

## ANALYSIS OF A NEW CLASS OF PREDATOR-PREY MODEL

RONALD E. MICKENS

Department of Physics  
Clark Atlanta University  
Atlanta, Georgia 30314, USA

**Abstract.** We formulate predator-prey models of two interacting populations such that in the absence of the predator, the prey population has a single stable fixed-point. The prey reproduction term is of a form commonly used to model viral infection rather than logistic or exponential functions. The location of all biological meaningful fixed-points are determined along with an analysis of their (linear) stability properties. From the behavior of the flow field, the global properties of the solutions are deduced.

1. **Introduction.** The first serious mathematical model of the interactions between two populations was the predator-prey model derived by Lotka and Volterra [1, 3, 5]. However, this model has a major fault, namely, it is structurally unstable [5]. This follows from the fact that the pair of ordinary differential equations, involved in the model, possesses a first-integral [2, 3]; consequently, small changes in the initial conditions can lead to regions of the phase-space far from the trajectory determined by the original initial conditions. Also, since the fixed-point corresponding to both populations being present is a center, one can add small terms to the model differential equations and obtain completely different dynamical behaviors for the two populations [2, 3]. Previous extensions of the Lotka-Volterra equations give models for which the relevant fixed-point is stable [2] or to the situation where a single stable limit-cycle exists [3].

The main purpose of this paper is to introduce a class of mathematical models in which we modify the prey reproduction term, such that in the absence of the predator, the prey population equation has a single stable fixed-point. This prey reproduction is of a type commonly used for the modeling of viral infections [2]. In section 2, we describe the model and provide biological relevant explanations for the various terms which appear in it. In particular, we show that the prey population has a nonstandard birthrate behavior. Section 3 gives the mathematical analysis of the model. Finally, in Section 4, a summary is provided of the obtained results, further extensions of the model are given and discussed, and a possible generalization to the case where limit-cycles may exist is indicated.

2. **The Model.** The Lotka-Volterra equations are [1, 3, 5]

$$\frac{dx}{dt} = ax - bxy, \quad \frac{dy}{dt} = -cy + dxy, \quad (2.1)$$

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where  $(a, b, c, d)$  are positive constants, and  $x$  and  $y$ , respectively, represent the prey and predator populations. Our proposed model for predator-prey interactions is

$$\frac{dx}{dt} = \frac{\Lambda}{1 + k_2 y} - k_1 x - k_3 x y, \quad (2.2a)$$

$$\frac{dy}{dt} = -a_1 y + a_2 x y, \quad (2.2b)$$

where the parameters  $(\Lambda, k_1, k_2, k_3, a_1, a_2)$  are all non-negative. Note that the predator equation takes the same form as in the Lotka-Volterra equations. The major modification occurs in the prey birthrate term where the exponential growth is replaced by the expression

$$ax \longrightarrow \frac{\Lambda}{1 + k_2 y} - k_1 x. \quad (2.3)$$

This means that in the absence of the predator the evolution of the prey is determined by the equation

$$\frac{dx}{dt} = \Lambda - k_1 x. \quad (2.4)$$

Consequently, in this case, for all positive initial conditions,  $x(t)$  goes monotonically to the value  $\bar{x}^{(1)}$  where

$$\bar{x}^{(1)} = \frac{\Lambda}{k_1}. \quad (2.5)$$

The prey reproduction term, given by the first two expressions on the right-side of Eq. (2.2a), is a form commonly used to model viral infection [2]. In the absence of the predator, the prey is reproduced at a constant rate  $\Lambda$  and dies at the rate  $k_1 x$ . When the predator is also present, the effective production or birthrate,  $\Lambda$ , is reduced by a factor of  $(1 + k_2 y)$ . In other words, the presence of the predator stresses the prey with the net result being a lower birthrate.

Finally, for a single population model, the rate of change of the population can be written as [5]

$$\frac{dx}{dt} = B(x)x, \quad (2.6)$$

where  $B(x)$  is the birthrate function. In the case of Eq. (2.4), we have

$$B(x) = \frac{\Lambda}{x} - k_1. \quad (2.7)$$

Consequently, for this type of single population model, the effective birthrate becomes unbounded as  $x \rightarrow 0$  and goes to negative values for  $x > \bar{x}^{(1)} = \Lambda/k_1$ . As expected,  $B(x)$  is zero at  $x = \bar{x}^{(1)}$ .

**3. Analysis of the Model.** We now examine the main features of Eqs. (2.2) of interest, namely, the fixed-points and their (linear) stability properties. Biologically relevant states are those fixed-points for which the values are non-negative. One fixed-point is

$$(\bar{x}^{(1)}, \bar{y}^{(1)}) = \left( \frac{\Lambda}{k_1}, 0 \right), \quad (3.1)$$

and corresponds to the equilibrium population state of the prey when no predators are present. A second fixed-point is determined by the positive solutions of

$$\frac{\Lambda}{1 + k_2 y} - k_1 \bar{x} - k_3 \bar{x} \bar{y} = 0, \tag{3.2a}$$

$$-a_1 \bar{y} + a_2 \bar{x} \bar{y} = 0. \tag{3.2b}$$

From Eq. (3.2b), it follows that

$$\bar{x}^{(2)} = \frac{a_1}{a_2}. \tag{3.3}$$

Substitution of this into Eq. (3.2a) and simplifying the resulting expression gives for  $\bar{y}^{(2)} = \bar{y}$

$$\bar{y}^2 + \left( \frac{k_1 k_2 + k_3}{k_2 k_3} \right) \bar{y} - \left( \frac{k_1 a_2}{k_2 k_3 a_1} \right) (\bar{x}^{(1)} - \bar{x}^{(2)}) = 0. \tag{3.4}$$

The coefficient of the term linear in  $\bar{y}$  is positive. However, the constant term is negative. This can be shown by the following argument. In the presence of the predator, the equilibrium value of the prey population should be smaller than in their absence, i.e.,

$$\bar{x}^{(2)} < \bar{x}^{(1)}. \tag{3.5}$$

Hence, the constant term in Eq. (3.4) is negative. These two results mean that the two roots,  $\bar{y}_1$  and  $\bar{y}_2$ , of Eq. (3.4), satisfy the two conditions [4, 5]

$$\bar{y}_1 + \bar{y}_2 < 0, \quad \bar{y}_1 \bar{y}_2 < 0. \tag{3.6}$$

Thus, one root is positive and the second is negative. Consequence, only one biologically relevant state exists, since the predator population has to be positive, and its value is

$$2\bar{y}^{(2)} = - \left( \frac{k_1 k_2 + k_3}{k_2 k_3} \right) + \sqrt{\left( \frac{k_1 k_2 + k_3}{k_2 k_3} \right)^2 + \left( \frac{4k_1 a_2}{k_2 k_3 a_1} \right) (\bar{x}^{(1)} - \bar{x}^{(2)})}. \tag{3.7}$$

The linear stability of the fixed-point,  $(\bar{x}^{(2)}, \bar{y}^{(2)})$ , can be determined by solving for the characteristic values of the equation [2, 3]

$$\det(A - rI) = 0, \tag{3.8}$$

where  $I$  is the  $2 \times 2$  unit matrix, and  $A$  is the  $2 \times 2$  matrix whose elements are

$$A_{ij} \equiv \frac{\overline{\partial h_i}}{\partial x_j}; \quad i = 1, 2; \quad j = 1, 2; \tag{3.9}$$

where  $(x_1, x_2) = (x, y)$  and  $h_1(x, y)$  and  $h_2(x, y)$  are given respectively by the right-sides of Eqs. (2.2a) and (2.2b); the over-bar indicates evaluation of these expressions at  $x = \bar{x}^{(2)}$  and  $y = \bar{y}^{(2)}$ . Carrying out the indicated operations gives

$$r^2 + b_1 r + b_1 b_3 = 0, \tag{3.10}$$

$$b_1 = k_1 + k_3 \bar{y}^{(2)}, \quad b_2 = \frac{\Lambda k_2}{(1 + k_2 \bar{y}^{(2)})^2} + k_3 \bar{x}^{(2)}, \quad b_3 = a_2 \bar{y}^{(2)}, \tag{3.11}$$

where  $(b_1, b_2, b_3)$  are all positive. It follows that the two roots,  $r_1$  and  $r_2$ , of Eq. (3.10) satisfy the inequalities

$$r_1 + r_2 = -b_1 < 0, \quad r_1 r_2 = b_1 b_3 > 0. \tag{3.12}$$

Thus, if  $\tau_1$  and  $\tau_2$  are real, they are of the same sign and that sign is negative. However, if  $\tau_1$  and  $\tau_2$  are complex, then  $\tau_1 = \tau_2^*$ , and the real parts are negative. In either case, the conclusion is that the fixed-point  $(\bar{x}^{(2)}, \bar{y}^{(2)})$  is linearly unstable. A similar analysis shows that  $(\bar{x}^{(1)}, \bar{y}^{(1)})$  is linearly unstable.

The local behavior of the steady states or fixed-points along with a calculation of the null-clines can be used to reconstruct the global behavior of the solutions  $x(t)$  and  $y(t)$  in the two-dimensional phase-space  $(x, y)$ ; see section 5.9 of reference [1]. The null-clines are curves along which the trajectories in phase-space,  $y(x)$ , have either zero slope or unbounded slopes [1, 4]. The null-clines for Eqs. (2.2) are given by the curves:

$$\frac{dy}{dx} = 0 : \begin{cases} * \text{ along the } x\text{-axis;} \\ * \text{ along the line } x = \frac{a_1}{a_2}. \end{cases} \quad (3.13)$$

$$\frac{dy}{dx} = \infty : \text{ along the curve } y_\infty(x), \quad (3.14)$$

where  $y_\infty(x)$  is defined implicitly by the relation

$$x = \frac{\Lambda}{[k_1 + k_3 y_\infty(x)][1 + k_2 y_\infty(x)]}. \quad (3.15)$$

The null-clines divide the first quadrant ( $x \geq 0, y \geq 0$ ) into four regions, in each of which the slope  $dy/dx$  has the same sign. Careful study of the resulting flow-field shows that the fixed-point,  $(\bar{x}^{(2)}, \bar{y}^{(2)})$ , is globally stable; see again section 5.9 of reference [1]. Likewise,  $(\bar{x}^{(1)}, \bar{y}^{(1)})$  is globally unstable.

**4. Discussion.** The above predator-prey model, given by Eqs. (2.2), may be easily generalized. We have also studied two additional models where the predator equation remains the same as in Eq. (2.2b), but the prey equation is changed to one of the following two forms:

$$\frac{dx}{dt} = \Lambda - k_1 x - \frac{k_3 xy}{1 + k_4 y}, \quad (4.1)$$

or

$$\frac{dx}{dt} = \frac{\Lambda}{1 + k_2 y} - k_1 x - \frac{k_3 xy}{1 + k_4 y}. \quad (4.2)$$

The third term on the right-sides of these equations is a commonly used predator-prey interaction form which can be justified on the basis of realistic ecological conditions [3, 5]. By carrying out an analysis, exactly like that done in section 3, it was found that in both cases the corresponding fixed-point, at  $(\bar{x}^{(2)}, \bar{y}^{(2)})$ , was globally stable with respect to initial conditions in the first quadrant of the  $(x, y)$  phase-space. Hence, our major conclusion is that the replacement of the exponential growth term in the Lotka-Volterra equations by an expression used to model viral infections [2] and the similar replacement of the interaction term by a predator limiting effect function [5], the third term in either Eqs. (4.1) or (4.2), leads to a robust set of model equations for which the ecologically interesting fixed-point,  $(\bar{x}^{(2)}, \bar{y}^{(2)})$ , is globally stable.

Currently, we are studying a predator-prey model in which Eq. (2.2b) is modified to the form

$$\frac{dy}{dt} = b_1 y \left( 1 - \frac{b_2 y}{x} \right), \quad (4.3)$$

where  $(b_1, b_2)$  are positive parameters. Related predator-prey systems having this structure for the predator dynamics can give rise to limit-cycles; see section 3.4 of reference [5].

#### REFERENCES

- [1] L. Edelstein-Keshet, "Mathematical Models in Biology," McGraw-Hill, New York, 1988.
- [2] D. Kirschner & G. F. Webb, *A model for treatment strategy in the chemotherapy of AIDS*, Bulletin of Mathematical Biology, 58 (1996), 367–390.
- [3] R. May, "Theoretical Ecology, Principles and Applications," Saunders, Philadelphia, 1976.
- [4] R. E. Mickens, "Oscillations in Planar Dynamic Systems," World Scientific, Singapore, 1996.
- [5] J. D. Murray, "Mathematical Biology," Springer-Verlag, Berlin, 1989.

*E-mail address:* rohrs@math.gatech.edu