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SPECTRAL AND STABILITY QUESTIONS CONCERNING EVOLUTION OF NON-AUTONOMOUS LINEAR SYSTEMS

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The study of long time behaviour of quantum systems with time dependent Hamiltonians has received increasing attention in recent years. A number of numerical and theoretical studies are carried out for special systems (primarily rotors and oscillators) with time dependent external field ([BJLPN],[DGP],[JL],[SK]). In this article we survey recent analytical results about such systems driven by stationary ergodic external fields.

The basic set up consists of a separable Hilbert space $\mathcal{H}$ with a given time independent (unperturbed), essentially self adjoint operator H_0 acting on $\mathcal{H}$. The time dependent as well as random perturbations of H_0 are best modelled by considering a continuous map H_1 taking values in the set of essentially self adjoint operators on $\mathcal{H}$, defined on a compact metric space Ω. On Ω, a flow (i.e. a one parameter group of homeomorphisms $T_t : \Omega \to \Omega$) along with a flow invariant regular Borel probability measure μ is given. We remark that given a single time dependent perturbation (i.e. a self adjoint operator $H_1(t)$, $t \in \mathbb{R}$), one can take Ω to be the "hull" of this function with the natural translation flow defined on it. Thus our set up is fairly general. Now $H_\omega = H_0 + H_1(\omega)$ yields a family of Hamiltonians parametrized by points of Ω and the evolution of this (stochastic and non-autonomous) system is given by the Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = H_{T_t(\omega)}\psi. \tag{1}$$

Let $U_H(\omega, t)$ be the unitary evolution operator (propogator) determined by this equation (at this point we shall assume its existence), thus $U_H(\omega, 0) = I$-the identity operator and for each $\omega \in \Omega$ the curve $t \to U(\omega, t)$ satisfies equation (1). Furthermore the map $U : \Omega \times \mathbb{R} \to \mathcal{U}(\mathcal{H})$ satisfies the following *cocycle identity*:

$$U_H(\omega, t+s) = U_H(T_t(\omega), s)U_H(\omega, t) \qquad (\omega \in \Omega,\ t, s \in \mathbb{R}), \tag{2}$$

where $\mathcal{U}(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$ with the weak operator topology.

In the above set up (i.e. in the non-autonomous case) the appropriate Hilbert space in which the unitary evolution takes place is $L^2(\Omega, \mathcal{H}, \mu) \equiv \mathcal{H} \otimes L^2(\Omega, \mu)$-the space of $\mathcal{H}$ valued square integrable functions on Ω. The evolution is given by the unitary one parameter group $\{V_t^H\}_{t\in\mathbb{R}}$ defined as follows:

$$V_t^H f(\omega) = U(\omega, t)^{-1} f(T_t(\omega)), \quad f \in L^2(\Omega, \mathcal{H}, \mu). \tag{3}$$

The infinitesimal generator of this representation is called the *quasi energy operator* and the stability properties of the quantum system are studied in terms of the spectral properties of $\{V_t^H\}_{t\in\mathbb{R}}$. A quantum system is considered to be *stable* if the spectrum of this representation is discrete pure point and *unstable* otherwise.

The spectral properties of $\{V_t^H\}_{t\in\mathbb{R}}$ are closely related to the ergodic properties of certain *skew-product* flows generated by appropriate cocycle (the appropriate cocycle may not always be U_H). We shall illustrate this with some simple examples.

Example 1 : Let $\mathcal{H} = \mathbf{C}$, $H_0 = 0$ and $\Omega = \mathbb{T}^2$ be the 2-torus with the rotation flow having irrational winding number α:

$$T_t(\omega) \equiv T_t(\theta_1, \theta_2) = (\theta_1 + t, \theta_2 + \alpha t) \text{ mod } 1.$$

The invariant measure is the normalized Lebesgue measure on Ω. The perturbation H_1 is given by a map $h : \mathbb{T}^2 \to \mathbf{C}$. The unitary representation V_t^h acting on $L^2(\mathbb{T}^2, \mathbf{C})$ can be explicitly written down as

$$V_t^h f(\omega) = e^{(i \int_0^t h(T_s(\omega))ds)} f(T_t(\omega)), \quad f \in L^2(\mathbb{T}^2, \mathbf{C}).$$

Notice that $V_t^h f = e^{i\lambda t} f$ for some $f \in L^2(\mathbf{S}^1)$ if and only if the cocycle $e^{i \int_0^t h(T_s(\omega))ds}$ is "cohomologous" to the "constant cocycle" $e^{i\lambda t}$. Thus to show that $\{V_t^h\}_{t\in\mathbb{R}}$ has only continuous spectrum, it is enough to show that certain families of functional equations have no solutions. This in turns is related to showing that the skew-product flow

$$T_t^h(\theta, \omega) = (i \int_0^t h(T_s(\omega))ds + \theta \text{ mod } 1, T_t(\omega))$$

has "certain dynamical properties". Another variant of this example is due to J. Bllisard ([B]) where $\mathcal{H} = L^2(\mathbf{S}^1)$, $H_0 = -i\alpha\frac{d}{d\theta}$, $\Omega = \mathbf{S}^1$ with the periodic rotation flow with unit speed and the perturbation involves a series of "kicks", more precisely

$$H = -i\alpha \frac{d}{d\theta} + V(\theta) \sum_{n\in\mathbb{Z}} \delta(t-n),$$

where V is a periodic function with period 1. Since the base flow is periodic, the spectral behaviour is completely determined by the Floquet operator $\tilde{U} \equiv U_H(\cdot, 1)$, which can be written down as

$$\tilde{U} f(\theta) = e^{-iV(\theta)} f(\theta - \alpha).$$

We refer to [B] for a fairly complete description of various types of spectral behaviour depending on smoothness and variational properties of V and the diophantine properties of α.

This illustrates the general method to reduce spectral problems to dynamical problems about skew-product extensions. In fact, further question of whether the continuous spectrum has any absolutely continuous or singular continuous component can also be decided by the dynamical features of the skew-product flow. In the above (one dimensional) example all of these spectral features can be analyzed by means of the standard Fourier series techniques and the "small divisor argument".

Example 2 (Forced linear harmonic oscillator) : In this case $\mathcal{H} = L^2(\mathbb{R})$-the space of square integrable functions with respect to the usual Lebesgue measure, $H_0 = \frac{\hat{p}^2}{2} + \frac{(\nu\hat{q})^2}{2}$ where $\hat{p} = -i\frac{d}{dq}$ is the momentum operator, $\hat{q}$ is the position operator and ν is the natural frequncy. Given a flow $(\Omega, \{T_t\}_{t\in\mathbb{R}}, \mu)$ and an external field, $F : \Omega \to \mathbb{R}^2$, the perturbed i.e. forced Hamiltonian is given by

$$\mathbf{H}_\xi = \mathbf{H}_0 + \langle JF(\xi), \begin{pmatrix} q \\ p \end{pmatrix} \rangle \tag{4}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\langle\ ,\ \rangle$ is the formally defined usual inner product of two dimensional vectors. Again the unitary propogator (i.e. the cocycle) $U_H \equiv U_F$ can be explicitly written down as

$$U_F(\omega, t) = e^{-it\mathbf{H}_0} \exp\Big(-i \Big\langle \int_0^t R(\nu s)^{-1} JF(T_s(\omega)ds, \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} \Big\rangle \Big), \tag{5}$$

where $R(\theta) = \begin{pmatrix} cos\ \theta & sin\ \theta \\ -sin\ \theta & cos\ \theta \end{pmatrix}$. It turns out (see [NJ]) that the spectral properties of $\{V_t^F\}_{t\in\mathbb{R}}$ are determined by the following "twisted skew-product" flow on $\mathbb{R}^2 \times \mathbf{S}^1 \times \Omega$:

$$T_t^F(v, \theta, \omega) = \big(v + R(\theta) \int_0^t R(\nu s)^{-1} JF(T_s(\omega)ds,\ (\theta - \nu s) \text{ mod } 1,\ T_s(\omega)\big). \tag{6}$$

This flow is an $\mathbb{R}^2$ extension of the product flow: $\hat{T}_t(\theta, \omega) = ((\theta - \nu s)\ mod\ 1, T_t(\omega))$ on $\mathbf{S}^1 \times \Omega$ by the "twisted cocycle" $(\theta, \omega, t) \to R(\theta) \int_0^t R(\nu s)^{-1} JF(T_s(\omega)ds$. This map satisfies a modified (or twisted) version of the cocycle identity. In general, in forced oscillation problems the spectral behaviour is determined by the dynamics of such twisted cocycles where modification or the twist in the cocycle condition is determined by the nature of H_0. Here, we shall not discuss this issue in more detail.

Again, in this example finding eigenvalues is related to solving certain functional equations and this can be done under appropriate diophantine conditions involving ν and α and smoothness assumptions of F (see [NJ]). On the other hand ergodicity of the twisted skew-product flow on $\mathbb{R}^2 \times \mathbf{S}^1 \times \Omega$ guarentees that the spectrum is only continuous. Furthermore in a suitable class of external fields F's (the class of closure of "twisted coboundaries") under suitable "rigidity" and "resonance" assumption one can get a handle on obtaining only singular continuous spectrum. Without introducing additional technicalities, the results can be best summerized for uniquely ergodic flows as follows (see [N2]).

1 Theorem. *Consider the forced harmonic oscillator with the following assumptions:*

(1) *The flow $(\Omega, \{T_t\}_{t\in\mathbb{R}}, \mu)$ is uniquely ergodic, and μ is not supported on a single orbit.*
(2) *The flow $(\Omega, \{T_t\}_{t\in\mathbb{R}}, \mu)$ be weakly rigid, i.e. $T_{q_n} \to Id$- the identity map for some sequence $q_n \to \infty$, (here the convergence is in the weak topology on the set of all μ preserving Borel auomorphisms of Ω).*
(3) *let the system satisfy the strong resonance condition, i.e. the product flow $(\mathbf{S}^1 \times \Omega, \hat{T}_t, d\theta \times \mu)$ be ergodic and weakly rigid.*

Then the set

$$C_{sing} = \{F \in C(\Omega, \mathbb{R}^2) \mid \{V_t^F\}_{t\in\mathbb{R}} \textit{ has only singular continuous spectrum}\}$$

is residual in $C(\Omega, \mathbb{R}^2)$-the space of $\mathbb{R}^2$ valued continuous functions on Ω with suprimum metric.

Notice that when $\Omega = \mathbb{T}^2$ with the rotation flow with winding number α, then all of the above assumptions hold if $1, \nu, \alpha$ are rationally independent.

Even though $\mathcal{H}$ was infinite dimensional in the previous examples, the underlying (twisted) cocycle was taking values in abelian groups (e.g. $\mathbf{S}^1$ and $\mathbb{R}^2$). Now we consider finite dimensional (i.e. $\dim\mathcal{H} < \infty$) but genuinely non-abelian problems where methods of Control Theory play a crucial role.

Emaple 3 (The Rabi Oscillator) : Here $\mathcal{H} = \mathbf{C}^2$, $H_0 = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$, where $\lambda \in \mathbb{R}$ is a fixed parameter. The perturbed system is given by

$$i\frac{d\psi}{dt} = \begin{pmatrix} \lambda & f(T_t(\omega)) \\ f(T_t(\omega))^* & -\lambda \end{pmatrix} \psi, \qquad \psi \in \mathbf{C}^2. \tag{7}$$

This is the Schrödinger equation which describes the dynamics of a spin 1/2 particle moving under external (time dependent) random process $f : \Omega \to \mathbf{C}$. When $\mathcal{H}$ is finite dimensional (say $\mathcal{H} = \mathbf{C}^n$), one can study more general families of linear system of a given specific form. A general set up in this case is as follows: One is given a flow $(\Omega, \{T_t\}_{t\in\mathbf{R}}, \mu)$, and a Lie algebra L of a connected Lie group G. A continuous map $A : \Omega \to L$ determines a linear system

$$x' = A(T_t\omega)x, \quad x \in G, \; \omega \in \Omega. \tag{8}$$

Notice that in the case of Rabi oscillator $G = SU(2, \mathbf{C})$. Let X_A be the fundamental matrix solution (i.e. the cocycle) determined by (8). This defines a skew product flow on $G \times \Omega$ by

$$T_t^A(g, \omega) = (X_A(\omega, t)g, T_t(\omega)). \tag{9}$$

Notice that the product measure $\eta \times \mu$ is invariant under the skew-product flow, (η being a left Haar measure on G). Again ergodicity of this flow implies that the associated spectral measure of $\{V_t^A\}_{t\in\mathbb{R}}$ is purely continuous provided G is compact and does not fix any 'ray' in the projective space $P^1(\mathbf{C})$, [N2].

Observe that now we are studying cocycles X_A arising from (8) where A is of special form. This constraint on the class of linear systems is captured in terms of a given compact, convex constraining subset Q of the Lie algebra L. we shall consider linear systems arising from Q valued continuous maps on ω. Notice that in the case of Rabi oscillator $Q = \{\begin{pmatrix} -i\lambda & r \\ -r^* & i\lambda \end{pmatrix} \mid r \in \mathbf{C}\}$- a two dimensional subset of a three dimensional Lie algebra. Thus in this case we are dealing with the question of generic spectral behaviour in "very thin" classes of systems (or cocycles).

To prove generic dynamical and spectral results, we shall require that this constraining set Q to have the SAP (strong accessibility property). This property arises from Control Theoretic considerations. To describe this condition, let $L(Q)$ be the Lie subalgebra of L generated by Q, and let $L_0(Q)$ be the ideal of $L(Q)$ generated by the difference set $Q - Q = \{x - y \mid x \in Q, \; y \in Q\}$. Then subset Q has the SAP if $L_0(Q) = L$. The reader can verify that this property holds in the example of Rabi oscillator.

Here we shall restrict ourselves to a weaker problem of producing *minimal* skew-product flows $(G \times \Omega, \{T_t^A\}_{t\in\mathbb{R}})$. The following theorem (see [NS]) shows that this problem can be generically solved under the SAP condition and mild assumptions on the base flow.

2 Theorem. *Let the flow $(\Omega, \{T_t\}_{t\in\mathbf{R}})$ be aperiodic and minimal. Let G be a compact connected Lie group with Lie algebra L and $Q \subseteq L$ such that Q is compact and convex and has SAP. Then the set*

$$C_{min}(\Omega, Q) = \{A \in C(\Omega, Q) : (G \times \Omega, \{T_t^A\}_{t\in\mathbf{R}}) \text{ is minimal}\}.$$

is a residual subset of $C(\Omega, Q)$.

The problem of replacing minimality by ergodicity in the above theorem is much more difficult. However this can be done in some special cases, e.g. when the flow is the irrational rotation flow on the 2-torus and $G = SU(2, \mathbf{C})$. Thus in the case of Rabi oscillators generic nature of the spectrum is purely continuous.

A "non compact" version of the technique used to prove Theorem 2 also allows us to prove the following [N3].

3 Theorem. *Consider the above set up with $G = SL(2, \mathbb{R})$ and $Q \subset sl(2, \mathbb{R})$ with the assumptions*

(1) *the flow $(\Omega, \{T_t\}_{t\in\mathbb{R}}, \mu)$ is ergodic and $Supp(\mu) = \Omega$;*
(2) *Q is closed, convex and has SAP.*

Then the set

$$C_{pos}^{\mu} = \{A \in C(\Omega, Q) \mid \lambda_\mu^+(A) > 0\}$$

is a dense subset of $C(\Omega, Q)$, where $\lambda_\mu^+(A)$ ($\equiv \lim_{t\to\infty} \frac{1}{t} ln||X_A(\omega, t)||$ a.e. ω) is the largest Lyapunov exponent of the cocycle X_A with respect to μ.

The proof uses work of S. Kotani to reduce the problem to proving density of "unbounded cocycles". Thus fixing a point $\omega_0 \in \Omega$ and a compact set $K \subset SL(2, \mathbb{R})$ we need to show that given any $A_0 \in C(\Omega, Q)$ and $\varepsilon > 0$, we can find a ε perturbation A of A_0 (in the suprimum metric) such that the trajectory $X_A(\omega_0, t)$ goes outside K for some $t > 0$. This can be thought as a control problem (on Lie group G) of steering the trajectory (of the system $x' = A(T_t(\omega_0)x)$) into the set $G\backslash K$, by the control function A, under the constraints of keeping A Q-valued and ε close to A_0. This can be done using the SAP property and unimodularity of $SL(2, \mathbb{R})$, (see [N3] for details).

We end the article by pointing out in particular that Theorem 3 applies to the linear systems of the form

$$x' = \begin{pmatrix} 0 & 1 \\ q(T_t(\omega)) - \nu & 0 \end{pmatrix} x,$$

which arise from the spectral equation $L_\omega y = \nu y$ where $L_\omega = -\frac{d^2}{dt^2} + q(T_t(\omega))$ is the family of one-dimensional Schrödinger operators and $x = \begin{pmatrix} y \\ y' \end{pmatrix}$.

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