 \setminus \delta$ and $\mathcal{N}$ are homeomorphic.

Obviously $\mathcal{M} = (\mathcal{N} \cup \{(0,0,0,0)\} \times \mathrm{Cl}(D_1) \times \{1\})$ is a compact connected surface. It only rests to show that it is homeomorphic to $\mathbb{S}^2$.

To do this, we consider a triangulation T_1 on $\mathbb{P}^2$ (see e.g. [3]) so that δ consists only of vertexes and edges of T_1. [A possible way to construct such a triangulation is to triangulate $\mathcal{M}$ first so that one of the corresponding faces, Q, lies in $\mathcal{M} \setminus \mathcal{N}$ and, since there is no restriction in assuming that $\mathcal{M} \setminus Q$ and $\mathbb{P}^2 \setminus U$ are homeomorphic, use this to triangulate $\mathbb{P}^2 \setminus U$ so that $\mathrm{Bd}(U)$ consists exactly of an even number of vertexes and an even number of edges of the induced triangulation; now it is easy to extend this triangulation to the whole $\mathbb{P}^2$ and get the desired T_1]. The triangulation T_1 can be now carried onto $\mathcal{N}$ and then extended in a natural way to a triangulation T_2 on $\mathcal{M}$ having as many new faces as the number of contained edges in δ.

Let Q_i, E_i and V_i denote the number of faces, edges and vertexes of the triangulation T_i, $i = 1, 2$, and let $E_\delta = V_\delta$ be the number of edges and vertexes of T_1 included in δ. Let $\chi(\mathcal{M})$ and $\chi(\mathbb{P}^2)$ denote the Euler characteristics of $\mathcal{M}$ and $\mathbb{P}^2$. Then

$$\chi(\mathcal{M}) = Q_2 - E_2 + V_2 = (Q_1 + 2E_\delta) - (E_1 + 3E_\delta) + (V_1 + E_\delta + 1) = \chi(\mathbb{P}^2) + 1 = 2.$$

Thus $\mathcal{M}$ is homeomorphic to $\mathbb{S}^2$ as we desired to proved. □

3. **Proof of the first statement of Theorem.** This part of the proof is rather simple. Write $\gamma = \Phi_u(\mathbb{R})$. If γ is a critical point or the image of a periodic orbit then $\mathbb{P}^2 \backslash \gamma$ has either one or two components, each of them having γ as its boundary, and the statement follows.

Next assume that neither of the above cases hold. Let $U = \mathbb{P}^2 \backslash \omega_\Phi(u)$ and decompose it into its connected components as $U = \bigcup_{j \in J} O_j$. Since $\omega_\Phi(u) \cap \gamma = \emptyset$ because of [7], there is $O := O_k \supset \gamma$. We are going to prove that $\mathrm{Bd}(O) = \omega_\Phi(u)$ and $\mathbb{P}^2 \backslash O$ is connected.

Assume that $\mathbb{P}^2 \backslash O = \omega_\Phi(u) \cup \bigcup_{j \neq k, j \in J} O_j$ is not connected. Then $\mathbb{P}^2 \backslash O = C \cup B$ with $C \neq \emptyset \neq B$ and $\mathrm{Cl}(C) \cap B = C \cap \mathrm{Cl}(B) = \emptyset$, and each of the sets O_j and $\omega_\Phi(u)$

must be included either in B or C. Say $B = (\bigcup_{j\in J_1} O_j) \cup \omega_\Phi(u)$ and $C = \bigcup_{j\in J_2} O_j$ where $k \notin J_1 \cup J_2$ and $J_1 \cup J_2 \cup \{k\} = J$. As $\mathrm{Bd}(O_j) \subset \omega_\Phi(u)$ for any $j \in J_2$ and $\mathrm{Cl}(C) \cap B = \emptyset$, we arrive to a contradiction.

We must prove now that $\mathrm{Bd}(O) = \omega_\Phi(u)$. Since $\gamma \subset O$ and $\omega_\Phi(u) \cap O = \emptyset$ we have $\omega_\Phi(u) \subset \mathrm{Bd}(O)$. Conversely, $\mathrm{Bd}(O) = \mathrm{Bd}(O_k) = \mathrm{Cl}(O_k)\backslash O_k \subset (O_k \cup \omega_\Phi(u))\backslash O_k = \omega_\Phi(u)$. This finishes the proof.

4. **Proof of the second statement of Theorem.** We must distinguish two cases:

1. O includes a nonorientable Jordan curve δ.
2. O includes only orientable Jordan curves.

Let us first assume that O includes a nonorientable Jordan curve δ. Use the proposition in Section 2 to construct an embedding $i : \mathbb{P}^2\backslash\delta \to \mathbb{S}^2$. Let $S = \mathbb{S}^2\backslash i(\mathbb{P}^2\backslash\delta)$. Since δ is nonorientable, there is a neighborhood $U \subset O$ of δ such that $U\backslash\delta$ is connected; hence, $O\backslash\delta$ is connected and so are $i(O\backslash\delta)$ and $V = i(O\backslash\delta) \cup S$. Since $\mathbb{P}^2\backslash O$ is connected, $i(\mathbb{P}^2\backslash O) = \mathbb{S}^2\backslash V$ is connected as well. Thus V is simply connected and we can use Vinograd's theorem (see also [1]) to construct a smooth flow $\Psi : \mathbb{R} \times \mathbb{S}^2 \to \mathbb{S}^2$ having $\mathrm{Bd}(V)$ as the ω-limit set of one of its orbits.

Let $F : \mathbb{S}^2 \to \mathbb{R}^3$ be the smooth vector field associated to Ψ. Multiply F by an appropriate scalar smooth function λ so that the new vector field $F_1 = \lambda F$ equals F on a neighborhood of $\mathrm{Bd}(V)$ and equals zero on an open set including S. If Ψ_1 is the flow associated to F_1 then $\mathrm{Bd}(V)$ is still one of its ω-limit sets but now the restriction $\Psi_1|_{\mathbb{R}\times(\mathbb{S}^2\backslash S)}$ is a flow on $\mathbb{S}^2\backslash S$ as well.

Let us consider now the composition

$$\Phi_1 : \mathbb{R} \times (\mathbb{P}^2\backslash\delta) \overset{\mathrm{Id}\times i}{\to} \mathbb{R} \times (\mathbb{S}^2\backslash S) \overset{\Psi_1}{\to} \mathbb{S}^2\backslash S \overset{i^{-1}}{\to} \mathbb{P}^2\backslash\delta,$$

which is a smooth flow on $\mathbb{P}^2\backslash\delta$ having $i^{-1}(\mathrm{Bd}(V)) = \mathrm{Bd}(O) = \Omega$ as one of its ω-limit sets. As $\Phi_1(t,u) = u$ for any $t \in \mathbb{R}$ and any u belonging to some neighborhood of δ, Φ_1 can be trivially extended to a smooth flow on the whole $\mathbb{P}^2$ having Ω as one of its ω-limit sets. This concludes the proof in Case 1.

Let us now assume that there are only orientable Jordan curves into O. See $\mathbb{P}^2$ as $\mathbb{S}^2/\sim$ and recall that $p : \mathbb{S}^2 \to \mathbb{P}^2$ denotes the quotient map. We next show that p maps bijectively each of the connected components of $p^{-1}(O)$ onto O. If we call one of them A, this implies in particular that $p^{-1}(O)$ has exactly two connected components, the other being $-A$, where we define $-Z = \{-u : u \in Z\}$ for any $Z \subset \mathbb{S}^2$.

First we prove that $p|_A$ is injective. Assume the opposite. Then we could find points u and $-u$ both belonging to A. Construct an arc $C \subset A$ having u and $-u$ as its endpoints. It is not restrictive to assume that C contains no other pair of antipodal points [otherwise take an (open) arc $D \subset C$ small enough having no such pairs, include it in a maximal arc $E \subset C$ with the same property and replace C by the closure of E]. Moreover, we can assume that there is a homeomorphism $\pi : [0,1] \times (0,1) \to U \subset A$ such that $\pi([0,1] \times \{1/2\}) = C$ and $\pi(0,t) = -\pi(1,1-t)$ for any $t \in [0,1]$. Then $p(C)$ is a Jordan curve in O having $p(U)$, a Möbius band, as a neighborhood. Thus $p(C)$ is nonorientable, a contradiction.

Let us prove now that $p(A) = O$. Otherwise the set $p(A)$ has a boundary point b in O (because O is connected). Notice that, since $p(A)$ is open (because p is an open map), $b \notin p(A)$. Find a sequence $(v_n)_n$ of points of $p(A)$ converging to b and take points u_n in A with $p(u_n) = v_n$; it is not restrictive to assume that $(u_n)_n$ converges, say to a. Since $p(a) = b$, $a \notin A$. Now it suffices to take a very small connected neighborhood U of a and note that $p(A \cup U) \subset O$, contradicting the definition of A.

Now we prove that A is simply connected. To see it suppose the opposite to find disjoint nonempty closed sets C_1, C_2 with $\mathbb{S}^2 \setminus A = C_1 \cup C_2$. We can for example assume (as $-A$ is connected) that $-A \subset C_2$. Since $A \cup C_1$ is open, $D := C_2 \cap (\mathbb{S}^2 \setminus ((-A) \cup (-C_1)))$ is closed. Thus both $p(C_1)$ and $p(D)$ are closed; moreover, we clearly have $p(C_1) \cap p(D) = \emptyset$. Further, it is easy to check that $p(C_1 \cup D) = \mathbb{P}^2 \setminus p(A) = \mathbb{P}^2 \setminus O$ and hence, by the hypothesis, $p(C_1 \cup D)$ is connected. Then $D = \emptyset$ and $C_2 \subset (-A) \cup (-C_1)$. From this, $\mathbb{S}^2 = (A \cup C_1) \cup ((-A) \cup (-C_1))$. But $-A \subset C_2$ implies $(-A) \cap C_1 = \emptyset$. Then $(A \cup C_1) \cap ((-A) \cup (-C_1)) = C_1 \cap (-C_1)$ is both open and closed, so $A \cup C_1$ and $(-A) \cup (-C_1)$ are in fact disjoint. Since $\mathbb{S}^2$ is connected, we arrive to a contradiction.

Since A is simply connected, we can reason as in [1] to construct a smooth vector field $F : \mathbb{S}^2 \to \mathbb{R}^3$ vanishing outside A and such that its associated flow has an orbit in A having $\mathrm{Bd}(A)$ as its ω-limit set. Define $G : \mathbb{S}^2 \to \mathbb{R}^3$ by $G(u) = F(u) - F(-u)$. Notice that this new vector field coincides with F on A and has the additional property that its associated flow Ψ satisfies $\Psi(t, u) = -\Psi(t, -u)$ for any $t \in \mathbb{R}$ and any $u \in \mathbb{S}^2$.

Let $\Phi : \mathbb{R} \times \mathbb{P}^2 \to \mathbb{P}^2$ be defined by $\Phi(t, [u]) = [\Psi(t, u)]$. It is a simple task to check that Φ is a well defined smooth flow on $\mathbb{P}^2$. Take an appropriate point $u_0 \in A$ so that $\mathrm{Bd}(A) = \omega_\Psi(u_0)$. Then we clearly have $\Omega = \omega_\Phi([u_0])$, which finishes the proof.

REFERENCES

[1] F. Balibrea and V. Jiménez López, *A characterization of the ω-limit sets of planar continuous dynamical sistems*, J. Diff. Eq. **145** (1996), 469–488.

[2] M. W. Hirsch, *Differential topology*, Springer-Verlag, New York, 1988.

[3] L. C. Kinsey, *Topology of surfaces*, Springer-Verlag, New York, 1993.

[4] V. Jiménez López and G. Soler López, *Transitive open sets in manifolds*, preprint, Universidad de Murcia, 2000.

[5] I. Nikolaev and E. Zhuzhoma, *Flows on 2-dimensional manifolds*, Lecture Notes in Mathematics **1705**, Springer-Verlag, Berlin, 1999.

[6] J. Stallings, *The piecewise linear structure of Euclidean space*, Proc. Camb. Phil. Soc. **58** (1962), 481–488.

[7] R. A. Smith and S. Thomas, *Transitive flows on two-dimensional manifolds*, J. London Math. Soc. **37** (1988), 569–576.

[8] R. E. Vinograd, *On the limiting behavior of an unbounded integral curve*, Moskov. Gos. Univ. Uč. Zap. 155, Mat. **5** (1952), 94–136. (Russian)

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