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OSCILLATORY THEOREMS OF N-TH ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This paper is to obtain sufficient conditions under which the functional differential equation $x^{(n)}(t) + F(t, x(g_1(t)),\ x(g_2(t)), \ldots, x(g_m(t)) = h(t),\ n > 2$ has oscillatory, nonoscillatory or almost oscillatory solutions.

1. Introduction.

We are here concerned with the oscillatory and nonoscillatory behavior of solutions of higher order functional differential equations of the form

$$x^{(n)}(t) + F(t, x(g_1(t)),\ x(g_2(t)), \ldots, x(g_m(t)) = h(t),\ n > 2 \tag{1.1}$$

where $g_i, h : [t_0, \infty) \to R,\ F : [t_0, \infty) \times R^m \to R$ are continuous, and $g_i(t) \to \infty$ as $t \to \infty,\ i = 1, 2, \ldots m$.

We consider only solutions of (1.1) which are non-trivial and defined for all $t \geq t_0$. Such a solution is said to be oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. A nonoscillatory solution is said to be strongly monotone if it tends to zero as $t \to \infty$ together with its derivatives up to $n-1$, where n is the order of the differential equation. Equation (1.1) is called oscillatory if all its solutions are oscillatory and almost oscillatory if all the solutions are either oscillatory or strongly monotone. We will use the following lemma in this paper.

Lemma 1. *(Kiguradze [8]) If $u(t)$ is an n-times differentiable function on R^+ of constant sign, $u^{(n)}(t)$ is of constant sign and not identically zero in any interval $[t_1, \infty)$ and $u(t)u^{(n)}(t) \leq 0$, then*

i) *there exists a $t_2 \geq t_1$ such that the functions $u^{(k)}(t),\ k = 1, 2, \ldots, n-1$, are of constant sign on $[t_1, \infty)$,*

ii) *there is an integer $l, 0 \leq l \leq n-1$ with $n - \ell$ is odd, such that for $t \geq t_2$*

$$u(t)u^{(k)}(t) > 0,\ k = 0, 1, \ldots, l$$
$$(-1)^{n-k-1}u(t)u^{(k)}(t) > 0,\ k = l, \ldots, n-1$$

and if $1 \leq l \leq n-1$

$$|u^{(l-k-1)}(t)| \geq \frac{t - t_2}{k+1}|u^{(l-k)}(t)| \quad \textit{for}\ \ k = 0, 1, \ldots, l-1$$

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2. Oscillatory behavior.

Theorem 1. *Suppose that*

(a) $x_1 F(t, x_1, x_2, \ldots, x_m) > 0$ *for* $x_1 x_i > 0,\ i = 2, 3, \ldots, m$

(b) $|F(t, x_1, x_2, \ldots, x_m)| \leq |F(t, y_1, y_2, \ldots, y_m)|$ *for* $|x_i| \leq |y_i|,\ x_i y_i > 0,\ i = 1, 2, \ldots, m$

(c) *There is an oscillatory function* $\phi(t)$ *such that*

$$\lim_{t\to\infty} \phi^{(i)}(t) = 0 \ \textit{ for } \ i = 0, 1, \ldots, n-1 \ \textit{ and } \ \phi^{(n)}(t) = h(t).$$

If

$$\int^{\infty} t^{n-1} F(t, a, a, \ldots, a) dt = \infty \ \textit{ for every } \ a > 0, \tag{2.1}$$

then every bounded solution $x(t)$ of (1.1) is oscillatory when n is even, and every solution $x(t)$ of (1.1) is almost oscillatory when n is odd.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) such that $\lim_{t\to\infty} x(t) \neq 0$ when n is odd. Without loss of generality, suppose that $x(t) > 0$ and $x(g_i(t)) > 0,\ i = 1, 2, \ldots, m$, for $t \geq t_1$. Setting $x(t) = y(t) + \phi(t)$ it follows from (1.1) that

$$y^{(n)}(t) = -F(t, x(g_1(t)),\ x(g_2(t)), \ldots, x(g_m(t))) \leq 0 \tag{2.2}$$

for $t \geq t_1$. It is easy to see that $y(t)$ is eventually positive. Then $y(t)y^{(n)}(t) \leq 0$ and therefore by Lemma 1, in view of the fact that $y(t)$ is bounded, it follows that there is a $t_2 \geq t_1$ such that for $t \geq t_2$

$$(-1)^{j-1} y^{(j)}(t) > 0,\ j = 1, 2, \ldots, n-1 \ \text{ for } n \text{ even} \tag{2.3}$$

and

$$(-1)^{j} y^{(j)}(t) > 0,\ j = 1, 2, \ldots, n-1 \ \text{ for } n \text{ odd}. \tag{2.4}$$

In both cases we have

$$\lim_{t\to\infty} y^{(j)}(t) = 0,\ j = 1, 2, \ldots, n-1. \tag{2.5}$$

We now integrate (1.1) $(n-1)$ times from t to ∞ and use (2.5) to arrive at

$$(-1)^n y'(t) = \frac{1}{(n-2)!} \int_t^{\infty} (s-t)^{n-2} F(s, x(g_1(s)), \ldots, x(g_m(s))) ds.$$

Integrating the above equation between T and $t, t > T \geq t_2$, we get

$$\begin{aligned}(-1)^n [y(t) - y(T)] &= \frac{1}{(n-2)!} \int_T^t \int_r^{\infty} (s-r)^{n-2} F(s, x(g_1(s)), \ldots, x(g_m(s))) ds\, dr \\ &\geq \frac{1}{(n-1)!} \int_T^t (s-T)^{n-1} F(s, x(g_1(s)), \ldots, x(g_m(s))) ds.\end{aligned} \tag{2.6}$$

Let n be even. Then we see from (2.3) that $y'(t) > 0$ and so $\lim_{t\to\infty} y(t) = y(\infty) > 0$ exists. Furthermore $\lim_{t\to\infty} x(t) = y(\infty) > 0$. Using (b), (taking a greater T if necessary), it is clear that

$$F(t, x(g_1(t)), \ldots, x(g_m(t))) \geq F(s, \frac{1}{2}y(\infty), \ldots, \frac{1}{2}y(\infty)) \text{ for } t \geq T \tag{2.7}$$

Using (2.7) in (2.6) we obtain

$$y(t) - y(T) \geq \frac{1}{(n-1)!}\int_T^t (s-T)^{n-1} F(s, \frac{1}{2}y(\infty), \ldots, \frac{1}{2}y(\infty))ds \tag{2.8}$$

Letting $t \to \infty$ in (2.8) we get a contradiction to (2.1). Therefore $x(t)$ must be oscillatory.

Let n be odd. Then from (2.4) $y'(t) < 0$ and so $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = y(\infty) \geq 0$. By our assumption $y(\infty) > 0$. This however proceeding as above results in a contradiction with (2.1). Therefore either $x(t)$ is oscillatory or else it satisfies $\lim_{t\to\infty} x(t) = 0$. If the latter holds, then because of (c) and (2.5) we have $\lim_{t\to\infty} x^{(i)}(t) = 0, i = 1, 2, \ldots, n-1$. This clearly completes the proof.

In the next theorem we show that if

$$\int^{\infty} t^{n-1}|h(t)|dt < \infty \tag{2.9}$$

is satisfied and (2.1) fails to hold for some $a > 0$, then there exists a bounded nonoscillatory solution $x(t)$ of (1.1).

Theorem 2. *Let (a)–(c) and (2.9) hold. If (2.1) does not hold, then (1.1) has a bounded nonoscillatory solution such that* $\lim_{t\to\infty} x(t) \neq 0$.

Proof. Let $a > 0$, and let $T \geq 0$ be so large that

$$\int_T^{\infty} t^{n-1} F(t, a, a, \ldots, a)dt < \frac{a}{8}(n-1)!, \tag{2.10}$$

and

$$\int_T^{\infty} t^{n-1}|h(t)|dt < \frac{a}{8}(n-1)!. \tag{2.11}$$

Set $T_0 = \inf_{1\leq i\leq m} \{g_i(t) : t \geq T\}$ and let C be the space of continuous functions on $[T_0, \infty)$. Let X be a subset of C defined by

$$X = \{x \in C : a/2 \leq x \leq a\}.$$

It is obvious that X is bounded, closed, and convex. We now define an operator S on X by

$$(Sx)(t) = \begin{cases} \Phi(t) & \text{if } t \geq T \\ \Phi(T) & \text{if } t \leq T \end{cases}$$

where

$$\Phi(t) = \frac{3a}{4} + \frac{(-1)^{n-1}}{(n-1)!}\int_t^\infty (s-t)^{n-1}F(s, x(g_1(s)),\ x(g_2(s)), \ldots, x(g_m(s)))ds$$
$$- \frac{(-1)^n}{(n-1)!}\int_t^\infty (s-t)^{n-1}h(s)ds.$$

(i) $\mathcal{S}$ maps X into itself: In fact, for any $x \in X$, we have

$$\left|(\mathcal{S}x)(t) - \frac{3a}{4}\right| \leq \frac{1}{(n-1)!}[\int_T^\infty s^{n-1}|F|ds + \int_T^\infty s^{n-1}|h|ds].$$

So using (2.10) and (2.11), it follows from the above inequality that

$$\left|(\mathcal{S}x)(t) - \frac{3a}{4}\right| \leq \frac{a}{4}$$

i.e.,

$$\frac{a}{2} \leq (\mathcal{S}x)(t) \leq a.$$

(ii) $\mathcal{S}$ is continuous: To see this, let $\{x_k\}$ be a cauchy sequence in X, and let $\lim_{n\to\infty} \|x_k - x\| = 0$. Clearly $x \in X$. To prove the continuity of $\mathcal{S}$, we easily see that

$$\|(\mathcal{S}x_k)(t) - (\mathcal{S}x)(t)\| \leq \int_T^\infty s^{n-1}G_k(s)ds$$

where

$$G_k(s) = s^{n-1}|F(s, x_k(g_1(s)), \ldots, x_k(g_m(s))) - F(s, x(g_1(s)), \ldots, x(g_m(s)))|.$$

It is obvious that $\lim_{t\to\infty} G_k(s) = 0$ and $G_k(s) \leq 2s^{n-1}F(s, a, \ldots, a)|$. The above relations (2.10) and the Lebesque dominated convergence theorem gives us

$$\lim_{n\to\infty} \|\mathcal{S}x_n - \mathcal{S}x\| = 0.$$

which implies that $\mathcal{S}$ is continuous.

$\mathcal{S}X$ is precompact: To prove this, since $\mathcal{S}X$ is uniformly bounded, we only need to show that it is an equicontinuous family of functions on $[T_0, \infty)$.

Let $x \in X$ and $t_2 > t_1$. Then

$$|(\mathcal{S}x)(t_2) - (\mathcal{S}x)(t_1)| \leq \frac{2}{(n-1)!}[\int_{t_1}^\infty s^{n-1}F(s, a, \ldots, a)ds + \int_{t_1}^\infty s^{n-1}|h(s)|ds] \quad (2.12)$$

Since for any $\varepsilon > 0$ there exists a $T_1 > T$ such that

$$\int_{T_1}^\infty F(s, a, \ldots, a)ds < \frac{\varepsilon}{4}$$

and $\int\limits_{T_1}^{\infty} s^{n-1}|h(s)|ds < \frac{\varepsilon}{2}$, we see that if $t_1 \geq T_1$, then

$$|(\mathcal{S}x)(t_2) - (\mathcal{S}x)(t_1)| < \varepsilon \quad \text{for all } \; x \in X. \tag{2.13}$$

Suppose now $T_0 \leq t_1 < t_2 \leq T_1$. Then

$$\begin{aligned}(\mathcal{S}x)(t_2) - (\mathcal{S}x)(t_1) = &\int\limits_{t_2}^{\infty} |(s-t_2)^{n-1} - (s-t_1)^{n-1}|F\, ds \\ &+ \int\limits_{t_2}^{\infty} |(s-t_2)^{n-1} - (s-t_1)^{n-1}|\, |h(s)|ds \\ &+ \int\limits_{t_1}^{t_2} (s-t_1)^{n-1}F\, ds + \int\limits_{t_1}^{t_2} |(s-t_1)^{n-1}|h(s)|ds.\end{aligned}$$

In view of the inequality

$$|(s-t_2)^{n-1} - (s-t_1)^{n-1}| \leq (n-1)(t_2-t_1)s^{n-2}$$

we get

$$\begin{aligned}(\mathcal{S}x)(t_2) - (\mathcal{S}x)(t_1) =&(n-1)(t_2-t_1) \int\limits_{t_2}^{\infty} s^{n-1}F\, ds \\ &+ \int\limits_{t_2}^{\infty} s^{n-1}|h(s)|ds \\ &+ \int\limits_{t_1}^{t_2} s^{n-1}F\, ds + \int\limits_{t_1}^{t_2} s^{n-1}|h(s)|ds.\end{aligned}$$

Clearly, for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|(\mathcal{S}x)(t_2) - (\mathcal{S}x)(t_1)| < \varepsilon, \;\; |t_2 - t_1| < \delta, \quad \text{for all } \; x \in X.$$

This means that the interval $[T_0, \infty)$ can be divided into a finite number of subintervals on which every $(\mathcal{S}x)(t), x \in X$, has oscillation less than ε. Thus $\mathcal{S}X$ is an equicontinuous family on $[t_0, \infty)$, and hence it is a compact subset of X.

It follows from the Shauder's fixed point theorem that there exists a $x \in X$ such that $\mathcal{S}(x) = x$. It is clear that x is a bounded nonoscillatory solution of (1.1).

Corollary 1. *Let (a)–(c) and (2.9) hold. Then a necessary and sufficient condition in order that every bounded solution of (1.1) be oscillatory if n is even and be either oscillatory or such that*

$$\lim_{t \to \infty} x(t) = 0$$

if n is odd is that

$$\int^{\infty} t^{n-1}F(t, a, a, \ldots, a)dt = \infty \quad \textit{for every } a > 0.$$

Example 1. Consider the equation

$$x'''(t) + \frac{6}{t^4}x^3(t) = \frac{3t^2 - 3t + 1}{t^4(t-1)^3}, \tag{2.14}$$

so that $q(t) = 6t^{-4}, f(y) = y^3$, $h(t) = 3t^2 - 3t + 1/t^4(t-1)^3$, and $g(t) = t - 1$. All conditions of Theorem 2 are satisfied and so the equation (2.14) has a bounded nonoscillatory solution. In fact $x(t) = 1 + t^{-1}$ is a bounded solution of (2.14).

Remark. If F is replaced by $-F$ in the equation (1.1) then one can show that Theorem 1, Theorem 2, and Corollary 1 hold when the words even and odd are interchanged in the conclusions.

So far we have been concerned with the bounded solutions of (1.1). To treat all the solutions it is natural to impose more conditions on the function F that appears in (1.1). We assume that there are continuous functions $Q : [t_0, \infty) \to [0, \infty)$ and $G : R^m \to R$, $x_1G(x_1, x_2, \ldots, x_m) > 0$ for $x_1x_i > 0$, $i = 2, 3, \ldots, m$ such that

$$|F(t, x_1, x_2, \ldots, x_m) \geq Q(t)|G(x_1, x_2, \ldots, x_m)|. \tag{2.15}$$

In addition, there is a nondecreasing function $H : R \to R$, $uH(u) > 0$ for $u \neq 0$, which is superlinear in the sense that

$$\int^{\infty} \frac{1}{|H(u)|}du < \infty \tag{2.16}$$

such that

$$\exists C > 0 \text{ such that } \left|\frac{G(x_1, x_2, \ldots, x_m)}{H(u)}\right| > C, \ |x_i| \geq |u|, \ i = 1, 2, \ldots, m \tag{2.17}$$

Theorem 3. *Suppose that*

(1) *There is a continuously differentiable function $g(t)$ such that $g(t) \leq \min\{t, g_i(t)\}$,*

$i = 1, 2, \ldots, m$, $g(t) \to \infty$, and $g'(t) \geq 0$.

(2) *There is an oscillatory function $\rho(t)$ such that $\rho^{(n)}(t) = h(t)$ and $\rho^{(i)}(t) \to 0$, $i = 1, 2, \ldots, m$.*

(3)
$$\int^{\infty} t^{n-1}Q(t)dt = \infty.$$

Then every solution of (1.1) is oscillatory if n is even, and is almost oscillatory if n is odd.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). We may assume that $x(t)$ is eventually positive. Proceeding as in the proof of Theorem 1, one can easily see that there is a $t_1 \geq t_0$ such that $y(t) = x(t) - \rho(t) > 0$ and $y^{(n)}(t) \leq 0$ for all $t \geq t_1$. Since $y(t)y^{(n)}(t) \leq 0$ and $y^{(n)}(t)$ is not identically zero on any half-line of the form $[t_*, \infty)$ for some $t_* \geq t_0$, by application Lemma 1, there are a $t_2 \geq t_1$ and an integer number $l \in \{0, 1, 2, \ldots, n-1\}$ with $(-1)^{n-l-1} = 1$ such that for $t \geq t_2$

$$\begin{aligned} y^{(i)}(t) &> 0, \quad i = 1, 2, \ldots, l \\ (-1)^{i-1}y^{(i)}(t) &> 0, \quad i = l, \ldots, n-1. \end{aligned} \tag{2.18}$$

Now since

$$F(t, a, a, \ldots, a) \geq Q(t)G(a, a, \ldots, a)$$

and

$$\int^{\infty} t^{n-1}Q(t)dt = \infty,$$

it follows from Theorem 1 that every bounded solution of (1.1) is oscillatory if n is even, and is almost oscillatory if n is odd. Therefore it is enough to show that every unbounded solution $x(t)$ of (1.1) is oscillatory. Suppose that $x(t)$ is an unbounded nonoscillatory solution of (1.1). Clearly this means that the number l associated with $y(t)$ is greater than or equal to one when n is even, and is greater than or equal to two when n is odd. Suppose that $l \in \{1, 2, \ldots, n-1\}$. Then by Taylor's theorem for $s \geq t \geq t_2$,

$$y^{(l)}(t) = \sum_{j=0}^{n-l-1} \frac{y^{(l+j)}(s)}{j!}(t-s)^j - \int_t^s \frac{(t-u)^{n-l-1}}{(n-l-1)!} y^{(n)}(u)du.$$

Using (2.18) and (2.15) it follows from the above inequality that

$$y^{(l)}(t) \geq \int_t^{\infty} \frac{(u-t)^{n-l-1}}{(n-l-1)!} Q(u)G(x(g_1(u)), \ldots, x(g_m(u))))du. \tag{2.19}$$

On the other hand, since $y(t)$ is positive and increasing, and $\rho(t) \to \infty$ as $t \to \infty$ there is a $t_3 \geq t_2$ such that for every $\lambda \in (0,1)$ and for every $t \geq t_3$,

$$x(g_i(t)) \geq \lambda y(g_i(t)), \quad i = 1, 2, \ldots, m$$

and therefore by (1),

$$x(g_i(t)) \geq \lambda y(g(t)), \quad i = 1, 2, \ldots, m.$$

In view of the last inequality and the fact that $\lim_{t\to\infty} y(t) = \infty$, taking a larger t_3 if necessary, it follows from (2.17) that

$$\frac{G(x(g_1(t)), \ldots, x(g_m(t)))}{H(\lambda y(g(t)))} \geq c > 0, \tag{2.20}$$

where c is a constant

Now for $t \geq t_3$ using (2.20) in (2.19) we have

$$y^{(l)}(t) \geq \int_t^{\infty} \frac{(u-t)^{n-l-1}}{(n-l-1)!} Q(u)H(\lambda y(g(u)))du$$

and so

$$y^{(l)}(g(t)) \geq \int_t^{\infty} \frac{(g(u)-g(t))^{n-l-1}}{(n-l-1)!} Q(u)H(\lambda y(g(u)))du. \tag{2.21}$$

Suppose that $l = 1$. Then from (2.21) with the help of the fact that $g(t)$ is nondecreasing we obtain

$$\frac{y'(g(t))}{H(\lambda y(g(t)))} \geq \int_t^s \frac{(g(u)-g(t))^{n-l-1}}{(n-l-1)!} Q(u)du. \tag{2.22}$$

Multiplying both sides of (2.22) by $g'(t)$ and integrating from t_3 to ∞ we see that

$$\int_{\lambda y(g(t_3))}^{\infty} \frac{du}{H(u)} \geq \int_{t_3}^{\infty} \frac{(g(u)-g(t_3))^{n-l-1}}{(n-l-1)!} Q(u)du, \tag{2.23}$$

which contradicts to (2.16) and (3). Thus $l = 1$ is not possible. On the other hand if $l \geq 2$ then we integrate (2.21) multiplied by $g'(t)$ from t_3 to t and see that

$$y^{(l-1)}(g(t)) \geq \frac{(g(t)-g(t_3))^{n-l-1}}{(n-l-1)!} \int_t^{\infty} Q(u)H(\lambda y(g(u)))du.$$

Repeating this procedure one can easily see that

$$y'(g(t)) \geq \frac{(g(t)-g(t_3))^{n-2}}{(n-2)!} \int_t^{\infty} Q(u)H(\lambda g(u))du.$$

Multiplying the above inequality by $g'(t)/H(\lambda g(t))$ and integrating between the limits t_3 and ∞ we again arrive at the inequality (2.23) and therefore we conclude that l cannot be greater than or equal to 2. This last observation clearly completes the proof.

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