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A THEOREM ON CHAOTIC DYNAMICS AND ITS APPLICATION TO DIFFERENTIAL DELAY EQUATIONS

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ABSTRACT. We announce results obtained in [3]. Let X be a Banach space, let $D \subset X$ and let $f : D \to X$ be a compact map. As a consequence of a generalized version of the Lefschetz fixed point theorem we get a result on the existence of a compact invariant set $S \subset D$ such that the dynamics generated by f restricted to S is semiconjugated to a subshift of finite type and S contains infinitely many periodic trajectories. The result is applied in a proof of the existence of symbolic dynamics of systems generated by scalar differential delay equations with a sine-like right hand side.

The aim of this note is to announce two main results of the paper [3]. First of them is Theorem 1, a result on the existence of a kind of chaotic dynamics for a compact map in a Banach space. Its weaker form was proved in [5]. In order to formulate the theorem we need to introduce several definitions.

Let X and X' be topological spaces and let (C, E) and (C', E') be pairs of subspaces in X and, respectively X'. For a map $f : C \to X'$ define the *induced map*

$$f^{\#} := f^{\#}_{(C,E),(C',E')} : C/E \to C'/E'$$

by

$$f^{\#}([x]) := \begin{cases} [f(x)] & \text{if } x \in C \setminus E,\ f(x) \in C' \setminus E', \\ [E'] & \text{if } x \in E \text{ or } f(x) \in (X' \setminus C') \cup E'. \end{cases}$$

Here C/E and C'/E' denote the quotient spaces. Recall that C/E is obtained by collapsing E to one point (denoted by $[E]$) if $E \neq \emptyset$ and $C/\emptyset$ is defined as $C \cup \{*\}$, where $* := [\emptyset]$ is a point outside of C. If $x \in C \setminus E$ then $[x]$ denotes the corresponding point in C/E.

In the case $(C, E) = (C', E')$ the map $f^{\#}$ was introduced in [4], where the condition of continuity of this map appeared in the definition of the index pair in the discrete-time Conley index theory. On can prove (see [3]) that if f is continuous and

$$f(E) \cap C' \subset E',$$
$$C' \cap \overline{f(C) \setminus C'} \subset E'$$

then the induced map $f^{\#} : C/E \to C'/E'$ is continuous.

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Let $\Sigma_r := \{1, \dots, r\}^{\mathbb{Z}}$ is the space of two-sided sequences of points from the set $\{1, \dots, r\}$ and let $\sigma : \Sigma_r \to \Sigma_r$ be the shift map

$$\sigma(\dots, s_{-1}, s_0, s_1, \dots) := (\dots, s_0, s_1, s_2, \dots).$$

σ is called the *full shift* on r symbols. Let $A = (A_{ij})_{i,j=1,\dots,r}$ be an $r \times r$ matrix, $A_{ij} = 0$ or 1. Define

$$\Sigma_A := \{\{s_n\}_{n\in\mathbb{Z}} \in \Sigma_r : A_{s_n s_{n+1}} = 1 \forall n \in \mathbb{Z}\}.$$

$\sigma_A : \Sigma_A \to \Sigma_A$, the restriction of σ is called the *subshift of finite type* generated by the matrix A. The matrix A is called *irreducible* if A^k has all entries nonzero for some positive integer k. If A is irreducible then σ_A generates a dynamical system on Σ_A which is chaotic in the Devaney's sense: the set of periodic points of σ_A is dense in Σ_A, there exists a dense trajectory in Σ_A and there is sensitive dependence on initial data (compare [1]).

Let again X be a topological space, $D \subset X$, $f : D \to X$, and let $S \subset D$ be invariant under f. Define the *set of trajectories* of f in S as

$$\text{traj}(f, S) := \{\{x_n\}_{n\in\mathbb{Z}} \in S^{\mathbb{Z}} : f(x_n) = x_{n+1} \forall n \in \mathbb{Z}\}.$$

f induces the map $f^{\mathbb{Z}} : \{x_n\}_n \mapsto \{f(x_n)\}_n$ on $S^{\mathbb{Z}}$. Its restriction $f^{\mathbb{Z}}_S : \text{traj}(f, S) \to \text{traj}(f, S)$ is the shift map,

$$f^{\mathbb{Z}}_S(\dots, x_{-1}, x_0, x_1, \dots) = (\dots, x_0, x_1, x_2, \dots).$$

In the following result $\widetilde{H}$ denotes the reduced Alexander-Spanier cohomology functor over the field of rational numbers $\mathbb{Q}$.

Theorem 1. *Let X be a Banach space, for $p = 1, \dots, r$ let (C_p, E_p) be a pair of closed absolute neighborhood retracts (ANRs) in X, and let $C_p \cap C_q = \emptyset$ provided $p \neq q$. Assume that for some n and every p*

$$\widetilde{H}^n(C_p/E_p) = \mathbb{Q}, \quad \widetilde{H}^i(C_p/E_p) = 0 \quad \forall i \neq n.$$

Let $f : \bigcup_{p=1}^r C_p \to X$ be a compact continuous map and let $A = (A_{ij})_{i,j=1,\dots,r}$ be an irreducible matrix. If for every p and q such that $A_{pq} = 1$ the induced map

$$f^{\#}_{pq} := f^{\#}_{(C_p,E_p),(C_q,E_q)} : C_p/E_p \to C_q/E_q$$

is continuous and the homomorphism

$$\widetilde{H}^*(f^{\#}_{pq}) : \widetilde{H}^*(C_p/E_p) \to \widetilde{H}^*(C_q/E_q)$$

is an isomorphism then there exists a compact set $S \subset C_1 \cup \dots \cup C_r$ invariant with respect to f and a continuous surjective map $\phi : \text{traj}(f, S) \to \Sigma_A$ such that

$$\sigma_A \circ \phi := \phi \circ f^{\mathbb{Z}}_S.$$

Moreover, the counterimage of each K-periodic sequence in Σ_A contains at least one K-periodic trajectory of f.

The above result is a consequence of a generalized version of the Lefschetz Fixed Point Theorem (also stated in [3] and in a less general setting in [5]). In Figure 1, it is shown a planar map for which Theorem 1 applies with $r = 4$, $n = 1$, and

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{1}$$

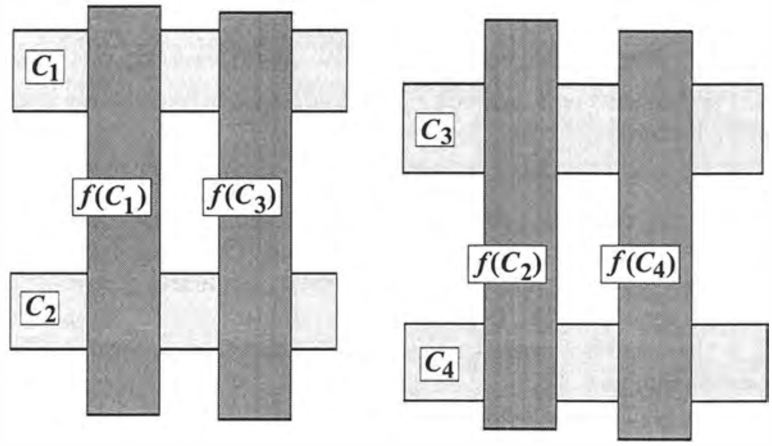

FIGURE 1. An example of a map f for which Theorem 1 applies. The sets $E_1, \dots, E_4$ and their images are equal to the horizontal sides of the rectangles.

Now we are going to formulate Theorem 2, the second main result of [3]. It is devoted to the existence of complicated solutions for some differential-delay equations of the form

$$\dot{x}(t) = g(x(t-1)) \tag{2}$$

for a continuous $g : \mathbb{R} \to \mathbb{R}$.

Assume that $T > 0$ and g is a T-periodic function. A solution $y : \mathbb{R} \to \mathbb{R}$ of (2) is called a *periodic solution of the second kind* if there exist $k \in \mathbb{Z}$ and $\omega > 0$ (called a *generalized period*) such that $y(t+\omega) = y(t) + k \cdot T$ for $t \in \mathbb{R}$. We will also consider solutions which follow the prescribed sequence of numbers. This kind of behaviour was precisely described in [2] by a sequence of the following definitions. Let $y : \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{Z}$, and $J \subset \mathbb{R}$ be an interval. One says that the function y *oscillates* about k on J if $y(J) \cap \mathbb{Z} = \{k\}$. Let $k, l \in \mathbb{Z}$, $k \neq l$. One says that y *wanders* from k to l on J if there exist subintervals J_k, J_{kl}, J_l of J with $J = J_k \cup J_{kl} \cup J_l$, with $J_k \leq J_{kl} \leq J_l$, and such that y oscillates about k on J_k, y oscillates about l on J_l, and

$$y(J_{kl}) \cap \mathbb{Z} = \{m \in \mathbb{Z} : \min\{k,l\} < m < \max\{k,l\}\}.$$

Given a sequence $l = (\dots, l_{-1}, l_0, l_1, \dots) \in \mathbb{Z}^{\mathbb{Z}}$, one says that y *follows the level sequence* l if there exist intervals J_j $(j \in \mathbb{Z})$ such that

$$\begin{gathered}
\sup_{j\in\mathbb{Z}} (\sup J_j - \inf J_j) < \infty, \\
\textstyle\bigcup_{j\in\mathbb{Z}} J_j = \mathbb{R}, \ \inf J_j < \inf J_{j+1}, \\
\sup J_j < \sup J_{j+1} \ (j \in \mathbb{Z}), \\
y \text{ wanders from } l_j \text{ to } l_{j+1} \text{ on } J_j \ (j \in \mathbb{Z}).
\end{gathered}$$

In order to state the theorem we define a kind of piecewise-linear sine-like function $\check{s} : \mathbb{R} \to \mathbb{R}$ by

$$\check{s}(x) := \begin{cases} x & \text{for } |x| \leq 1/2, \\ -x+1 & \text{for } x \in (1/2, 3/2), \\ \text{2-periodic} & \text{elsewhere.} \end{cases}$$

Theorem 2. *Let $\alpha := (9/e)(\log(9) - 1)$. For every equation (2), where g is close enough to $-\alpha\check{s}$ in the C^0-norm and g is also close enough to $-\alpha\check{s}$ in the C^1-norm, except for a small neighborhood of the set $1/2 + \mathbb{Z}$, and for every level sequence $l = (\dots, l_{-1}, l_0, l_1, l_2, \dots)$ with $l_0 = -1$, $|l_{k+1} - l_k| = 2$ $(k \in \mathbb{Z})$, there exists a solution y which follows this sequence. If l is such that the sequence $\{s_n\}$ defined by $s_n := l_{n+1} - l_n$ $(n \in \mathbb{Z})$ is periodic with minimal period $K \in \mathbb{N}$, then there exists a periodic solution y of the second kind which follows l. If, in this case, P is the minimal generalized period of y then y satisfies $y(t + P) = y(t) + l_K - l_0$ for all $t \in \mathbb{R}$.*

The choice of the class of equation for which Theorem 2 is valid was given in [2], where the existence of some heteroclinic solutions connecting constant solutions $1 + 2k$ and $1 + 2(k + 1)$ $(k \in \mathbb{Z})$ for these equations was proved. After the modulo-2 reduction, there exists a section of the semiflow generated by the considered equation in a neighborhood of the constant solution $1 + 2k$ such that its Poincaré map f behaves (in a generic case) like in Figure 1; now the horizontal direction is equal to some infinite-dimensional Banach space (hence the horizontal intervals correspond to balls) and the vertical direction is equal to $\mathbb{R}$. In [2], a further analysis on f delivered the existence of solutions which follow the prescribed level sequence. A different proof of that fact and proofs of the other conclusions of Theorem 2 is given in [3]. Its main idea can be briefly summarized as follows. Theorem 1 implies the existence of a semiconjugacy of the Poincaré map f to the shift Σ_A (with the matrix A given in (1)) on some compact invariant set and the existence of some periodic trajectories. A further reduction to the shift on two symbols Σ_2 and a suitable analysis on the obtained trajectories lead to the required result.

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