PROCEEDINGS OF THE INTERNATIONAL CONFERENCE ON
DYNAMICAL SYSTEMS AND DIFFERENTIAL EQUATIONS
May 18 – 21, 2000, Atlanta, USA pp. 398–405

DYNAMICS OF GENERALIZED EULER TOPS WITH CONSTRAINTS

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Abstract. We study the dynamics of the generalized Euler tops on various groups subject to left-invariant nonholonomic constraints. We blend together reconstruction and quasi-periodic Floquet theories to analyze the qualitative dynamics and discuss differences with the free rigid body motion. We also discuss some examples of particular interest.

1. **Introduction.** In this paper we analyze the dynamics of various generalizations of the Euler top with some of its body angular velocity components set equal to zero. This system is G-invariant, where the Lie group G is the configuration space of the top. The nonholonomic constraints imposed on the body are left-invariant. The *unconstrained* generalized Euler top can be shown to be noncommutatively integrable for a wide selection of Lie groups, and hence the system evolves on a torus or on a cylinder of dimension lower than half that of the phase space.

We recall briefly the notions of reduction of mechanical systems. For a mechanical system with symmetry, the process of reduction neglects the directions along the group variables and thus provides a system with fewer degrees of freedom. In many important examples, the reduced system is integrable. Switching back to the original system is called reconstruction. If the symmetry group is abelian, then the reconstruction may be performed explicitly. The process of reconstruction in general, when the symmetry group is non-abelian, involves integration of a linear non-autonomous differential equation on a Lie group (see [12], [13] for details).

The reduced dynamics of generalized constrained Euler tops was studied in [5] and [9]. In this paper we study the dynamics in the full phase space. Our results here extend those obtained in our earlier publication [16]. As in the unconstrained case, we study the dynamics in a situation when the reduced dynamics in the Lie algebra is quasi-periodic. The reduced dynamics therefore has its own symmetry group, the m-dimensional torus, where m is the number of frequencies of a generic reduced trajectory. However, there is no analogue of non-commutative integrability

1991 *Mathematics Subject Classification.* Primary: 37J35, 37J60, 70K43; Secondary: 34C20, 70K70.

*Research patrially supported by a University of Michigan Rackham Fellowship.

†Research partially supported by NSF grant DMS–9803181, AFOSR grant F49620-96-1-0100, and an NSF group infrastructure grant at the University of Michigan.

in the nonholonomic setting. The question then arises of what is the dynamics in the full phase space.

We show that the constrained Euler top, depending on the inertia tensor, may or may not allow a bigger symmetry group, the product of G and the m-dimensional torus. If this bigger symmetry group exists, then the dynamics reduced by the action of this new group consists of equilibria only. The full dynamics in this case is either quasi-periodic with $(\operatorname{rank} G + m)$ frequencies or unbounded on a cylinder. The preference for the quasi-periodic/cylindrical dynamics is completely determined by the structure of the Lie group G. If the bigger symmetry group cannot be constructed, the dynamics is governed by equations with a small parameter. We show that the dynamics in the full phase space is approximated by the quasi-periodic dynamics on time intervals which are much longer than ones expected from the standard perturbation theory arguments. Finally we discuss some examples.

2. **Generalized Euler Tops.** In this section we introduce a dynamical system on a Lie group called the generalized Euler top. Consider a Lie group G with Lie algebra $\mathfrak{g}$. Let $\mathbb{I}$ be a symmetric positive-definite *inertia tensor*. The Lagrangian of the generalized top is the left-invariant quadratic form on $\mathfrak{g}$ defined by $L = \frac{1}{2}\langle \Omega, \mathbb{I}\Omega \rangle$. We then define the momentum by $p = \partial_\Omega L = \mathbb{I}\Omega$, where $\Omega \in \mathfrak{g}$ is the *angular velocity* of the generalized top. This system is G-invariant. The group G acts on the configuration space by left shifts. The dynamics of the generalized rigid body is governed by the following equations:

$$\dot{p}_b = C^c_{ab} I^{ad} p_c p_d, \tag{1}$$

$$\dot{g} = g\Omega, \tag{2}$$

where I^{ad} are the components of the inverse inertia tensor. For the unconstrained rigid body the indices in (1) range from 1 to n, where $n = \dim G$. Equations (1) are called the *Euler-Poincaré* equations (see [12]). These equations decouple from the full set of equations of motion. For a wide selection of algebras these equations are integrable and demonstrate quasi-periodic dynamics (see [14] and references therein). If the Euler-Poincaré equations are integrable, the dynamics of the generalized top is quasi-periodic in the full phase space as well (see [14]). This follows from analysis of the *reconstruction equation* (2). Equations (1) for $G = SO(3)$ become the well known Euler equations for a rigid body in three-dimensional space.

3. **Constrained Tops.** We now consider a generalized top with left-invariant nonholonomic constraints $\Omega^k = 0, k = l+1, \ldots, n$. Any set of constraints imposed on the angular velocity of the body can be written down this way if the appropriate basis in $\mathfrak{g}$ is chosen. The constraints are nonholonomic if and only if the subspace of $\mathfrak{g}$ spanned by the unconstrained directions is not a subalgebra. As in section 2, the system here is G-invariant, and the group action on the configuration space is free. This is important for conclusion of theorems 1 and 2.

The equations of motion of the constrained top are equations (1) and (2), but now the indices in (1) range from 1 to l, *i.e.* they label the directions of the unconstrained components of the angular velocity (see [3]). For $G = SO(3)$ with a single constraint $\Omega^3 = 0$ this system, generally called the *Suslov problem*, was introduced in [15]. A similar problem, the *Chaplygin sleigh*, (see [4]), may be viewed as a generalized constrained top on the group $SE(2)$.

Various generalization of the Suslov problem are considered in [5] and [9]. Fedorov and Kozlov [5] consider the generalized Suslov problem on $SO(n)$ while Jovanović [9] addresses the problem on various six-dimensional Lie groups. These authors study the *reduced dynamics, i.e.,* the dynamics on a subspace $\mathcal{L}^*$ of $\mathfrak{g}^*$ governed by equations (1). The main result of these papers is that the flow on $\mathcal{L}^*$ is quasi-periodic. Fedorov and Kozlov [5] suggested calling equations (1) on $\mathcal{L}^*$ the *Euler-Poincaré-Suslov* equations.

In this paper we study the dynamics in the full phase space $\mathcal{L}^* \times G$ with quasi-periodic reduced dynamics. Our proofs are based on the theorems introduced in the next section.

4. **Reconstruction.** In order to understand the dynamics of the constrained generalized Euler top in the full phase space we need to be able to solve the reconstruction equation equation (2). As we mentioned before, the reduced dynamics, *i.e.* the dynamics on $\mathcal{L}^*$ governed by Euler-Poincaré-Suslov equations (1), is assumed to be quasi-periodic. This means that for each trajectory, there exists a map $\Phi : T^m \to \mathcal{L}^*$ and a new independent variable τ such that the flow in $\mathcal{L}^*$ becomes

$$p_j(\tau) = \Phi_j(\omega_k \tau + \alpha_k), \qquad j = 1, \dots, l, \qquad k = 1, \dots, m.$$

In the simplest case $m = 0$, and the reduced dynamics consists of equilibria only. Consider such an equilibrium. A trajectory $\gamma(t)$ in the full phase space is called a *relative equilibrium* if $\pi \circ \gamma(t)$ is an equilibrium of the reduced flow, where $\pi : \mathcal{L}^* \times G \to \mathcal{L}^*$ is the projection. Consider also a group orbit through the above equilibrium of the reduced system. The properties of the relative equilibria are given by the following theorem.

Theorem 1. *The group orbit through the equilibrium of the reduced system is foliated by tori T^p, $p \leq$ rank G, with irrational flow on each torus, or by copies of $\mathbb{R}$.*

This theorem first appeared in Field [6] and recently was extended to the case of non-compact Lie groups in [1].

Now consider the case when the reduced dynamics is quasi-periodic and does not consist of equilibria. A trajectory $\gamma(t)$ is called a *relative quasi-periodic orbit* if $\pi \circ \gamma(t)$ is a quasi-periodic orbit. The closure of the reduced trajectory is a torus whose dimension equals the number of (independent) frequencies. In the case of periodic reduced trajectory ($m = 1$) we call such orbit a *relative periodic orbit.*

We first study the reconstruction process when the reconstruction equation is *reducible, i.e.* there is a (quasi-periodic) substitution $g = ha(\tau)$ that transforms (2) into the equation

$$\frac{dh}{d\tau} = h\eta, \tag{3}$$

where η is a time-independent Lie algebra element.

Theorem 2. *If the reconstruction equation is reducible, then the group orbit through the m-dimensional invariant torus of the reduced flow is foliated by tori T^p, $p \leq$ (rank $G + m$), with irrational flow on each torus, or by copies of $\mathbb{R}$ with unbounded linear flow.*

Proof. The reconstruction equation after the substitution $g = ha(t)$ becomes (3). Since η is a fixed Lie algebra element, the flow $h(t) = h_0 \exp(t\eta)$ is either quasi-periodic or unbounded linear. In the quasi-periodic case, by theorem 1 for a generic

η, $\overline{h(t)}$ is a rank G-dimensional torus in G. The corresponding trajectory of the system, $(p(t), g(t))$, is thus quasi-periodic. Generically, it has (rank $G+m$) frequencies, where m is the number of frequencies of the quasi-periodic Lie algebra element $\Omega(t)$ in (2). □

The next theorem gives a geometric criterion for reducibility of the reconstruction equation.

Theorem 3. *The reconstruction equation is reducible if and only if the full dynamics is $T^m \times G$-invariant.*

Proof. Since the reduced dynamics is quasi-periodic, we can introduce the action-angle variables (F, ϕ) in $\mathcal{L}^*$ and a new independent variable τ, in which the reduced flow equations become

$$\frac{dF}{d\tau} = 0, \qquad \frac{d\phi}{d\tau} = \omega. \tag{4}$$

The full dynamics is given by (4) supplemented with the reconstruction equation

$$\frac{dg}{d\tau} = g\Omega(\tau).$$

Suppose that the full dynamics is $T^m \times G$-invariant. We then can perform reduction with respect to the group $T^m \times G$. The reduced dynamics consists of equilibria only (given by the first of equations (4)), and thus, by theorem 1, the dynamics in $T^m \times G$ is either quasi-periodic or given by an unbounded linear flow. In particular, there exist new variables in the group G such that the $T^m \times G$-reconstruction equation becomes

$$\frac{d\phi}{d\tau} = \omega, \qquad \frac{dh}{d\tau} = h\eta,$$

where η is a *fixed* element of the Lie algebra $\mathfrak{g}$. The G-reconstruction equation is therefore reducible.

Suppose now that equation (2) is reducible. Then we can choose new coordinates h on the group G such that the reconstruction equation becomes (3) The flow in the full phase space thus is governed by equations (3) and (4). These equations are $T^m \times G$-invariant. □

An important special case of theorem 3 is the reconstruction from a reduced periodic orbit. In this case the reconstruction equation is always reducible, which follows from Floquet theory. The original system is then $S^1 \times G$-invariant, and a generic quasi-periodic motion has (rank $G + 1$) frequencies. This result was obtained by Krupa [11] and Field [7].

In the noncompact case, the preference for the torus or for $\mathbb{R}$ in theorems 1 and 2 is completely determined by the group G. In particular, if $G = SE(n)$, then the group dynamics is generically quasi-periodic on a maximal torus for even n, and is represented by unbounded linear flow for odd n. See [1] for details and other examples.

Remark 1. Of course the inertia tensor contains complete information about the dynamics of the top. In particular, it determines in principle the $T^m \times G$-invariance and reducibility of the reconstruction equation. However, there are no adequate methods that allow one to tell when this reducibility occurs. To work around this difficulty we use below the *effective reducibility approach.* It is interesting to notice that the presence of the symplectic structure in the theory of the unconstrained rigid body allows one to construct a bigger symmetry group and to establish reducibility of the reconstruction equation. The reduced dynamics consists of equilibria only. This is why the unconstrained n-dimensional rigid body is an integrable system. See Mischenko and Fomenko [14] and, for example, Bloch *et al.* [2] for details and theory of noncommutative integrability. The absence of a symplectic structure and a suitable notion of commuting integrals in nonholonomic mechanics prevents us from showing that the constrained problem is $T^m \times G$-invariant.

5. **The Case of Irreducible Reconstruction Equation.** If the reconstruction equation is irreducible, we are able in certain situations to study the group dynamics using the *effective reducibility approach* of Jorba *et al.* [8]. Suppose that, for the reduced motion $p(t)$ under consideration, the reconstruction equation becomes

$$\frac{dg}{d\tau} = g(A + \varepsilon R(t, \varepsilon)), \tag{5}$$

where A is a fixed and $R(t, \varepsilon)$ is a quasi-periodic Lie algebra elements. Then, if the set of diophantine conditions

$$|\lambda_j - \lambda_l + i(k, \omega)| \geq \frac{c}{|k|^\gamma}, \qquad l, j = 1, \dots, n, \tag{6}$$

is satisfied, where λ_j are the eigenvalues of A regarded as a linear operator in a suitable representation of the group G, the reconstruction equation in suitable variables $h = ga(t)$ becomes

$$\frac{dh}{d\tau} = h(A^*(\varepsilon) + \varepsilon R^*(t, \varepsilon))$$

with R^* exponentially small in $-(1/\varepsilon)^{1/\gamma}$. See [8] for details. The influence of the quasi-periodic terms on the group dynamics is therefore negligible on time intervals of length $\sim \exp(1/\varepsilon)$. The group flow determined by the reconstruction equation may be approximated by a quasi-periodic flow on a time interval of length $\sim \exp(1/\varepsilon)$, which is considerably longer than the time span of order $1/\varepsilon$ that would be expected from standard perturbation theory arguments.

6. **Examples.** Here we apply the approach exposed in sections 4 and 5 to generalized constrained problems on various matrix Lie groups.

6.1. **The n-Dimensional Suslov Problem.** Fedorov and Kozlov [5] consider an n-dimensional Suslov problem, *i.e.* they consider the motion of an n-dimensional rigid body with a diagonal inertia tensor $I_{ij} = \delta_{ij} I_j$, $I_1 > I_2 > \cdots > I_n$, subject to the constraints $\Omega_{ij} = 0$, $i, j > 2$, where $\Omega \in so(n)$ is the body angular velocity. This system is $SO(n)$-invariant, where the group $SO(n)$ acts on the configuration space by left shifts. Fedorov and Kozlov prove that the reduced system on $so(n)$ has $n-1$ quadratic integrals $F_0(\Omega) = h/2, F_1(\Omega) = c_1, \dots, F_{n-2}(\Omega) = c_{n-2}$, where $F_0(\Omega)$ is the positive-definite energy integral. The common level surface of these integrals is diffeomorphic to the disjoint union of $(n-2)$-dimensional tori if $h, c_1, \dots, c_{n-2}$ are all positive and satisfy the condition

$$\frac{c_1}{I_2 - I_3} + \cdots + \frac{c_{n-2}}{I_2 - I_n} < h.$$

The flow on these tori in the appropriate angular coordinates is governed by the equations

$$\frac{d\phi_1}{d\tau} = \omega_1, \ \ldots, \ \frac{d\phi_{n-2}}{d\tau} = \omega_{n-2}, \tag{7}$$

where τ is a new independent variable introduced by $d\tau = \Omega_{12}dt$. The solutions of equations (7) are quasi-periodic motions on tori. The reduced Suslov problem is therefore integrable. The frequencies $\omega_1, \ldots, \omega_{n-2}$ are the same for all the tori and depend only on the values of $I_1, \ldots, I_n$. In particular, if the trajectories are closed (periodic) on one torus, then they are closed on the rest of the tori as well. Explicitly,

$$\omega_s = \sqrt{f(I_{s+2})}, \qquad f(z) = \frac{(I_1 - z)(I_2 - z)}{(I_1 + z)(I_2 + z)}.$$

The function $f(z)$ is decreasing on the interval $[0, I_2]$ and takes values between 0 and 1. Therefore the set of $(I_1, \ldots, I_n)$ for which the diophantine conditions (6) fail is of zero measure.

The non-zero components of the body angular velocity are:

$$\Omega_{12} = \frac{1}{I_1 + I_2}\sqrt{h - \sum_{s=1}^{n-2}\left(\frac{c_s}{I_2 - I_{s+2}}\sin^2\phi_s + \frac{c_s}{I_1 - I_{s+2}}\cos^2\phi_s\right)},$$

$$\Omega_{1,s+2} = \sqrt{\frac{c_s}{(I_1 + I_{s+2})(I_2 - I_{s+2})}}\sin\phi_s,$$

$$\Omega_{2,s+2} = \sqrt{\frac{c_s}{(I_2 + I_{s+2})(I_1 - I_{s+2})}}\cos\phi_s.$$

As discussed in section 4, the full dynamics is quasi-periodic if the reconstruction equation is reducible. If it is irreducible, the effective reducibility approach can be applied in the following two cases.

Case 1. Here we consider the motions with one dominant component of the angular velocity. Put $c_s = \varepsilon b_s$, $s = 1, \ldots, n-2$. In this case $|\Omega_{l,s+2}| \ll |\Omega_{12}|$, $l = 1, 2$, $s = 1, \ldots, n-2$. Then the reconstruction equation takes the form of equation (5) where A is a constant skew-symmetric matrix with entries $A_{12} = -A_{21} = 1$ and $A_{ij} = 0$ for all other pairs of indices i and j. Assume that the diophantine conditions (6) are satisfied. As we mentioned before, this is true for a generic inertia tensor. Applying the effective reducibility approach (see section 5), we thus conclude that the trajectories of the Suslov problem may be approximated by quasi-periodic curves on the time interval of length $\sim \exp(1/\varepsilon)$.

Case 2. Now consider the case when $c_s = \varepsilon b_s$, $s = 2, \ldots, n-2$, *i.e.* we have motions with three dominant components, Ω_{12}, Ω_{13}, and Ω_{23}, of the angular velocity. Then the reconstruction equation is of the form

$$\frac{dg}{d\tau} = g(B(t) + \varepsilon Q(\tau, \varepsilon)),$$

where $B(t)$ is a $2\pi/\omega_1$-periodic function and $Q(\tau, \varepsilon)$ is a quasi-periodic function. We first find a periodic substitution $k = ga(t)$ which transforms the equation $dg/d\tau = gB(t)$ into $dk/d\tau = kA$, where A is a constant matrix. The reconstruction equation thus becomes (5). Assume that the appropriate diophantine conditions hold. Then, as above, we find a quasi-periodic substitution $h = kb(t)$ which makes quasi-periodic terms in the reconstruction equation exponentially small in $-(1/\varepsilon)^{1/\gamma}$. Thus, in the case of three dominant components of the angular velocity, we observe the same behavior as in the case of one dominant component. Summarizing, we have:

Theorem 4. *Consider the n-dimensional Suslov problem with quasi-periodic reduced flow. In the case of one or three dominant components of the angular velocity, the dynamics of the n-dimensional Suslov problem may be approximated by quasi-periodic dynamics on the time interval of length* $\sim \exp(1/\varepsilon)$.

6.2. **The Chaplygin Sleigh and the System on $SE(3)$.** The Chaplygin sleigh is a system consisting of a blade that has a single contact point with a horizontal plane. We assume here that the contact point of the blade coincides with the center of mass. The blade can slide freely along the direction of its sharp edge and can rotate around the vertical line through the contact point, but cannot move in the direction orthogonal to itself. If (x, y) are the coordinates of the contact point, ϕ is the angle between the blade and a fixed direction in the plane, and (u, v) are the components of the velocity of the blade along and orthogonal to the blade, respectively, then the Lagrangian and the constraint become

$$L = \frac{1}{2}\left(m(u^2 + v^2) + J\dot{\phi}^2\right), \qquad v = 0.$$

Note that here u and v are components of velocity along two noncommuting vector fields. This makes the constraint nonholonomic. In the above, m is the mass and J is the moment of inertia of the blade. The configuration space for this system is the group $SE(2)$. Both the Lagrangian and the constraint are $SE(2)$-invariant. The reduced dynamics is trivial and therefore, by theorem 1, the full dynamics is either periodic (maximal tori in $SE(2)$ are one-dimensional), or given by an unbounded flow. As mentioned in section 4, typically the dynamics is periodic. Physically this depends on whether or not the the initial value of the angular velocity of the blade vanishes.

Consider now a system on the group $SE(3)$ that generalizes the Chaplygin sleigh. Consider a body moving in the absence of external forces in a three-dimensional space subject to the nonholonomic constraints $v^2 = v^3 = \Omega^1 = 0$, where v^i and Ω^i are the components of the linear and angular velocity in the body frame. One can think of this system as a simple model of an arrow flying in the air. The constraints are produced by the two orthogonal fins of the arrow which prevent lateral motion and rotation due to the viscosity of the air. One can check that in this case the reduced dynamics is trivial as well. The dynamics in the full phase space, as in the previous system, is either periodic (maximal tori of $SE(3)$ are one-dimensional as well), or is represented by a linear flow. However, in this system we generically observe unbounded (nonperiodic) motions (see section 4 for the explanation).

6.3. **A System on $SE(3)$ with Nontrivial Reduced Dynamics.** Jovanović [9] shows that a system on $SE(3)$ with the Lagrangian

$$L = \frac{1}{2}\langle A\Omega_+, \Omega_+\rangle + \langle C\Omega_+, \Omega_-\rangle + \langle B\Omega_-, \Omega_-\rangle, \qquad (\Omega_+, \Omega_-) \in se(3),$$

and the constraint

$$\sigma_+\Omega_+^1 + \sigma_-\Omega_-^1 = 0$$

exhibits quasi-periodic integrable reduced dynamics. In the above equations, A, B, and C are diagonal matrices, and σ_+ and σ_- are real numbers.

As before, we need to distinguish the cases of reducible and irreducible reconstruction equation. If it is reducible, then, because of the properties of the group $SE(3)$, the generic trajectory in the full phase space is represented by an unbounded linear flow. Thus, generically we will not see invariant tori in the full phase space. The same conclusion is true for the case of an irreducible reconstruction equation. We emphasize that this conclusion follows from the topological structure of a generic commutative subgroup of $SE(3)$.

Acknowledgments. We would like to thank Professors A. Derdzinski, M.G. Forest, and J. Marsden for helpful discussions.

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