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NORMAL FORM OF DUFFING-VAN DER POL OSCILLATOR UNDER NONAUTONOMOUS PARAMETRIC PERTURBATIONS

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Abstract. A new generalization of Poincarés normal form theory to nonautonomous differential equations is applied to the Duffing-van der Pol oscillator under a nonautonomous bounded parametric perturbation. The nonautonomous normal form is calculated for parameter values corresponding to the pitchfork scenario. To accomplish the enormous computational effort the computer algebra program MAPLE is used. The normal form can be used to study nonautonomous bifurcation phenomena.

1. **Preliminaries.** Henry Poincaré founded the normal form theory for autonomous differential equations $\dot{x} = f(x)$ near a rest point in his thesis in 1879. If the eigenvalues $\lambda_1, \dots, \lambda_n$ of the linearization $\dot{x} = Df(x_0)x$ at the rest point x_0 satisfy the *nonresonance condition*

$$\lambda_j \neq \sum_{i=1}^{n} \ell_i \lambda_i \,, \tag{1.1}$$

$j \in \{1, \dots, n\}$, $\ell_i \in \mathbf{N}_0 = \{0, 1, \dots\}$, $\sum_{i=1}^n \ell_i \geq 2$, then the differential equation can be formally linearized. In Siegmund [4] Poincarés normal form result is extended to nonautonomous differential equations along an arbitrary reference solution. We cite a specialized version of the general result in Siegmund [4] which is adequate for our purposes. Consider a differential equation

$$\boxed{\dot{x} = Ax + F(t, x)} \tag{1.2}$$

with $A \in \mathbf{R}^{d \times d}$, $d \in \{1, 2, \dots\}$ and $F : \mathbf{R} \times B_\varepsilon(0) \to \mathbf{R}^d$ a C^1 function which is C^K in x with partial derivatives which are continuous in (t, x) and

$$F(t, 0) \equiv 0 \,, \quad D_x F(t, 0) \equiv 0 \,,$$

where $K \geq 2$ and $B_\varepsilon(0) = \{x \in \mathbf{R}^d : \|x\| < \varepsilon\}$. Assume that A is blockdiagonal

$$A = \mathrm{diag}(A_1, \dots, A_n)$$

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with $1 \leq n \leq d$ and the eigenvalues of A_i all have the same real part $\lambda_i \in \mathbf{R}$. Writing the differential equation (1.2) coordinate-wise and Taylor-expanding the nonlinearity we get

$$\begin{aligned} x_1 &= A_1 x_1 + \sum_{\ell \in \mathbf{N}_0^n \,:\, 2 \leq |\ell| \leq K} \tfrac{1}{\ell!} D_x^\ell F_1(t,0) \cdot x^\ell + O(\|x\|^{K+1}) \\ &\;\;\vdots \\ x_n &= A_n x_n + \sum_{\ell \in \mathbf{N}_0^n \,:\, 2 \leq |\ell| \leq K} \tfrac{1}{\ell!} D_x^\ell F_n(t,0) \cdot x^\ell + O(\|x\|^{K+1}) \end{aligned}$$

where $\ell = (\ell_1, \ldots, \ell_n) \in \mathbf{N}_0^n$ is a multi-index, $|\ell| = \ell_1 + \cdots + \ell_n$ and $F_1, \ldots, F_n$ are the components of F corresponding to the block structure of A. The abbreviations D_x^ℓ, x^ℓ and $\ell!$ stand for $D_{x_1}^{\ell_1} \cdots D_{x_n}^{\ell_n}$, $x_1^{\ell_1} \cdots x_n^{\ell_n}$ and $\ell! = \ell_1! \cdots \ell_n!$, respectively.

The following theorem is a special case of a theorem in Siegmund [4] and it yields a transformation which eliminates a Taylor-component $\frac{1}{\ell!} D_x^\ell F_i(t,0) \cdot x^\ell$ provided a corresponding nonresonance condition holds. Note that the resulting transformation is periodic in t, if the differential equation is, hence it is independent of t if the differential equation is autonomous.

Theorem 1.1 (Normal Form Theorem). *Consider the nonautonomous differential equation (1.2). Let $j \in \{1, \ldots, n\}$ be an index and $\ell \in \mathbf{N}_0^n$, $2 \leq |\ell| \leq K$, a multi-index. Assume that the nonresonance condition (1.1) holds for the real parts λ_i of the eigenvalues of the blocks A_i. Then a local C^K equivalence H exists which eliminates the j-th Taylor component $\frac{1}{\ell!} D_x^\ell F_j(t,0) \cdot x^\ell$ belonging to the multi index ℓ and leaves fixed all other Taylor coefficients up to order $|\ell|$.*

I.e., (1.2) is locally C^K equivalent to a differential equation

$$\boxed{\dot{x} = Ax + G(t,x)}\,,$$

with $G(t,0) \equiv 0$, where G is defined on a set $\mathbf{R} \times B_\varepsilon(0)$ with some $\varepsilon > 0$ and for all $\kappa \in \mathbf{N}_0^n$ with $1 \leq |\kappa| \leq |\ell|$ and all $i \in \{1, \ldots, n\}$ the identity

$$D_x^\kappa G_i(t,0) \equiv \begin{cases} D_x^\kappa F_i(t,0), & \text{for } \kappa \neq \ell \text{ or } i \neq j \\ 0, & \text{for } \kappa = \ell \text{ and } i = j \end{cases}$$

holds. The local near-identity C^k equivalence $H : \mathbf{R} \times B_{\varepsilon'}(0) \to B_\varepsilon$, $(t,x) \mapsto x + h(t,x)$, $\varepsilon' > 0$, is defined through

$$h_i(t,x) = \begin{cases} 0 & \text{if } i \neq j\,, \\ \int_t^\infty e^{A_j(t-s)} \frac{1}{\ell!} D_x^\ell F_j(s,0) \cdot [e^{A_1(s-t)} x_1]^{\ell_1} \cdots [e^{A_n(s-t)} x_n]^{\ell_n}\, ds, & \\ \quad \text{if } i = j \text{ and } \lambda_j > \sum_{i=1}^n \ell_i \lambda_i\,, & \\ -\int_{-\infty}^t e^{A_j(t-s)} \frac{1}{\ell!} D_x^\ell F_j(s,0) \cdot [e^{A_1(s-t)} x_1]^{\ell_1} \cdots [e^{A_n(s-t)} x_n]^{\ell_n}\, ds, & \\ \quad \text{if } i = j \text{ and } \lambda_j < \sum_{i=1}^n \ell_i \lambda_i\,, & \end{cases}$$

where $h_1, \ldots, h_n$ are the components of h corresponding to the block structure of A.

2. **Main Result.** We consider the prototypical *Duffing-van der Pol oscillator*

$$\ddot{y} = \alpha y + \beta \dot{y} - y^3 - y^2 \dot{y}\,. \tag{2.3}$$

For $\alpha < 0$ fixed and β the bifurcation parameter, the system (2.3) exhibits a Hopf bifurcation for $\beta = 0$. For $\beta < 0$ fixed and α the bifurcation parameter, it undergoes a pitchfork bifurcation at $\alpha = 0$. Hoping to stimulate the development of a nonautonomous bifurcation theory (see also L. Arnold [1, Chap. 8, 9]) we consider system (2.3) under the influence of a nonautonomous parametric perturbation. Let α be replaced by $\alpha+\sigma\xi(t)$, where $\xi : \mathbf{R} \to [-1,1]$ is a bounded measurable function and σ is an intensity parameter. With $x = \binom{y}{\dot{y}}$ the perturbed version of (2.3) is

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & \beta \end{pmatrix} x + (\alpha + \sigma\xi(t)) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ -x_1^3 - x_1^2 x_2 \end{pmatrix} . \tag{2.4}$$

The linearization of (2.4) at $x = 0$ is

$$\dot{v} = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} v + \sigma\xi(t) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} v .$$

We choose $\beta = -1$ and treat (α, σ) as a small two-dimensional parameter (the pitchfork scenario if $\xi \equiv 0$).

As a first step we diagonalize the linear part of (2.4) at $\alpha = \sigma = 0$ yielding (writing again x for the new coordinate $T^{-1}x$ with $T = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$)

$$\dot{x} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x + [\alpha x_c - \alpha x_s + \sigma x_c \xi(t) - \sigma x_s \xi(t) - x_c^3 + 2x_c^2 x_s - x_c x_s^2] \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \tag{2.5}$$

where $x = \binom{x_c}{x_s}$ consists of the *center coordinate* x_c and the *stable coordinate* x_s. By adding the trivial equations $\dot{\alpha} = 0$, $\dot{\sigma} = 0$ (which we will omit for notational convenience) we can apply the Normal Form Theorem to eliminate all nonresonant terms up to an order $K \geq 2$. The four components $(x_1, x_2, \alpha, \sigma)$ of the linearized system have eigenvalue real parts $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = 0$ and $\lambda_4 = 0$.

We follow the very successful scheme proposed by Elphick et al. [2] for simultaneously obtaining the normal form, eliminating the stable variables from the center equation and determining the center manifold in the autonomous situation (see also L. Arnold [1, Chap. 8.4] for the random case). We seek for a transformation

$$x \mapsto x + h(t, x, \alpha, \sigma) = \begin{pmatrix} x_c + h^c(t, x_c, x_s, \alpha, \sigma) \\ x_s + h^s(t, x_c, x_s, \alpha, \sigma) \end{pmatrix} \tag{2.6}$$

which transforms (1.2) into

$$\dot{x}_c = g^c(t, x_c, \alpha, \sigma) + O((|x_c| + |x_s|)^{N+1} + (|\alpha| + |\sigma|)^{M+1}) ,$$

$$\dot{x}_s = -x_s + g^s(t, x_c, x_s, \alpha, \sigma) + O((|x_c| + |x_s|)^{N+1} + (|\alpha| + |\sigma|)^{M+1}) ,$$

with $2 \leq N, M \leq K$. We therefore eliminate all nonresonant monomials of the form

$$f_{pquv}(t) \cdot x_1^p x_2^q \alpha^u \sigma^v \tag{2.7}$$

in the x_c- and the x_s-equation with a sequence of transformations $x_c \mapsto x_c + h^c_{pquv}(t) \cdot x_1^p x_2^q \alpha^u \sigma^v$ resp. $x_s \mapsto x_s + h^s_{pquv}(t) \cdot x_1^p x_2^q \alpha^u \sigma^v$ for $2 \leq p + q \leq N$ and $0 \leq u + v \leq M$. The composition of these transformations yields (2.6).

The step by step strategy to construct the sequence of transformations is as follows: Start with $u + v = 0$, eliminate terms (2.7) with $2 \leq p + q \leq N$. Note that $u + v + p + q$ has to be ≥ 2. In the next step eliminate all terms with $u + v = 1$ and $1 \leq p + q \leq N$, then $u + v = 2$, $0 \leq p + q \leq N$ and so forth to $u + v = M$. The computational effort for these calculations is enormous and could only be accomplished by using the computer algebra program MAPLE. To clarify the algorithm and the application of the Normal Form Theorem we calculate the first steps manually.

Consider the x_c-equation. The only terms (2.7) with $u+v=0$ are $-x_c^3$, $2x_c^2x_s$ and $-x_cx_s^2$. We can consider them in arbitrary order, since they are all of order 3 and a transformation which eliminates one of them does not change terms of order ≤ 3. The first term $-x_c^3$ corresponds to $j=1$ (first equation) and $\ell=(p,q,u,v)=(3,0,0,0,)$. To apply the normal form theorem we have to check the nonresonance condition

$$\lambda_1 \neq 3\lambda_1 + 0\lambda_2 + 0\lambda_3 + 0\lambda_4 \, .$$

Since $0 \neq 0$ is wrong, the nonresonance condition does not hold, the term $-x_c^3$ is resonant and we can not eliminate it. The nonresonance condition

$$\lambda_1 \neq 2\lambda_1 + 1\lambda_2 + 0\lambda_3 + 0\lambda_4$$

for $2x_c^2x_s$ holds and the normal form theorem yields the transformation

$$\begin{aligned} H^c_{2100}(t,x_c,x_s,\alpha,\sigma) &= x_c + \int_t^\infty e^{\lambda_1(t-\tau)} \cdot 2 \cdot [e^{\lambda_1(\tau-t)}x_c]^2 \cdot [e^{\lambda_2(\tau-t)}x_s]\, d\tau = \\ &= x_c + 2x_c^2x_s e^t \int_t^\infty e^{-\tau}\, d\tau \; = \; x_c + 2x_c^2 x_s \, . \end{aligned}$$

This transformation eliminates $2x_c^2x_s$ in the x_c-equation and produces terms of order ≥ 3, which we ignore. Also $-x_cx_s^2$ is nonresonant and can be eliminated with the transformation

$$H^c_{1200} = x_c - \tfrac{1}{2}x_cx_s^2 \, .$$

Consider now the x_s-equation, the term $2x_c^2x_s$ is resonant, but $-x_c^3$ and $-x_cx_s^2$ can be eliminated and (2.5) is equivalent to

$$\dot{x} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} \alpha x_c - \alpha x_s + \sigma x_c\xi(t) - \sigma x_s \xi(t) - x_c^3 \\ \alpha x_c - \alpha x_s + \sigma x_c \xi(t) - \sigma x_s\xi(t) - 2x_c^2x_s \end{pmatrix} .$$

Now the terms with a α or σ are considered. We refrain from writing down the calculations and use the computer algebra program MAPLE. The result for $N=3$ and $M=2$ is: The truncated transformation $h=\binom{h^c}{h^s}$ (without the $O((|x_c|+|x_s|)^{N+1}+(|\alpha|+|\sigma|)^{M+1})$ terms) has the form:

$$\begin{aligned} h^c &= \left[-\alpha + 2\alpha^2 - \sigma\xi_1 + 2\alpha\sigma(\xi_1+\xi_2) + 2\sigma^2\xi_3\right] x_s - \frac{1}{2}x_cx_s^2 + 2x_c^2x_s \, , \\ h^s &= \left[-\alpha + 2\alpha^2 - \sigma\xi_4 + 2\alpha\sigma(\xi_4+\xi_5) + 2\sigma^2\xi_6\right] x_c - x_cx_s^2 + x_c^3 \, . \end{aligned}$$

The truncated nonautonomous normal forms are:

$$\begin{aligned} \dot{x}_c &= g^c &&= [\alpha + \sigma\xi - \alpha^2 - \alpha\sigma\xi_1 - \sigma^2\xi\xi_1]x_c - x_c^3 \, , \\ \dot{x}_s &= -x_s + g^s &&= [-1 - \alpha - \sigma\xi + \alpha\sigma\xi_4]x_s + 2x_c^2x_s \, . \end{aligned} \tag{2.8}$$

In these equations the coefficients are as follows:

$$\xi_1(t) = \int_t^\infty e^{t-\tau}\xi(\tau)\, d\tau \, ,$$

$$\xi_2(t) = \int_t^\infty \xi_1(\tau)\, d\tau \, ,$$

$$\xi_3(t) = \int_t^\infty \xi(\tau)\xi_1(\tau)\, d\tau \, ,$$

$$\xi_4(t) = \int_{-\infty}^{t} e^{-(t-\tau)} \xi(\tau)\, d\tau \ ,$$

$$\xi_5(t) = \int_{-\infty}^{t} \xi_4(\tau)\, d\tau \ ,$$

$$\xi_6(t) = \int_{-\infty}^{t} \xi(\tau)\xi_4(\tau)\, d\tau \ .$$

The scalar center equation (2.8) can now be utilized for the development of a nonautonomous bifurcation theory of the original two-dimensional system (2.4).

REFERENCES

[1] L. Arnold, *Random Dynamical Systems*, Springer, Berlin - Heidelberg (1998).
[2] C. Elphick, E. Tirapegui, M. E. Brachet, P. H. Coullet, and G. Iooss, *A simple global characterization of normal forms for singular vector fields.* Physica D **29**, 95-127 (1987).
[3] S. Siegmund, *Spektral-Theorie, glatte Faserungen und Normalformen für Differentialgleichungen vom Carathéodory-Typ.* PhD thesis, University of Augsburg, Germany (1999).
[4] S. Siegmund, *Normal Forms for Nonautonomous Differential Equations*, Journal of Differential Equations (to appear).

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