PROCEEDINGS OF THE INTERNATIONAL CONFERENCE ON
DYNAMICAL SYSTEMS AND DIFFERENTIAL EQUATIONS
May 18 – 21, 2000, Atlanta, USA pp. 366–370

ATTRACTORS AND KERNELS: LINKING NONLINEAR PDE SEMIGROUPS TO HARMONIC ANALYSIS STATE-SPACE DECOMPOSITION

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Abstract. The global attractor of an infinite-dimensional nonlinear dynamical system (whose solution semigroup S is almost periodic on a uniformly convex state-space H) is shown to coincide with K.Jacobs' set of reversible vectors H_r. This connects a body of work from abstract harmonic analysis to the qualitative theory of nonlinear PDEs, and offers a new avenue to analyzing global attractors.

1. Introduction

Infinite dimensional nonlinear dynamical systems that arise from partial differential equations (PDEs) and boundary value problems (BVPs) are often studied in a fairly standard manner (see [8] and, for comparison, [6]). First, a BVP (or initial value problem) is stipulated along with relevant function spaces or subspaces which reflect the initial conditions and in which solutions are sought. Those spaces are usually Hilbert or Banach spaces. Next, under suitable assumptions, one shows that the problem is well-posed; existence and uniqueness of a solution is demonstrated. The solution can then be expressed via a one-parameter semigroup of continuous (in general, nonlinear) maps on the space or subspace. With an aim toward finite-dimensionally capturing and characterizing the infinite dimensional system's asymptotic essence, the existence of absorbing sets is then investigated. Those are sets into which all solution trajectories (or orbits) eventually enter under the action of the semigroup. If they are present, then the possible existence of local or global attractors is considered. When there is a global attractor, such as for dissipative systems, its nature becomes a crucial issue; is it finite dimensional, and can its dimension be closely estimated. This stems in part from an interest in whether or not the solution's long-term behavior can be represented in a practical way via finite dimensional approximations. A result that offers an alternative approach to characterizing attractors is thus of interest.

This paper shows that, under appropriate conditions, the global sttractor for a nonlinear PDE/BVP coincides with (what has been long known in abstract harmonic analysis as) the subset of reversible (or recurrent, or minimal) vectors arising in K. Jacobs' state-space decomposition ([4], [5], [2]). This link between almost periodic semigroups and qualitative methods for nonlinear partial differential equations seems to have heretofore not been sufficiently appreciated.

2. Definitions, Assumptions, and Facts

We discuss the case of a one-parameter semigroup $S = \{S(t) : t \geq 0\}$ of continuous (nonlinear) maps of a uniformly convex Banach space H into itself. (In particular, H could be a Hilbert space or a suitable part of a Hilbert space in which the

AMS 2000 *Mathematics Subject Classification*. 35B41, 47H20, 37L05, 43A65.
Key words and phrases. Attractor, Almost periodic semigroup, Nonlinear semigroup, Jacobs' decomposition, Reversible vectors, Suskevic kernel.

action of S is confined). This setting encompasses several examples of infinite dimensional dynamical systems from nonlinear PDEs [8]. However, we avoid stating results in their greatest generality in an effort to make the presentation more readily understood by individuals from both specialization areas: nonlinear dynamical systems/DEs and abstract harmonic analysis. Readers wishing further elaboration and extensions are directed to the references [7], [2], and [1] as starting points.

2.1. Definitions, etc. from dynamical systems and differential equations.

A subset A of H is an attractor provided (i) A is (completely) S-invariant: $S(t)A = A$ for all $t \geq 0$, and (ii) A attracts every orbit in a neighborhood of A: there exists a neighborhod U of A such that $d(S(t)u_0, A) \to 0$ as $t \to \infty$ for all u_0 in U. As usual here $d(x,A) = \inf\{d(x,y): y \in A\}$ and $d(x,y) = \| x - y \|$. The attractor A (uniformly) attracts $B \subset U$ means $d(S(t)B, A) \to 0$ as $t \to \infty$, where $d(B_0, B_1) = \sup_{x \in B_0} \inf\{d(x,y): y \in B_1\}$. For A to be the global attractor for S means that A is a compact attractor that attracts the bounded sets of H. A global attractor is necessarily unique, and is maximal under inclusion among the bounded attractors and bounded completely S-invariant sets.

We denote the (forward, or positive) orbit (or trajectory) of a point x_0 in H by $Sx_0 = \{S(t)x_0 : t \geq 0\}$, and the ω-limit set of x_0 (resp., A) by $\omega(x_0) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)x_0}$ (resp., $\omega(A) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)A}$). Note that $z \in \omega(A) \Leftrightarrow$ there exists a sequence $\{z_n\} \subset A$ and a sequence $\{t_n\} \subset \mathbb{R}^+ = [0,+\infty)$ with $t_n \to \infty$ such that $S(t_n)z_n \to z$ as $n \to \infty$.

A subset B of an open set $U \subset H$ is said to be absorbing (or absorbs the bounded sets of U) provided the orbit of any bounded subset of U enters B after a certain time: for all bounded $B_0 \subset U$, there exists $t_1 = t_1(B_0) \in \mathbb{R}^+$ such that $S(t)B_0 \subset B$ for all $t \geq t_1$. Finally, the semigroup S is asymptotically compact if for every bounded sequence $\{x_k\}$ in H and every sequence $\{t_k\}$ in $\mathbb{R}^+$ with $t_k \to \infty$, the set $\{S(t_k)x_k : k \in \mathbb{N}\}$ is relatively compact in H.

The following result is from [8].

Theorem 2.1. *If S is asymptotically compact and if there exists an open set U in H containing a bounded absorbing (in U) set B, then $A = \omega(B)$ is a compact attractor which attracts the bounded sets of H, and is the global attractor for S.*

2.2. Definitions, etc. from abstract harmonic analysis.

For the remainder of this paper, we shall assume that the semigroup S is almost periodic, which is to say that S is equicontinuous and each orbit Sx_0 is relatively compact. The reader should be forewarned that some other presentations do not include the equicontinuity as a criteria of almost periodicity but rather keep it as a separate condition; our usage here agrees with [7]. Note that if S is dissipative (meaning if $d(S(t)x,S(t)y) \leq d(x,y)$ for all $x,y \in H$ and all $t \in \mathbb{R}^+$) it is equicontinuous. With S being almost periodic, it then follows that the closure $\overline{S}$ of S in H^H, where the latter is endowed with the product topology (i.e., the topology of pointwise convergence), is again a semigroup. Moreover it is a compact topological semigroup which is to say it has jointly continuous "product"([7], Prop. 1.1). Further, S being almost periodic implies that S is asymptotically compact, and that $\overline{Sx} = \overline{S}x$ for any x in H.

By the structure theorems for compact semigroups (e.g., [1], [2], or [7]), $\overline{S}$ thus has a minimal (two-sided) ideal $K = K(\overline{S})$ called the (Suskevic) kernel of $\overline{S}$. Further, since S is Abelian, K is a compact topological group. We say that a point x in H is reversible (or recurrent, or minimal) if $y \in \overline{S}x$ implies $x \in \overline{S}y$ and $\overline{S}x = \overline{S}y$. Equivalently, x is reversible if $\overline{S}x = \overline{S}Tx$ for each $T \in \overline{S}$. Denote the set of all reversible points by H_r. We immediately observe that $\overline{S}H_r \subseteq H_r$.

The following proposition is from [7] (where it is established for any topological space H).

Proposition 2.2. *Let x be in H. The following are equivalent.*

(i) $x \in H_r$;

(ii) *For each* $T \in \overline{S}$ *there exists a* $R \in \overline{S}$ *such that* $RTx = x$;

(iii) *There exists and idempotent map E in the kernel K such that* $Ex = x$.

(An idempotent map E satisfies the property that $E = E^2$.)

Corollary 2.3. $H_r = \{EH\colon E \in K, E = E^2\}$.

Proof. Immediate from the equivalence (i) ⇔ (iii) of the proposition. □

The next result is a special (sub-)case of Theorem 2.8 in [7].

Theorem 2.4. *For an almost periodic (nonlinear) semigroup S acting on a uniformly convex Banach space H, the following are equivalent.*

(i) *The kernel K is a compact topological group;*

(ii) *K contains a unique idempotent E.*

Corollary 2.5. *If furthermore S is Abelian, then* $H_r = EH$ *is a closed subset of H and* $K = \overline{S}E$.

Proof. If $x_0 \in \overline{H_r}$ and $\{x_n\}$ is a sequence in H_r with $\lim x_n = x_0$, then by Proposition 2.2, for each subscript n there exists an idempotent map E_n in K such that $E_nx_n = x_n$. Since S is Abelian, it follows that so is $\overline{S}$ which implies (see [7] or [2]) that K contains a unique idempotent E. Thus $E_n = E$ for all n, so that $Ex_n = x_n$. Hence $E\,x_0 = E\,(\lim x_n) = \lim Ex_n = \lim x_n = x_0$, and so (by Proposition 2.2) $x_0 \in H_r$ which shows that H_r is closed. That $H_r = EH$ follows from Corollary 2.3. Finally, $K = \overline{S}E$ is a consequence of the structure theory of compact topological (semi)groups (e.g., [2] or [1]). □

Remark 2.6. In [4] and [5], Konrad Jacobs first described the set H_r as part of his decomposition of the state-space H into H_r and H_0 (the latter being known as the set of flight, or dissipative, vectors). For elaboration in the case of a semigroup of linear operators, see [2] or [1]; for results applicable to semigroups of nonlinear maps, see [7].

The astute observer will detect a strong similarity between characterizations of H_r and H_0 and the parts into which an asymptotically compact semigroup S splits (see Remark I.1.5 in [8]). We believe that this is more than mere coincidence and are examining how to best depict parallels between the splitting of the semigroup and Jacobs' splitting of the state-space.

3. Main Result

Theorem 3.1. *If S is an Abelian almost periodic semigroup acting on a uniformly convex Banach space H, and if there exists an open set U in H containing a bounded absorbing (in U) set B, then (letting A be the global attractor provided by Theorem 2.1) we have $A = KH = H_r$.*

Proof. To show that $H_r = KH$ is now not hard. By Corollary 2.5, $K = \overline{S}E$ and $H_r = EH$. Moreover, as observed earlier, $\overline{S}H_r \subseteq H_r$ so we have that $KH = (\overline{S}E)H = \overline{S}(EH) = \overline{S}H_r \subseteq H_r$. But also (since $E \in K$) $H_r = EH \subseteq KH$.

That $A = H_r$ is established in the Theorem 2.8 and remarks following Proposition 3.1 of [7]. Alternatively, one can argue as follows. First observe that for any x in H, the orbit Sx is bounded (because it is relatively compact and hence—since H is a complete metric space—also precompact; it is a standard result that every precompact subset of a topological vector space is bounded). Also, with E as the unique idempotent map from Theorem 2.4, we have (by Corollary 2.3) $Ex \in H_r$. Next, since A is an attractor, let U_A be an open neighborhood of A such that for every u_0 in U_A, $S(t)u_0 \to u_0^* \in A$ as $t \to \infty$, and let V_ε be an absorbing ε-neighborhood of A contained in U_A. Since Sx is bounded, then for $t \geq t_{\varepsilon/2}(x)$ $d(S(t)x, A) \leq \varepsilon/2$ so $S(t)x \in V_\varepsilon$, and hence as $t \to \infty$ with $t \geq t_{\varepsilon/2}(x)$, $S(t)x \in A$.

Now let $x \in H_r$, so (by Proposition 2.2.(iii)) $Ex = x$. Let $\{S(t_\alpha)\}$ be a net in S with $t_\alpha \to \infty$ for which $\lim S(t_\alpha) = E$. (If necessary, set $S(t_\alpha) = E$ for all indices beyond some index α_0.) Then the set $\{S(t_\alpha)x\}$ is bounded (being a subset of the relatively compact orbit Sx) and so for $t_\alpha \geq t_{\varepsilon/2}(x)$ (as above), $d(S(t_\alpha)x, A) \leq \varepsilon/2$ implying that $S(t_\alpha)x \in V_\varepsilon$. Hence for $t_\alpha \geq t_{\varepsilon/2}(x)$ we have that $\lim S(t_\alpha)x \in A$. It then follows that $x = Ex = \lim S(t_\alpha)x \in A$. Therefore $H_r \subseteq A$. For the opposite inclusion, it is easy to show that A (being completely S-invariant) is completely $\overline{S}$-invariant. In particular, $EA = A$ which then yields that $A = \overline{S}A = \overline{S}(EA) \subseteq \overline{S}(EH) = (\overline{S}E)H = KH = H_r$. Thus $A = H_r$. □

Remark 3.2. The preceding alternative argument is essentially the one we presented at Y2K ICDSDE as a resolution to a 1995 conjecture. The latter, which we posed that spring to the University of Maryland's Applied Dynamics seminar, claimed that it should be possible to carry over substantial parts of Jacobs' machinery from the linear to the nonlinear case and that the image of the kernel would then correspond to the global attractor. At ICDSDE we thought our result was completely new, but a month later became aware of [7] from which the result follows (as noted above). In that paper the attractor is further characterized as a union of pairwise disjoint compact Abelian groups, and is shown to be connected whenever H is. However when H is a Hilbert space, an even more detailed characterization of the attractor $A = H_r$ for the nonlinear semigroup might also be obtainable (as is the case for a semigroup of linear operators).

Remark 3.3. Since Jacobs' decomposition prevails for a wider variety of circumstances than strong convergence in uniformly convex Banach space, Theorem 3.1 suggests that Jacobs' "machinery" offers an avenue to conclude the existence of a global attractor under other conditions than those of Theorem 2.1. In particular, a result similar to Theorem 3.1 should be achievable using weak topology, weak convergence, and with an assumption of (equicontinuity and) weak almost periodicity. Also, an extension to locally convex topological vector spaces H is also likely (analogous to what is known for the linear setting). Lastly, when linear operators can be used to approximate the nonlinear solutions (for instance, [3]), we anticipate that the more fully developed linear operator versions of Jacobs' "machinery" can be directly brought to bear in the analysis.

Acknowledgements. Thanks go to D. Cheban for helpful comments, and to A. S. Vatsala for the opportunity to present this work in the Y2K ICDSDE special ssession on Qualitative Behavior of Nonlinear Dynamic Systems.

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