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THE SPECTRAL GAP OF GRAPHS AND STEKLOV EIGENVALUES ON SURFACES

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Abstract. Using expander graphs, we construct a sequence $\{\Omega_N\}_{N\in\mathbb{N}}$ of smooth compact surfaces with boundary of perimeter N, and with the first non-zero Steklov eigenvalue $\sigma_1(\Omega_N)$ uniformly bounded away from zero. This answers a question which was raised in [10]. The sequence $\sigma_1(\Omega_N)L(\partial\Omega_n)$ grows linearly with the genus of Ω_N, which is the optimal growth rate.

1. Introduction

Let Ω be a compact, connected, orientable smooth Riemannian surface with boundary $\Sigma=\partial\Omega$ of length $L(\Sigma)$. The Steklov eigenvalue problem on Ω is

$$\Delta f=0 \text{ in } \Omega,\quad \partial_\nu f=\sigma f \text{ on } \Sigma,$$

where Δ is the Laplace-Beltrami operator on Ω and ∂_ν denotes the outward normal derivative along the boundary Σ. The Steklov spectrum of Ω is denoted by

$$0=\sigma_0<\sigma_1(\Omega)\leq\sigma_2(\Omega)\leq\cdots\nearrow\infty,$$

where each eigenvalue is repeated according to its multiplicity. In [10], the second author and I. Polterovich asked the following question:

Is there a sequence $\{\Omega_N\}$ of surfaces with boundary such that $\sigma_1(\Omega_N)L(\partial\Omega_N)$ goes to ∞ as N goes to ∞?

The goal of this paper is to give a positive answer to this question.

Theorem 1. *There exist a sequence $\{\Omega_N\}_{N\in\mathbb{N}}$ of compact surfaces with boundary and a constant $C>0$ such that for each $N\in\mathbb{N}$, genus$(\Omega_N)=1+N$, and*

$$\sigma_1(\Omega_N)L(\partial\Omega_N)\geq CN.$$

For each $\gamma\in\mathbb{N}$, consider the class $\mathcal{S}_\gamma$ of all smooth compact surfaces of genus γ with non-empty boundary and define

$$\sigma^\star(\gamma)=\sup_{\Omega\in\mathcal{S}_\gamma}\sigma_1(\Omega)L(\partial\Omega).$$

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Few results on $\sigma^\star(\gamma)$ are known. G. Kokarev proved [13] that for each genus γ,

$$\sigma^\star(\gamma) \leq 8\pi(\gamma+1). \tag{1}$$

See also [7] for a similar bound involving the higher Steklov eigenvalues $\sigma_k(\Omega)$, and [10] for a bound involving the number of boundary components. In [9], A. Fraser and R. Schoen proved that $\sigma^\star(0) = 4\pi$. In this case, the supremum is attained in the limit by a sequence of surfaces with their number of boundary components tending to infinity.

Corollary 2. *There exists a constant $C > 0$ such that for each $\gamma \in \mathbb{N}$, $\sigma^\star(\gamma) \geq C\gamma$.*

It follows from Inequality (1) that this linear growth rate is optimal. In the construction of $\{\Omega_N\}$ that we propose, the number of boundary components also tends to infinity. It would be interesting to know if this condition is necessary.

Remark 3. *On a closed surface M, the problem to bound the first non-zero normalized eigenvalue $\lambda_1(M)Area(M)$ of the Laplace operator is well known. In particular, Inequality (1) is similar to the celebrated Yang-Yau inequality* [16]*. The problem of constructing closed surfaces M with large normalized eigenvalue $\lambda_1(M)Area(M)$ has been considered by several authors. See for instance* [5, 2, 3, 4]*, where the eigenvalues of surfaces are also compared to those of graphs, mostly in the special case of coverings and Cayley graphs of the groups of their automorphisms. Our proofs also use techniques which are related to those of* [8].

Plan of the paper. In Section 2, we present the construction of a surface Ω_Γ which is obtained from a regular graph $\Gamma = (V, E)$ by gluing copies of a *fundamental piece* following the pattern of the graph Γ. In Section 3, we introduce the spectrum of the graph Γ and state a comparison result (Theorem 7) between $\lambda_1(\Gamma)$ and $\sigma_1(\Omega_\Gamma)$. This is then used, in conjunction with expander graphs, to prove Theorem 1. In Section 4, we present the comparison argument leading to the proof of Theorem 7.

Remark 4. *While this paper was in the final stage of its preparation, we learned that Mikhail Karpukhin* [12] *has also developed a method for constructing surfaces with a large normalized Steklov eigenvalue $\sigma_1 L$.*

2. Constructing surfaces from graphs

Let Γ be a finite connected regular graph of degree k. The set of vertices of Γ is denoted $V = V(\Gamma)$, the set of edges is denoted $E = E(\Gamma)$. The number of vertices of Γ is $|V(\Gamma)|$. We will construct a Riemannian surface Ω_Γ modelled on the graphs Γ from a fixed orientable Riemannian surface M_0 which we call the *fundamental piece* (See Figure 1) and which is assumed to satisfy the following hypotheses:

(1) The boundary of M_0 has $k+1$ components Σ_0, $B_1, \cdots, B_k$.
(2) Each of the boundary component is a geodesic curve of length 1.
(3) The component Σ_0 has a neighbourhood which is isometric to the cylinder $C_0 = \Sigma_0 \times [0, 1] \subset M_0$, and Σ_0 corresponds to $\{0\} \times \Sigma$.

The manifold Ω_Γ is obtained by gluing copies of the fundamental piece M_0 following the pattern of the graph Γ: to each vertex $v \in V$, there corresponds a isometric copy M_v of M_0. The edges emanating from a vertex $v \in V(\Gamma)$ are labelled $e_1(v), \cdots, e_k(v)$. The corresponding boundary components $B_1, \cdots, B_k$ are identified along these edges: if $v \sim w$ then there are $1 \leq i, j \leq k$ such that $e_i(v) = e_j(w)$ and the boundary component B_i of M_v is identified to the boundary

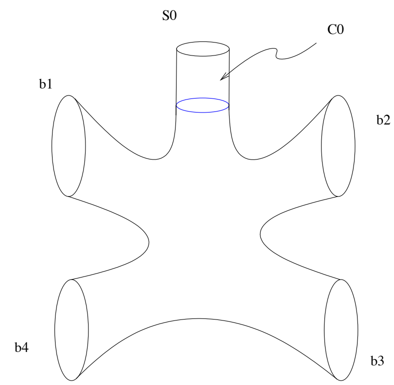

FIGURE 1. The fundamental piece M_0 for a graph Γ of degree 4.

component B_j of M_w. The manifold Ω_Γ has one boundary component Σ_v for each vertex $v \in V(\Gamma)$, each of them being isometric to Σ_0 with corresponding cylindrical neighbourhood $M_v \supset C_v \cong [0,1] \times \Sigma_0$

The following lemma shows that the genus of Ω_Γ grows linearly with the number of vertices of the graph Γ.

Lemma 5. *The genus of the surface* Ω_Γ *is*

$$\gamma(\Omega_\Gamma) = 1 + \left(\gamma(M_0) + \frac{k}{2} - 1\right)|V(\Gamma)|,$$

where $\gamma(M_0)$ *is the genus of the fundamental piece* M_0*, and* $|V(\Gamma)|$ *is the number of vertices of* Γ.

Remark 6. *Because the number of vertices of odd degree is always an even number,* $1 + (\gamma(M_0) + \frac{k}{2} - 1)|V(\Gamma)|$ *is an integer.*

Proof of Lemma 5. The genus γ and the Euler-Poincaré characteristic χ of a smooth compact orientable surface with b boundary components are related by the formula

$$\chi = 2 - 2\gamma - b.$$

Let $K : \Omega_\Gamma \to \mathbb{R}$ be the Gauss curvature. Since the boundary curves $\Sigma_0, B_1, \cdots, B_k$ are geodesics, it follows from the Gauss-Bonnet formula hat

$$\chi(\Omega_\Gamma) = \frac{1}{2\pi}\int_{\Omega_\Gamma} K = \frac{1}{2\pi}|V(\Gamma)|\int_{M_0} K = \chi(M_0)|V(\Gamma)| = (1 - 2\gamma(M_0) - k)|V|,$$

where we have used that the number of boundary components of M_0 is $k+1$. It follows that the genus of Ω_Γ is

$$\gamma(\Omega_\Gamma) = \frac{1}{2}(2 - \chi(\Omega_\Gamma) - |V|) = 1 + \left(\gamma(M_0) + \frac{k}{2} - 1\right)|V|.$$

□

3. Comparing eigenvalues on graphs to Steklov eigenvalues

Our main reference for spectral theory on graphs is [6]. The space

$$\ell^2(V(\Gamma)) = \{x : V(\Gamma) \to \mathbb{R}\}$$

is equipped with the norm defined by

$$\|x\|^2 = \sum_{v\in V(\Gamma)} x(v)^2.$$

The discrete Laplacian Δ_Γ acts on $\ell^2(V(\Gamma))$ and is defined by the quadratic form

$$q_\Gamma(x) = \sum_{v\sim w}\big(x(v)-x(w)\big)^2, \tag{2}$$

where the symbol $v \sim w$ means that the vertices v and w of Γ are adjacent, and the sum appearing in (2) is over all non-oriented edges of Γ. The discrete Laplacian Δ_Γ has a finite non-negative spectrum which we denote by

$$0 = \lambda_0 < \lambda_1(\Gamma) \leq \lambda_2(\Gamma) \leq \cdots \leq \lambda_{|V|-1}(\Gamma),$$

where each eigenvalue is repeated according to its multiplicity. The first non zero eigenvalue admits the following variational characterization:

$$\lambda_1(\Gamma) = \min\left\{\frac{q_\Gamma(x)}{\|x\|^2}\,\Big|\, x : V(\Gamma)\to\mathbb{R},\ \sum_{v\in V(\Gamma)} x(v) = 0\right\}. \tag{3}$$

In order to compare $\lambda_1(\Gamma)$ to the first non-zero Steklov eigenvalue of Ω_Γ, the following variational characterization will be used:

$$\sigma_1(\Omega_\Gamma) = \inf\left\{\int_{\Omega_\Gamma}|\nabla f|^2\,\Big|\, f\in C^\infty(\Omega_\Gamma),\ \int_{\partial\Omega_\Gamma} f = 0,\ \int_{\partial\Omega_\Gamma} f^2 = 1\right\}.$$

The main result of this paper will follow from the following estimate.

Theorem 7. *There exist constants $\alpha,\beta > 0$ depending only on the fundamental piece M_0 such that*

$$\alpha \leq \frac{\sigma_1(\Omega_\Gamma)}{\lambda_1(\Gamma)} \leq \beta$$

The proof of Theorem 7 will be presented in Section 4.

3.1. **Expander graphs and the proof of Theorem 1.** To prove Theorem 1, we will use expander graphs, through one of their many characterizations.

Definition 8. *A sequence of k-regular graphs $\{\Gamma_N\}_{N\in\mathbb{N}}$ is called a* family of expander graphs *if $\lim_{N\to\infty}|V(\Gamma_N)| = +\infty$ and $\lambda_1(\Gamma_N)$ is uniformly bounded below by a positive constant.*

See [11] for a survey of their properties and applications. Consider a fundamental piece M_0 of genus 0, with 5 boundary components, that is with $k = 4$. Let $\{\Gamma_N\}$ be a family of 4-regular expander graphs such that the number of vertices $|V(\Gamma_N)| = N$. The existence of this family of expander graphs follows from the classical probabilistic method [15]. It follows from Lemma 5 that the genus of Ω_{Γ_N} is

$$\gamma(\Omega_{\Gamma_N}) = 1 + N.$$

By definition, there is a constant $c > 0$ such that $\lambda_1(\Gamma_N) \geq c$ for each $N \in \mathbb{N}$. Since the boundary of Ω_{Γ_N} has N boundary components of length 1, Theorem 7 leads to

$$\sigma_1(\Omega_{\Gamma_N})L(\partial\Omega_{\Gamma_N}) \geq \alpha N\lambda_1(\Gamma_N) \geq c\alpha N.$$

This completes the proof of Theorem 1.

4. Proof of the comparison results

Let $f : \Omega_\Gamma \to \mathbb{R}$ be a smooth function. Given a vertex $v \in V(\Gamma)$, the function f_v is defined to be the restriction of f to the cylinder C_v. On each cylinder C_v, the function f_v admits a decomposition $f_v = \overline{f}_v + \tilde{f}_v$ where

$$\overline{f}_v(r) = \int_{\Sigma_v} f(r,x)\,dV_{\Sigma_v}(x)$$

is the average of f on the corresponding slice of C_v. It follows that for each $r \in [0,1]$,

$$\int_{\Sigma_v} \tilde{f}(r,x)\,dV_{\Sigma_v}(x) = 0.$$

The function $\overline{f}$ is defined to be $\overline{f}_v$ on each cylinder C_v, and similarly the function $\tilde{f}$ is defined to be $\tilde{f}_v$ on each C_v.

Let $f \in C^\infty(\Omega_\Gamma)$ be a Steklov eigenfunction corresponding to $\sigma_1(\Omega_\Gamma)$. The function

$$x = x_f : V(\Gamma) \to \mathbb{R}$$

is defined to be the average of f over the boundary component Σ_v. Since $|\Sigma_v| = 1$ for each vertex v, this is expressed by

$$x(v) = \int_{\Sigma_v} f\,dV_{\Sigma_v} = \overline{f}_v(0).$$

Because

$$\sum_{v\in V(\Gamma)} x(v) = \sum_v \int_{\Sigma_v} f\,dV_{\Sigma_v} = \int_\Sigma f\,dV_\Sigma = 0,$$

the function x can be used as a trial function in the variational characterization (3) of $\lambda_1(\Gamma)$. It follows from the orthogonality of $\overline{f}$ and $\tilde{f}$ on the boundary $\Sigma = \partial\Omega_\Gamma$ that

$$\begin{aligned}\int_{\partial\Omega_\Gamma} f^2\,dV_\Sigma &= \int_{\partial\Omega_\Gamma} (\overline{f}+\tilde{f})^2\,dV_\Sigma \\ &= \sum_{v\in V(\Gamma)} x(v)^2 + \int_{\partial\Omega_\Gamma} \tilde{f}^2\,dV_\Sigma \le \frac{1}{\lambda_1(\Gamma)} q_\Gamma(x) + \int_{\partial\Omega_\Gamma} \tilde{f}^2\,dV_\Sigma. \qquad (4)\end{aligned}$$

The two terms on the right-hand side of the previous inequality will be bounded above in terms of $\|\nabla f\|^2_{L^2(\Omega_\Gamma)}$. In order to bound $\int_{\partial\Omega_\Gamma} \tilde{f}^2\,dV_\Sigma$, it will be sufficient to consider the behaviour of $\tilde{f}$ locally on each cylinder C_v. More work will be required to bound $q_\Gamma(x)$.

4.1. Local estimate of smooth functions on cylindrical neighbourhoods.

On the model cylinder $C_0 = [0,1] \times \Sigma_0$, consider the following mixed Neumann-Steklov spectral problem:

$$\begin{aligned} \Delta f &= 0 \quad \text{on } (0,1)\times\Sigma_0, \\ \partial_n f &= 0 \quad \text{on } \{1\}\times\Sigma_0, \\ \partial_n f &= \mu f \text{ on } \{0\}\times\Sigma_0. \end{aligned} \qquad (5)$$

This problem is related to the sloshing spectral problem. See [1, 14] for details.

Lemma 9. *Let μ be the first non-zero eigenvalue of the sloshing problem (5). For any smooth function $f : \Omega_\Gamma \to \mathbb{R}$,*

$$\int_{\partial\Omega_\Gamma} \tilde{f}^2\, dV_\Sigma \leq \mu^{-1} \int_{\Omega_\Gamma} |\nabla f|^2. \tag{6}$$

Proof. The first non-zero eigenvalue of this problem is characterized by

$$\mu = \inf \left\{ \frac{\int_{(0,1)\times\Sigma_0} |\nabla f|^2}{\int_{\{0\}\times\Sigma_0} f^2} \; : \; f \in C^\infty([0,1]\times\Sigma_0), \int_{\{0\}\times\Sigma_0} f\, ds = 0 \right\}. \tag{7}$$

Since $\tilde{f}$ is orthogonal to constants on each boundary component Σ_v, it follows from (7) that

$$\int_{\partial\Omega_\Gamma} \tilde{f}^2\, dV_\Sigma \leq \mu^{-1} \sum_{v\in V(\Gamma)} \int_{C_v} |\nabla\tilde{f}|^2.$$

The proof of Lemma 9 is completed by observing that $\int_{C_v} |\nabla\tilde{f}|^2 \leq \int_{C_v} |\nabla f|^2$. Indeed, the gradient of the function $f_v = \overline{f_v} + \tilde{f}_v : C_v \to \mathbb{R}$ is

$$\nabla f_v = \frac{\partial}{\partial_r}(\overline{f_v} + \tilde{f}_v)\partial_r + \frac{\partial}{\partial_\theta}\tilde{f}_v\partial_\theta.$$

It follows that the Dirichlet energy of f_v is expressed by

$$\int_{C_v} |\nabla f_v|^2 = \int_{C_v} |\nabla\tilde{f}_v|^2 + \int_{C_v} \left(\overline{f_v}(r)'\right)^2 dr + 2\int_{C_v} \overline{f_v}(r)'\tilde{f}_v(r)'\, dr,$$

where

$$\int_{C_v} \overline{f_v}(r)'\tilde{f}_v(r)' = \int_0^1 \overline{f_v}(r)' \int_{\Sigma_v} \tilde{f}_v(r) = \int_0^1 \overline{f_v}(r)' \left(\int_{\Sigma_v} \tilde{f}_v(r)\right)' = 0.$$

□

4.2. Global estimate and graph structure.

Lemma 10. *There exists a positive constant C_0 depending only on the fundamental piece M_0 such that the following holds for any function f on Ω_Γ*

$$\sum_{v\sim w} \big(x(v) - x(w)\big)^2 \leq C_0 \int_{\Omega_\Gamma} |\nabla f|^2. \tag{8}$$

The proof of Lemma 10 is based on the following general estimate.

Lemma 11. *Let Ω be a smooth compact connected Riemannian surface with boundary. Let A and B be two of the connected components of the boundary $\partial\Omega$, both of length 1. There exists a constant $C > 0$ depending only on Ω such that any smooth function $f \in C^\infty(\Omega)$ satisfies*

$$\left|\int_A f - \int_B f\right|^2 \leq C \int_\Omega |\nabla f|^2 \tag{9}$$

In fact, we will use this estimate only for harmonic functions.

Proof of Lemma 11. Let $x = \int_A f$, $y = \int_B f$ be the average of f on the two boundary components A, B. Let

$$\langle f\rangle = \frac{1}{|\Omega|}\int_\Omega f,$$

be the average of f on the surface Ω. Finally, set $g = f - \langle f\rangle$. Now, since $\int_\Omega g = \int_\Omega (f - \langle f\rangle) = 0$,

$$\int_\Omega g^2 \leq \mu^{-1}\int_\Omega |\nabla g|^2 = \mu^{-1}\int_\Omega |\nabla f|^2,$$

where $\mu > 0$ is the first non zero Neumann eigenvalue of Ω. It follows that

$$\|f - \langle f\rangle\|^2_{H^1(\Omega)} = \|g\|^2_{H^1(\Omega)} = \|g\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{L^2(\Omega)} \leq (\mu^{-1}+1)\|\nabla f\|^2_{L^2(\Omega)}. \quad (10)$$

In other words, the Dirichlet energy of f controls how far f is from its average $\langle f\rangle$ in H^1-norm. This is essentially a version of the Poincaré Inequality. The restrictions of f to A and B are also close to the average $\langle f\rangle$ in L^2-norm. Indeed, it follows from the fact that the trace operators $\tau_a : H^1(\Omega) \to L^2(A)$ and $\tau_B : H^1(\Omega) \to L^2(B)$ are bounded, using the Cauchy-Schwartz inequality, that

$$|x - \langle f\rangle| = |\int_A (f - \langle f\rangle)| = |\int_A (\tau_A(g))| \leq \|\tau_A(g)\|_{L^2(A)} \leq \|\tau_A\|\|g\|_{H^1(\Omega)},$$

and similarly $|y - \langle f\rangle| \leq \|\tau_B\|\|g\|_{H^1(\Omega)}$, where $\|\tau_A\|$ and $\|\tau_B\|$ are the operator norms. These two inequalities together lead to

$$|x-y| \leq |x - \langle f\rangle| + |y - \langle f\rangle| \leq (\|\tau_A\| + \|\tau_B\|)\|g\|_{H^1(\Omega)}.$$

In combination with (10) this imply

$$|x-y|^2 \leq (\|\tau_A\| + \|\tau_B\|)^2(\mu^{-1}+1))\int_\Omega |\nabla f|^2.$$

One can take $C = (\|\tau_A\| + \|\tau_B\|)^2(\mu^{-1}+1))$. The proof is completed. □

Proof of Lemma 10. For each adjacent vertices $v \sim w$ of the graph Γ, we apply Lemma 11 to the surface $M_v \cup M_w$ with $A = \Sigma_v$ and $B = \Sigma_w$ to get

$$\big(x(v) - x(w)\big)^2 \leq C\int_{M_v\cup M_w} |\nabla f|^2.$$

Since the graph Γ is regular of degree k, it follows that

$$\sum_{v\sim w}\big(x(v) - x(w)\big)^2 \leq C\sum_{v\sim w}\int_{M_v\cup M_w} |\nabla f|^2 = Ck\int_{\Omega_\Gamma} |\nabla f|^2,$$

where the last equality holds since the sum is over all edges, so that each copy of the fundamental piece M_v appears exactly k times. □

4.3. **The proof of Theorem 7.**

The upper bound. Let f be a Steklov eigenfunction corresponding to the first non-zero Steklov eigenvalue $\sigma_1(\Omega_\Gamma)$. Combining the local estimate obtained in Lemma 9 and the global estimate of Lemma 10 with Inequality (4) leads to

$$\int_{\partial\Omega_\Gamma} f^2\, dV_\Sigma \leq \left(\frac{C_0}{\lambda_1(\Gamma)} + \frac{1}{\mu}\right)\int_{\Omega_\Gamma} |\nabla f|^2,$$

which of course can be rewritten

$$\sigma_1(\Omega) = \frac{\int_{\Omega_\Gamma}|\nabla f|^2}{\int_{\partial\Omega_\Gamma} f^2} \geq \left(\frac{C_0}{\lambda_1(\Gamma)} + \frac{1}{\mu}\right)^{-1} = \frac{\lambda_1(\Gamma)}{\lambda_1(\Gamma)/\mu + C_0} \geq \min\left\{\mu, \frac{1}{C_0}\right\}\frac{\lambda_1(\Gamma)}{\lambda_1(\Gamma)+1}.$$

Now, because we are on a regular graph of degree k, $\lambda_1 \leq k$, so that

$$\sigma_1(\Omega) \geq \frac{1}{k+1} \min\left\{\mu, \frac{1}{C_0}\right\} \lambda_1(\Gamma),$$

and taking $\beta = \frac{1}{k+1} \min\left\{\mu, \frac{1}{C_0}\right\}$, the proof of Theorem 7 is completed.

The lower bound. Let $x \in \ell^2(\Gamma)$ be a normalized eigenfunction corresponding to $\lambda_1(\Gamma)$. The function x satisfies

$$\sum_{v \in V(\Gamma)} x(v)^2 = 1 \quad \text{and} \quad \sum_{v \in V(\Gamma)} x(v) = 0.$$

Using x, a smooth function $f_x : \Omega_\Gamma \to \mathbb{R}$ is defined to be $x(v)$ on M_v away from collar neighbourhoods of the boundary components B_1, B_2, B_3, B_4, and to interpolate between $x(v)$ and $x(w)$ in a neighbourhood of the glued geodesic if $v \sim w$. The function f_x satisfies

$$\int_{\partial\Omega_\Gamma} f_x = \sum_{v \in V} x(v) = 0,$$

and can therefore be used in the variational characterization of $\sigma_1(\Omega_\Gamma)$. The estimate of the Rayleigh quotient is simple and follows [8, p. 290] verbatim. We will not reproduce it here.

References

[1] R. Bañuelos, T. Kulczycki, I. Polterovich and B. Siudeja, Eigenvalue inequalities for mixed Steklov problems, in *Operator Theory and its Applications*, Amer. Math. Soc. Transl. Ser. 2, 231, Amer. Math. Soc., Providence, RI, 2010, 19–34. MR 2758960

[2] R. Brooks, The first eigenvalue in a tower of coverings, *Bull. Amer. Math. Soc. (N.S.)*, **13** (1985), 137–140. MR 799796

[3] R. Brooks, The spectral geometry of a tower of coverings, *J. Differential Geom.*, **23** (1986), 97–107. MR 840402

[4] M. Burger, Estimation de petites valeurs propres du laplacien d'un revêtement de variétés riemanniennes compactes, *C. R. Acad. Sci. Paris Sér. I Math.*, **302** (1986), 191–194. MR 832070

[5] P. Buser, On the bipartition of graphs, *Discrete Appl. Math.*, **9** (1984), 105–109. MR 754431

[6] F. R. K. Chung, *Spectral Graph Theory*, CBMS Regional Conference Series in Mathematics, No. 92, American Mathematical Society, Providence, RI, 1997. MR 1421568

[7] B. Colbois, A. El Soufi and A. Girouard, Isoperimetric control of the Steklov spectrum, *J. Funct. Anal.*, **261** (2011), 1384–1399. MR 2807105

[8] B. Colbois and A.-M. Matei, On the optimality of J. Cheeger and P. Buser inequalities, *Differential Geom. Appl.*, **19** (2003), 281–293. MR 2013096

[9] A. Fraser and R. Schoen, Sharp eigenvalue bounds and minimal surfaces in the ball, `arXiv:1209.3789`, (2013).

[10] A. Girouard and I. Polterovich, Upper bounds for Steklov eigenvalues on surfaces, *Electron. Res. Announc. Math. Sci.*, **19** (2012), 77–85. MR 2970718

[11] S. Hoory, N. Linial and A. Wigderson, Expander graphs and their applications, *Bull. Amer. Math. Soc. (N.S.)*, **43** (2006), 439–561 (electronic). MR 2247919

[12] M. Karpukhin, Large Steklov and Laplace eigenvalues, in preparation.

[13] G. Kokarev, Variational aspects of Laplace eigenvalues on Riemannian surfaces, `arXiv:1103.2448`, (2011).

[14] N. N. Moiseev, Introduction to the theory of oscillations of liquid-containing bodies, in *Advances in Applied Mechanics, Vol. 8*, Academic Press, New York, 1964, 233–289. MR 0167074

[15] M. S. Pinsker, On the complexity of a concentrator, in *7th International Teletraffic Conference*, 1973, 318/1–318/4.

[16] P. C. Yang and S. T. Yau, Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **7** (1980), 55–63. MR 577325

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