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REMARKS ON ALMOST AUTOMORPHIC DIFFERENTIAL EQUATIONS

GASTON MANDATA N'GUEREKATA

Department of Mathematics
Morgan State University, Baltimore, MD 21221

Abstract. This paper gives a result of almost automorphy of solutions of the differential equation $x'(t) = Ax + f$ in a Banach space. It also gives some properties of weakly almost automorphic solutions of the associated homogeneous equation $x' = Ax$.

1. **Introduction.** Consider in a normed vector space X the differential equation

$$x'(t) = Ax(t) + f(t), t \in \mathbb{R}, \tag{1}$$

where $f \in L^\infty(\mathbb{R}, X)$.

A central question is the following: "Under what conditions are qualitative properties of the function $f(t)$ transmitted to the solutions of (1)?" This question was initially raised and solved by Bohr and Neugebauer for bounded solutions of (1) for $f(t)$ an almost periodic function with values in a finite dimensional space. Several authors, including J.A. Goldstein, S. Zaidman, W.M. Ruess, W.H. Summers, M. Zaki, and G.M. N'Guerekata, have conducted extensive investigations of the problem in infinite dimensional spaces.

A larger class of functions, namely almost automorphic functions (cf. Definition 2.1 below) has also been considered in works by Zaidman [6,7], Zaki [8] and N'Guerekata [2,3,4,5]. In our recent paper [2], we proved the following:

Theorem 1.1. *([2] page 362) Let X be a perfect Banach space, $A \in L(X)$ and $f : \mathbb{R} \to X$ an almost automorphi function. Then every bounded solution of (1) is an almost automorphic if we assume that there exists a finite dimensional subspace X_1 of X such that*

(α) $Ax(0) \in X_1$,
(β) $(e^{tA} - I)f(s) \in X_1$, for any $s, t \in \mathbb{R}$,
(γ) $e^{tA}u \in X_1$, for any $t \in \mathbb{R}$, and any $u \in X_1$.

In the present article we extend this result to general (not necessarily perfect) Banach spaces and discuss a property of weakly almost automorphic solution of the associated homogeneous equation.

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2. Some Preliminaries.

Let X be a Banach space equipped with the norm topology and X^* be its dual.

Definition 2.1. (Bochner) A strongly continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if from every sequence of real numbers (s'_n), we can extract a subsequence (s_n) such that $\lim_{n\to\infty} f(t+s_n) = g(t)$ and $\lim_{n\to\infty} g(t-s_n) = f(t)$ pointwise on $\mathbb{R}$.

Definition 2.2. [8] A weakly continuous function $f : \mathbb{R} \to X$ is said to be weakly almost automorphic if from every sequence of real numbers (s'_n), we can extract a subsequence (s_n) such that weak-$\lim_{n\to\infty} f(t+s_n) = g(t)$ and weak-$\lim_{n\to\infty} g(t-s_n) = f(t)$ pointwise on $\mathbb{R}$.

Remarks 2.3. If $f : \mathbb{R} \to X$ is almost automorphic, then

i) its range is relatively compact in X

ii) f is weakly almost automorphic

If $f : \mathbb{R} \to X$ is weakly almost automorphic, then

i) f is bounded in norm

ii) the function $< \phi, f(t) >: \mathbb{R} \to X$ is almost automorphic, for every $\phi \in X^*$.

We recall some known facts:

Proposition 2.4. [6] If A is an $n \times n$ matrix, and $f : \mathbb{R} \to \mathbb{R}^n$ is an almost automorphic function, then $Af(t)$ is also almost automorphic.

Proposition 2.5. [6,7] If $A \in L(X)$ is a bounded linear operator and $f :\to X$ is an almost automorphic function, then $Af(t)$ is also almost automorphic.

Proposition 2.6. [8] Let $f :\to X$ be an almost automorphic function and suppose that $F(t) = \int_0^\infty f(s)\, ds$ has a relatively compact range in X, then $F(t)$ is also almost automorphic. In a uniformly convex Banach space, the conclusion remains true if $F(t)$ is bounded in norm.

3. Main Results.

Theorem 3.1. *Let $A \in L(X)$ be a bounded linear operator and $f : \mathbb{R} \to X$ an almost automorphic function. Let $x(t)$ be a strong solution of (1). Assume that there exists a finite dimensional subspace X_1 of X such that*

(α) $Ax(0) \in X_1$,

(β) $e^{tA}u \in X_1, \forall t \in \mathbb{R}, \forall u \in X_1$

(γ) $(e^{tA} - I)f(s) \in X_1, \forall s, t \in \mathbb{R}$.

Then $x(t)$ is almost automorphic.

Proof. Consider the projection $P : X \to X_1$. We can write $X = X_1 \oplus N(P)$, where $N(P)$ is the null space for P. By letting $Q = I - P$, we can easily prove that $Qu = 0, \forall u \in X_1$, and $Q^2u = Qu, \forall u \in X$. Also, both P and Q are bounded linear operators.

Now let $x(t)$ be a solution of (1) as stated in the theorem. Then we obtain

$$x(t) = x_1(t) + x_2(t), t \in \mathbb{R},$$

where $x_1(t) = Px(t) \in X_1$ and $x_2(t) = Qx(t) \in N(P)$.

Since the range of $x(t)$ is relatively compact in X, so are the ranges of both $x_1(t)$ and $x_2(t)$ as one can easily observe.

Also, we have

$$\begin{aligned} x'(t) &= x_1'(t) + x_2'(t) \\ &= Ax_1'(t) + Ax_2'(t) + Pf(t) + Qf(t),\ t \in \mathbb{R}. \end{aligned} \tag{2}$$

$x(t)$ has the Lagrange integral representation

$$\begin{aligned} x(t) &= e^{tA}x(0) + \int_0^t e^{(t-s)A} f(s)\, ds \\ &= e^{tA}x(0) + \int_0^t f(s)\, ds + \int_0^t (e^{(t-s)A} - I) f(s)\, ds \end{aligned} \tag{3}$$

Using assumption (γ), we can say that $\int_0^t (e^{(t-s)A} - I) f(s)\, ds \in X_1$.

Applying Q to (3) we get, for every $t \in \mathbb{R}$:

$$\begin{aligned} x_2(t) &= Qe^{tA}x(0) + Q\int_0^t f(s)\, ds \\ &= Qe^{tA}x(0) + \int_0^t Qf(s)\, ds. \end{aligned}$$

Thus

$$\begin{aligned} x_2'(t) &= QAe^{tA}x(0) + Qf(t) = Qe^{tA}Ax(0) + Qf(t) \\ &= Qf(t),\ t \in \mathbb{R} \end{aligned}$$

since $Ax(0)$, and therefore $e^{tA}Ax(0)l$ is in X_1.

Now we observe that $Qf(t)$ is almost automorphic by **Proposition 2.5**, that is, $x_2'(t)$ is almost automorphic. We then deduce by **Proposition 2.6** that $x_2(t)$ is almost automorphic since its range is relatively compact in X.

Let us now apply P to (2); we then obtain X_1 the differential equation

$$x_1'(t) = PAx_1'(t) + PAx_2'(t) + P^2 f(t) + PQf(t),\ t \in \mathbb{R}.$$

where the function $g(t) = PAx_2(t) + P^2 f(t) + PQf(t)$ is almost automorphic.

The operator PA restricted to the subspace X_1 is a matrix and the fuction $x_1(t)$ is bounded since its range is relatively compact. So we deduce that it is almost automorphic by **Proposition 2.4**. Finally, $x(t)$ is almost automorphic as the sum of two almost automorphic functions. This completes the proof.

Theorem 3.2. *Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group of bounded linear opertors in X. Assume that the function $x(t) = T(t)x_0 : \mathbb{R} \to X$ is weakly almost automorphic for some $x_0 \in X$. Then*

$$\inf_{t\in\mathbb{R}} \|x(t)\| > 0, \text{ or } x(t) = 0 \forall t \in \mathbb{R}.$$

Proof. Assume that $\inf_{t\in\mathbb{R}} \|x(t)\| = 0$. Then there exists a sequence of real numbers (s_n') such that $s_n' \to \infty$ and $\|x(s_n')\| \to 0$, as $n \to \infty$.

Since $x(t)$ is weakly almost automorphic, there exists a subsequence (s_n) of (s_n') such that

$$\text{weak-}\lim_{n\to\infty} x(t + s_n) = y(t)$$

and

$$\text{weak-}\lim_{n\to\infty} y(t - s_n) = y(t)$$

pointwise on $\mathbb{R}$.

Consequently, for every $\phi \in X^*$, we get

$$\lim_{n\to\infty} < \phi, x(t+s_n) >=< \phi, y(t) >$$

and

$$\lim_{n\to\infty} < \phi, y(t-s_n) >=< \phi, x(t) >$$

for each $t \in \mathbb{R}$.

Write $x(t+s_n) = T(t+s_n)x_0 = T(t)T(s_n)$, $n = 1, 2, 3, ...$ We then get for each $t \in \mathbb{R}$, $\phi \in X^*$:

$$\lim_{n\to\infty} < \phi, T(t)x(s_n) >=< \phi, y(t) >,$$

so that $| < \phi, T(t)x(s_n) > | \le \|\phi\|\|T(t)\|\|x(s_n)\| \to 0$, as $n \to \infty$.

That is

$$< \phi, y(t) >= 0, \ \forall \phi \in X^*, \forall t \in \mathbb{R}.$$

This implies that $y(t) = 0$, $\forall t \in \mathbb{R}$, and consequently $x(t) = 0$, $\forall t \in \mathbb{R}$. This completes the proof.

Application 3.3 Consider in a Banach space X the differential equation

$$x'(t) = Ax(t), \ t \in \mathbb{R} \tag{4}$$

where A is a linear operator that generates a C_0-group of strongly continuous linear operators $T(t))_{t\in\mathbb{R}}$. Then every weakly almost automorphic mild solution $x(t)$ satisfies the property

$$\inf_{t\in\mathbb{R}} \|x(t)\| > 0, \ \text{ or } x(t) = 0, \ \forall t \in \mathbb{R}.$$

Proof. It suffices to observe that mild solutions $x(t)$ of (4) can be represented by $x(t) = T(t)x(0)$, $t \in \mathbb{R}$.

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E-mail address: gnguerek@morgan.edu