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EXISTENCE AND UNIQUENESS OF SINGLE SPIKE SOLUTION OF THE CARRIER-PEARSON PROBLEM

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Abstract. The Carrier-Pearson problem is a boundary value problem of the ordinary equation

$$\epsilon^2 u'' + u^2 - 1 = 0, -1 < x < 1,$$

with the boundary condition $u(-1) = u(1) = 0$. This paper presents an analytic proof of the existence and uniqueness of single spike solution of the boundary value problem.

1. **Introduction.** The boundary value problem,

$$\epsilon^2 u'' + u^2 - 1 = 0, \tag{1}$$

$$u(-1) = u(1) = 0, \tag{2}$$

where $u = u(x)$ is the unknown function, $-1 \leq x \leq 1$, and $0 < \epsilon \ll 1$ is a parameter, has been studied extensively since it was introduced by Carrier and Pearson in 1968. It was observed that there are two boundary layers at $x = \pm 1$, and an interior layer $x_0(\epsilon) \in (-1, 1)$. There may be denumerably many solutions to the boundary value problem, i.e., the solution may oscillate many times depending on the value of $u'(-1)$. In their book, Carrier and Pearson pointed out that the method of matched asymptotic expansions seemed to produce "spurious" solutions. This problem, in fact, is caused by the existence of transcendentally or exponentially small terms between the two boundary layers, actually, between the left boundary layer and interior layer. In order to better approximate the solution of (1)-(2), many researches presented several ways to formally construct the asymptotic solution between the left boundary layer and interior layer ([1]-[4], [6]). However, as for the existence of solutions to the boundary value problem, all the papers rely on only a qualitative study ([5]), which is not quite clear. This paper presents an analytic proof of the existence and uniqueness of such solutions. For the sake of simplicity, we only focus on the one-spike solutions. By one-spike or single spike solutions, we mean that the solutions of the boundary value problem (1)-(2) have only two wiggles, or only one minima on the left part and one maxima on the right part of the interval $[-1, 1]$. The result of the paper can be easily generalized to the multi-spike solutions.

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2. **Main result.** The main result of the paper is given in the following theorem:

Theorem 1. *For sufficiently small $\epsilon > 0$, there exists a unique single spike solution of the boundary value problem (1)-(2).*

The proof of the theorem will be given in the following sections.

3. **Boundedness and symmetry.** We first investigate some properties of the solution of (1)-(2) with a single spike. Consider equation (1) with the initial condition $u(-1) = 0$, $u'(-1) = \lambda < 0$. We claim that $u(x) > -1$ for all $x \in [-1, 1]$ by the argument by contradiction. Suppose that it is false. Then there would be a first point x_1 with $u(x_1) = -1$.It must be that $u'(x_1) \neq 0$ for otherwise, the solution would be $u(x) = -1$ for all x due to the uniqueness of the solution. Thus, $u'(x_1) < 0$, and hence, $u' < 0$ for all $x > x_1$ as long as $u < -1$. From equation (1), $u'' < 0$ as long as $u < -1$. This shows that the boundary condition $u(1) = 0$ would fail, a contradiction. Hence, the solution is bounded below. We will also prove that u is bounded above later.

Suppose that u is a solution with only one spike, there are two rest points in $(-1, 1)$. Let x_- and x_+ be the two points where u takes extrema, $u(x_-) = \min u$, $u(x_+) = \max u$ over interval $[-1, 1]$. Let $v(t) = u(x_- + t)$ and $w(t) = u(x_- - t)$ for $t \in [0, x_- + 1]$. Then, a simple substitution shows that both v and w satisfy equation (1) and initial conditions $v(0) = w(0) = u(x_-)$ and $v'(0) = w'(0) = 0$. By the uniqueness of solutions of the initial value problem, $v(t) = w(t)$ for all t as long as the solution exists. This shows that u is symmetric with respect to the line $x = x_-$ in $(-1, x_0)$, and that $x_0 - x_- = x_- + 1$, or $x_- = \frac{1+x_0}{2}$. Similarly, we can show that the solution is symmetric with respect to the line $x = x_+$ in $(x_0, 1)$ and $x_+ = \frac{1-x_0}{2}$. Since the single spike solution has only two rest points at x_- and x_+, $u' > 0$ for $x \in (x_-, x_+)$. Also, $x_+ - x_- = 1$ by the symmetry.

Since the equation is autonomous, we can transform the interval $[x_-, x_+]$ onto $[0, 1]$. This means that to prove Theorem 1, we only need to prove that there exists a unique increasing solution of equation (1) with boundary conditions

$$u'(0) = u'(1) = 0. \tag{3}$$

4. **Existence and uniqueness of the single spike solution.** Multiplying the both sides of (1) by u' and integrating the resulting equation, one gets

$$(\epsilon u')^2 = 2\left(u - \frac{u^3}{3} + c(\epsilon)\right) \tag{4}$$

where $c(\epsilon) = -u(0) + \frac{u^3(0)}{3}$ is determined by the boundary condition $u(0)$. Let $u(0) = \alpha$. We then consider the initial value problem (1) with

$$u(0) = \alpha, u'(0) = 0. \tag{5}$$

Theorem 2. *For sufficiently small $\epsilon > 0$, the initial value problem (1)-(5) has a unique increasing solution on $[0, 1]$.*

The remainder of the paper is a proof of Theorem 2.

If $\alpha = -1$, the initial value problem has the constant solution $u = -1$. If $\alpha < -1$, then $u''(x) < 0$ for sufficiently small $x > 0$, and hence, $u' < 0$, $u'' < 0$ as long as $u(x) < -1$ which cannot provide the solution with a spike. If $\alpha \geq 0$, the solution will not change sign between the $x = 0$ and the next critical point, which, of course, is not the solution we are looking for. Thus, we shall only consider $\alpha \in (-1, 0)$. Our goal is to prove that there exists only one α such that the solution of (1) - (5) satisfies $u' > 0$ for $x \in (0, 1)$ with $u'(1) = 0$, and $u(1) > 0$.

Define $f(u) = 3u - u^3$. From (4),

$$u' = \sqrt{\frac{2}{3}\frac{1}{\epsilon}}\sqrt{f(u) - f(\alpha)} \tag{6}$$

where $\alpha \in (-1, 0)$. Let $u'(x_+) = 0$ for an $x_+ > 0$. Then, $u(x_+) = \beta$ is the greatest zero point of the equation $f(u) - f(\alpha) = 0$, and α the mid-zero point of the equation, and therefore,

$$f(u) - f(\alpha) = (u - \mu)(u - \alpha)(\beta - u)$$

with $\mu \leq \alpha < 0 < \beta$. It is seen that

$$f(u) - f(\alpha) = -(u - \alpha)(u^2 + \alpha u + \alpha^2 - 3)$$

and

$$\beta = \frac{-\alpha + \sqrt{12 - 3\alpha^2}}{2} \tag{7}$$

It is also seen that $\beta > \sqrt{3}$ ($\alpha = 0$ implies $\beta = \sqrt{3}$ and $\mu = -\beta$) and β is a decreasing function of α because $\frac{d\beta}{d\alpha} = -\frac{1}{2} - \frac{3\alpha}{\sqrt{12-3\alpha^2}} < 0$ for $\alpha \in (-1, 0)$. From (6),

$$\int_\alpha^\beta \frac{du}{\sqrt{f(u) - f(\alpha)}} = \frac{\sqrt{2}}{\sqrt{3\epsilon}}.$$

For convenience, we define a function

$$I(\alpha) = \int_\alpha^\beta \frac{du}{\sqrt{f(u) - f(\alpha)}}. \tag{8}$$

Notice that this function is independent of ϵ. We then look for an α such that for given sufficiently small ϵ the equation

$$I(\alpha) = \frac{\sqrt{2}}{\sqrt{3\epsilon}} \tag{9}$$

has a solution. Thus, the existence and uniqueness of the increasing solution of the boundary value problem (1)-(3)is equivalent to the existence and uniqueness of the value of α, the solution of (9) for given $\epsilon > 0$. Since the function $f(u) - f(\alpha)$ has three distinct real zero points, the two limits of the improper integral $I(\alpha)$ are regular, and hence the improper integral converges and $I(\alpha)$ is well defined. In order to complete the proof of Theorem 2, we need the following lemma.

Lemma 1. *The function $I(\alpha)$ is continuous for $\alpha \in (-1, 0)$ and decreasing for sufficiently small $(\alpha + 1) > 0$.*

Proof. We define two functions $J(v)$ and $K(v)$ as follows:

$$J(v) = \int_v^0 \frac{du}{\sqrt{f(u) - f(\alpha)}},$$

and

$$K(v) = \int_0^v \frac{du}{\sqrt{f(u) - f(\alpha)}}.$$

Then, set

$$I(\alpha) = J(\alpha) + K(\beta).$$

We will prove that both $I(\alpha)$ and $J(\alpha)$ are continuous and decreasing for sufficiently small $(\alpha + 1) > 0$.

By the integration by parts,

$$\begin{aligned} J(\alpha) &= 2\int_{\alpha}^{0} \frac{1}{f'(u)} d\left(\sqrt{f(u) - f(\alpha)}\right) && (10) \\ &= \frac{2}{3}\left[\sqrt{-f(\alpha)} + 2\int_{\alpha}^{0} \sqrt{f(u) - f(\alpha)} \frac{-udu}{(1-u^2)^2}\right]. && (11) \end{aligned}$$

It is seen that $J(\alpha)$ is continuous and differentiable for $\alpha \neq \pm 1$. Differentiating $J(\alpha)$ with respect to α yields

$$J'(\alpha) = -\frac{1}{3} f'(\alpha)\left[\frac{1}{\sqrt{-f(\alpha)}} + 2\int_{\alpha}^{0} \frac{1}{\sqrt{f(u) - f(\alpha)}} \frac{-udu}{(1-u^2)^2}\right] \qquad (12)$$

from which $J'(\alpha) < 0$ for all $\alpha \in (-1, 0)$. Also, $f(\alpha) > f(-1) = -2$

$$\begin{aligned} \int_{\alpha}^{0} \frac{1}{\sqrt{f(u) - f(\alpha)}} \frac{-udu}{(1-u^2)^2} &\geq \int_{\alpha}^{0} \frac{1}{\sqrt{f(u) - f(-1)}} \frac{-udu}{(1-u^2)^2} \\ &= \int_{\alpha}^{0} \frac{1}{(u+1)^3\sqrt{2-u}} \frac{-udu}{(1-u)^2} \\ &\geq \frac{1}{4\sqrt{3}} \int_{\alpha}^{0} \frac{-udu}{(u+1)^3} \\ &= \frac{1}{4\sqrt{3}}\left[\frac{1}{2} - \frac{1}{1+\alpha} + \frac{1}{2(1+\alpha)^2}\right] \end{aligned}$$

From this we see that for sufficiently small $(1 + \alpha) > 0$

$$\begin{aligned} \int_{\alpha}^{0} \frac{1}{(u+1)^3\sqrt{2-u}} \frac{-udu}{(1-u)^2} &> \int_{\alpha}^{0} \frac{1}{(u+1)^3\sqrt{3}} \frac{-udu}{4} && (13) \\ &= \frac{1}{4\sqrt{3}} \int_{\alpha}^{0} \left[\frac{-1}{(u+1)^2} + \frac{1}{(1+u)^3}\right] du && (14) \\ &> \frac{1}{8(1+\alpha)^2}, && (15) \end{aligned}$$

and

$$\int_{\alpha}^{0} \frac{1}{(u+1)^3\sqrt{2-u}} \frac{-udu}{(1-u)^2} \leq \int_{\alpha}^{0} \frac{du}{(u+1)^3\sqrt{2}}$$

Thus, as $\alpha \to -1$, the leading term of the integral in $J(\alpha)$ is of order $O(\frac{1}{(1+\alpha)^2})$, hence, for sufficiently small $(\alpha + 1) > 0$,

$$\int_{\alpha}^{0} \frac{1}{\sqrt{f(u) - f(-1)}} \frac{-udu}{(1-u^2)^2} = O\left(\frac{1}{(1+\alpha)^2}\right)$$

and therefore,

$$J'(\alpha) < -\frac{f'(\alpha)}{16(1+\alpha)^2}. \tag{16}$$

Noting that $\alpha \in (-1, 0)$ implies $\beta \in (\sqrt{3}, 2)$, we rewrite $K(\beta)$ as

$$\begin{aligned}
K(\beta) &= \left(\int_0^{\sqrt{3}} + \int_{\sqrt{3}}^{\beta}\right) \frac{du}{\sqrt{f(u) - f(\beta)}} \\
&= \int_0^{\sqrt{3}} \frac{du}{\sqrt{f(u) - f(\beta)}} + 2\int_{\sqrt{3}}^{\beta} \frac{d\left(\sqrt{f(u) - f(\beta)}\right)}{f'(u)} \\
&= \int_0^{\sqrt{3}} \frac{du}{\sqrt{f(u) - f(\beta)}} - \frac{2\sqrt{f(\sqrt{3}) - f(\beta)}}{f'(\sqrt{3})} + \int_{\sqrt{3}}^{\beta} 2\sqrt{f(u) - f(\beta)} \frac{f''(u)}{[f'(u)]^2} du.
\end{aligned}$$

It then follows that $K(\beta)$ is continuous and differentiable with respect to β. Differentiating $K(\beta)$ with respect to β yields

$$\begin{aligned}
K'(\beta) &= \frac{f'(\beta)}{2} \int_0^{\sqrt{3}} [f(u) - f(\beta)]^{-\frac{3}{2}} du + \frac{f'(\beta)}{f'(\sqrt{3})\sqrt{f(\sqrt{3}) - f(\beta)}} \\
&\quad - \int_{\sqrt{3}}^{\beta} \frac{f''(u) f'(\beta) du}{[f'(u)]^2 \sqrt{f(u) - f(\beta)}}.
\end{aligned}$$

where

$$\begin{aligned}
&- \int_{\sqrt{3}}^{\beta} \frac{f''(u) f'(\beta) du}{[f'(u)]^2 \sqrt{f(u) - f(\beta)}} \\
&= -f'(\beta) \left\{ -\frac{2\sqrt{f(\sqrt{3}) - f(\beta)}}{[f'(\sqrt{3})]^3} - 2\int_{\sqrt{3}}^{\beta} \sqrt{f(u) - f(\beta)} \left[\frac{f''(u)}{(f'(u))^3}\right]' du \right\}.
\end{aligned}$$

This shows that $\frac{dK(\beta)}{d\beta} < 0$ and is bounded as $\beta \to 2$. In addition,

$$\begin{aligned}
\frac{d\beta}{d\alpha} &= \frac{1}{2}\left[-1 - \frac{3\alpha}{\sqrt{12 - 3\alpha^2}}\right] \\
&= -\frac{1}{2} \frac{12(1 - \alpha^2)}{\sqrt{12 - 3\alpha^2}(\sqrt{12 - 3\alpha^2} - 3\alpha)} < 0,
\end{aligned}$$

which is order of $O(1+\alpha)$. Although $\frac{dK}{d\alpha} > 0$ for $\alpha \in (-1, 0)$, $\frac{dK}{d\alpha} = O(1+\alpha)$. Since $J'(\alpha) < -\frac{f'(\alpha)}{16(1+\alpha)}$, it follows that $I'(\alpha) \sim J'(\alpha) < 0$ for sufficiently small $(1+\alpha) > 0$. This proves that $I(\alpha)$ is a decreasing function for sufficiently small $(\alpha + 1) > 0$. The proof of the Lemma is complete. □

Applying the Lemma, we prove Theorem 2. The conclusion of Theorem 1 is an immediate consequence of Theorem 2.

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