THE ADJOINT $L$-FUNCTION FOR $GL_5$

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Abstract. We describe two new Eulerian Rankin-Selberg integrals, using the same Eisenstein series defined on the group $E_8$, and cuspidal representations from $GL_5$ and $GSpin_{11}$, respectively. Connections with past work of Ginzburg, Bump-Ginzburg, Jiang-Rallis and others are described. We give some details of how to relate our two integrals via formal manipulations.

It is a fact of classical invariant theory that, if we consider the adjoint representation of $PGL_n(\mathbb{C})$ and the second fundamental representation of $PSp_{2n}(\mathbb{C})$, (often denoted $\Lambda^2_0$) then we find that the algebras of invariants for these two representations are isomorphic. This can be seen, for example, in the table on p. 260 of [15]. More precisely, each is a free algebra having $n-1$ generators of degrees 2, 3, 4, ..., $n$. It was observed in [10] that when such an isomorphism of algebras of invariants exists, it is an indication that if Rankin-Selberg constructions happen to exist for the $L$-functions corresponding to both cases, then they are likely to make use of the same Eisenstein series.

In the family at hand, there are already two examples for this: when $n = 3$, the adjoint $L$-function was constructed in [6] and the $L$-function for $\Lambda^2_0$ was constructed in [10], using the same Eisenstein series, which is defined on the exceptional group $G_2$. Similarly, for $n = 4$, the adjoint $L$-function was constructed in [1] and the $L$-function for $\Lambda^2_0$ was constructed in [2]. These constructions use the same Eisenstein series, which is defined on the exceptional group $F_4$.

In these notes we describe some recent progress towards the construction of Rankin-Selberg integrals corresponding to the case $n = 5$. As far as we know, these will be the first Rankin-Selberg integrals corresponding to $L$ group representations such that the algebra of invariants has more than three generators. This time, the Eisenstein series is defined on the exceptional group $E_8$.

Our first integral is of the form

$$\int_{Spin_{11}(F)\backslash Spin_{11}(\mathbb{A})} \varphi(g) \int_{U(F)\backslash U(\mathbb{A})} E(ug, s)\theta^\psi_\varphi(\ell(u)g)\psi_{U_F}(u)dudg,$$

where $U$ is the unipotent radical of a certain parabolic subgroup of $E_8$, and $\psi_{U_F}$ is a character of $U$. The group $Spin_{11}$ is embedded in the Levi part of this parabolic as the stabilizer of a certain character of a certain subgroup of $U$. Also, $\theta^\psi_\varphi$ is a theta function defined on the semidirect product of the metaplectic group $Sp_{32}(\mathbb{A})$ and

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the Heisenberg group $H_{33}(\mathbb{A})$ in 33 variables, while $\ell$ is a projection of $U$ onto $H_{33}$ and $j$ is an embedding of $Spin_{11}(\mathbb{A})$ into $Sp_{2\ell}(\mathbb{A})$. Finally, the function $\varphi(g)$ is a generic cusp form defined on the group $GSpin_{11}(\mathbb{A})$, while $E(g, s)$ is an Eisenstein series on $E_8(\mathbb{A})$. See section 2 for precise notation.

Our second integral is similar but simpler: using the same Eisenstein series and the same group $U$, we choose a character $\psi_U$ having stabilizer isomorphic to $SL_5$. The integral is

$$
(2) \quad \int_{SL_5(F) \backslash SL_5(\mathbb{A})} \varphi(g) \int_{U(F) \backslash U(\mathbb{A})} E(ug, s)\psi_U(u)dudg.
$$

This time $\varphi$ is a cusp form defined on $GL_5(\mathbb{A})$.

The relationship between our two integrals is another example of a phenomenon observed in [8]: if we replace $\varphi$ in (1) by an Eisenstein series induced from a cuspidal representation of the parabolic of $Spin_{11}$, having Levi part isomorphic to $GL_5$, then our $GL_5$ construction may be regarded, in a suitable formal sense, as a convergent sub-integral of our $Spin_{11}$ construction. We give some details in section 3. Incidentally, it is possible to relate the construction of [10] section 4 with that of [6] and the construction of [2] with that of [1] in a similar fashion.

In addition to the $L$-function attached to the second fundamental representation of $Sp_{2\ell}(\mathbb{C})$, there is another family of $L$-functions which seems to share a special affinity with the family of adjoint $L$-function. These are the ratios $\zeta_K(s)/\zeta_F(s)$ of Dedekind zeta functions, which may also be expressed as products of Artin $L$-functions. Here $K$ is a commutative algebra of degree $n$ over $F$.

This second observation goes back at least to [13]. The paper [14] of Jiang and Rallis makes the connection with the theory of integral representations. We refer the reader there for a more detailed discussion. What was noted in [14] is that, at least for values of $n$ up to 5, the two families seem to be related on a nuts-and-bolts level in the theory of integral representations, in addition to the relationships suggested by [13]. Indeed, the main Theorem of [14] may be formulated as stating that for $n = 3$ the ratio $\zeta_K(s)/\zeta_F(s)$ has an integral representation using the same Eisenstein series (in this case, defined on the group $G_2$) that was used in [6] to obtain a Rankin-Selberg integral for the adjoint $L$ function of $GL_3$. For the case $n = 2$, Jiang and Rallis point to [17, 16, 4], before noting that work of Wright and Yukie [18] constitutes evidence that a similar approach might work for $n = 4$ and 5. We expand on this observation slightly.

An integral representation for the ratio $\zeta_K(s)/\zeta_F(s)$ arises from a correspondence between algebras $K$ and $M(F)$ orbits of characters of $N(F)\backslash N(\mathbb{A})$, where $F = MN$ is a parabolic subgroup in a split reductive algebraic group $G$ defined over $F$. This $G$ is $Sp_4$ for $n = 2$, and $G_2$ for $n = 3$. For $n = 4$ and 5, a similar correspondence is defined in [18], with the role of $G$ being played by $F_4$ and $E_8$ respectively. (Actually, what Wright and Yukie attach to an orbit is not an algebra of degree $n$ but a field extension of degree at most $n$, but the overlap—field extensions of degree exactly $n$—is the main interest anyway.) Thus it may be regarded as further evidence of the Jiang-Rallis philosophy that Rankin-Selberg integrals representing the corresponding adjoint $L$-functions have been found in precisely these two groups.

Part of the heuristic similarity between adjoint $L$-functions and the products of Artin $L$-functions $\zeta_K(s)/\zeta_F(s)$ rests on the fact that the adjoint $L$-function may also be expressed as a quotient: $L(s, \pi \times \tilde{\pi})/\zeta_F(s)$, where $\tilde{\pi}$ is the contragredient
of \( \pi \). The same is true of the \( L \)-function for the representation \( \wedge^2 \), attached to an automorphic representation of \( \text{Spin}_{11}(\mathbb{A}) \): it is \( L(s, \pi, \wedge^2)/\zeta_F(s) \).

We now describe the content of this paper. In sections 1 and 2 we define our two integrals and state the results concerning their unfoldings. In section 3 we give some details of the formal manipulations that may be used to relate them. In section 4 we describe the part of the unramified computations for the \( GL_5 \) integral which is complete.

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1. The \( GL_5 \) Global Integral

The group \( E_8 \) has 8 simple roots and 120 positive roots. We number the simple roots

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\
\alpha_2 \\
\end{array}
\]

and refer to all other roots as octuples of integers in terms of this basis. For each root \( \alpha \) there is a one dimensional unipotent subgroup \( U_\alpha \). We will wish to make use of a family of isomorphisms \( x_\alpha : \mathbb{G}_a \to U_\alpha \). It will be important to know the coefficients \( n_{\alpha,\beta} \) such that

\[
[x_\alpha(r), x_\beta(s)] = x_{\alpha + \beta}(n_{\alpha,\beta}rs).
\]

The isomorphisms may be chosen so that all the coefficients are 0, 1, or \(-1\) as described in [5].

We now describe the elements of our Rankin-Selberg integral. First, let \( Q \) denote the maximal parabolic subgroup of \( E_8 \) whose unipotent radical contains the subgroup \( U_{\alpha_5} \). The derived group of its Levi part is isomorphic to \( SL_5 \times SL_4 \). We consider the Eisenstein series \( E(g, s) \) associated to the induced representation \( \text{Ind}_{E_8(\mathbb{A})}^E(\delta) \). Next, let \( P \) denote the parabolic subgroup whose unipotent radical, \( U_P \), contains the groups \( U_{\alpha_1} \) and \( U_{\alpha_8} \), and whose Levi part contains \( U_{\alpha_5} \) for the remaining values of \( i \). The derived group of this Levi part is isomorphic to \( \text{Spin}_{12} \).

We fix an additive character \( \psi \) of \( F(\mathbb{A}) \), and define a character of \( U_P(F) \) by

\[
\psi_U(x_\alpha(r)) = \begin{cases} 
\psi(r), & \alpha = \alpha_1, \alpha_8, 12343210, 01122221 \\
1, & \text{otherwise.}
\end{cases}
\]

The identity component of the stabilizer of this character in the Levi part of \( P \) is generated by the groups \( U_{\pm \alpha_i} \) for \( i = 2, 4, 5, 6 \). In particular, it is isomorphic to \( SL_5 \). Using the maps \( x_\alpha \) fixed above, we may pin down a specific isomorphism, which we use to identify elements of this subgroup of \( E_8 \) with \( 5 \times 5 \) matrices in \( SL_5 \).

The Fourier coefficient

\[
\int_{U(F) \backslash U(\mathbb{A})} E(ug, s)\psi_U(u)du
\]
is left $SL_5(F)$-invariant. Restricting the $g$ variable to $SL_5(\mathbb{A})$ we obtain a smooth automorphic function on $SL_5$ of moderate growth, which we now integrate against a cusp form, in the space of an irreducible automorphic cuspidal representation of $GL_5(\mathbb{A})$. Thus, our integral is

$$
\varphi(g) \int_{U(F)\backslash U(\mathbb{A})} \sum_{\delta} f(\bar{w}_0 u_0 h, s) \psi_U(u_0) du_0 dh.
$$

In regards to the integration over $U(F)\backslash U(\mathbb{A})$, we wish to make the following remark. In [11] and [7], a procedure is described which begins with a nilpotent orbit and associates to it a set of Fourier coefficients. Here, the “nilpotent orbits” are for the coadjoint action of a group on its Lie algebra over the algebraic closure. The details in [11, 7] are for classical groups and make use of the labeling of nilpotent orbits by partitions, but the same idea works for exceptional groups, where the most convenient labeling is by weighted Dynkin diagrams. The above integral defines a Fourier coefficient which is associated, by this procedure, to the nilpotent orbit in $\mathfrak{e}_8$ associated to the weighted Dynkin diagram

$$
\begin{array}{cccccccc}
2 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{array}
$$

As noted on p. 405 of [3], the stabilizer of a point in this orbit is a group of type $A_4$.

We unfold integral (3). The result is as follows. Let $\bar{w}_0$ denote the shortest element of the double coset $W_Q w_0 W_{SL_5}$, where $w_0$ is the longest element of the Weyl group. Let $V$ denote the maximal unipotent subgroup of $SL_5$ corresponding to our choice of positive roots. It is identified with the group of $5 \times 5$ upper triangular unipotent matrices. Let $V'$ to be the subgroup consisting of elements of the form

$$
\begin{pmatrix}
1 & v_1 & v_2 & v_3 & v_4 \\
1 & v_5 & v_6 & v_7 & \\
1 & -v_5 & v_2 & \\
1 & -v_1 & 1 & \\
1 & & & & \\
\end{pmatrix}
$$

Finally, let $U_0 = U \cap \bar{w}_0^{-1} U \cap \bar{w}_0 = \prod_{\alpha > 0; \alpha > 0} U_\alpha$. Then we have,

**Theorem.** For $\text{Re}(s)$ large, integral (3) is equal to

$$
I(W_\varphi, f, s) := \int_{V'(\mathbb{A}) \backslash SL_5(\mathbb{A})} W_\varphi(h) \int_{U_0(\mathbb{A})} f(\bar{w}_0 u_0 h, s) \psi_U(u_0) du_0 dh,
$$

where $W_\varphi$ is the Whittaker function of $\varphi$ and

$$
\delta_0 = x_00011110(1)x_01111000(-1)x_00111100(1)x_01121110(-1).
$$

From this it essentially follows that, for suitable data, the original integral is factorizable – i.e. that it may be expressed as an infinite product of corresponding local integrals.
2. The $GSpin_{11}$ Global Integral

Next we describe a similar construction, involving a cusp form in the space of an irreducible automorphic cuspidal generic representation of $GSpin_{11}(\mathbb{A})$.

We make use of the same parabolic $P$ as before. The derived group of the Levi part of $P$ is isomorphic to $Spin_{12}$ and so the Levi part itself is isomorphic to a quotient of $Spin_{12} \times GL_2$. Let $\psi_{U_P}$ be the character of $U_P$ defined by

$$
\psi_{U_P}(x_\alpha(r)) = \begin{cases} 
\psi(r) & \alpha = 01011111 \\
1 & \text{otherwise.}
\end{cases}
$$

The stabilizer of this character, in the Levi part, is isomorphic to $GSpin_{11}$.

The next element of our Rankin-Selberg integral is a theta function. To define it, we notice that a certain subgroup $U_H$ of $U_P$ has the structure of a Heisenberg group. Specifically, $U_H$ is the product of the subgroups $U_{\alpha}$ associated to roots $\alpha = n_1\alpha_1 + \cdots + n_8\alpha_8$, such that $n_8 = 0$ and $n_1 \neq 0$. Because of the Heisenberg group structure, $U_H/U_{22343210}$ is a 32 dimensional symplectic vector space, with the skew-symmetric bilinear form being given by commutator. By $Sp_{32}$ we shall mean the group of automorphisms of this symplectic vector space.

Above we noted a subgroup of the Levi part of $P$ which is isomorphic to $GSpin_{11}$. It acts on $U_H$ by conjugation. The subgroup of elements which fix the subgroup $U_{22343210}$, and hence correspond to elements of $Sp_{32}$, is precisely the derived group, isomorphic to $Spin_{11}$. This map of $Spin_{11}$ into $Sp_{32}$ is essentially the spin representation and, in particular, is an embedding. Thus $Spin_{11}$ is simultaneously identified with a certain subgroup of $E_8$, and with a certain subgroup of $Sp_{32}$.

Now, having fixed an additive character above, we obtain a Schrödinger-Weil representation $\omega_\phi$ of the semidirect product of $U_H(\mathbb{A})$ and $\widetilde{Sp}_{32}(\mathbb{A})$, where $\widetilde{Sp}_{32}(\mathbb{A})$ denotes the metaplectic double cover of $Sp_{32}(\mathbb{A})$. This representation has an automorphic realization by theta functions, $\theta^\psi_\phi$, where $\phi$ is a Schwartz function, and $\psi$ is the additive character fixed above. Furthermore, there is a map $j : Spin_{11}(\mathbb{A}) \rightarrow \widetilde{Sp}_{32}(\mathbb{A})$ such that the following diagram commutes:

$$
\begin{array}{ccc}
Spin_{11}(\mathbb{A}) & \rightarrow & Sp_{32}(\mathbb{A}) \\
\downarrow & & \downarrow \\
\widetilde{Sp}_{32}(\mathbb{A}) & & \\
\end{array}
$$

There is a natural projection from $U_P$ to $U_H$ which we denote $\ell$. Thus, a theta function $\theta^\psi_\phi$ may be evaluated at $\ell(u)j(h)$ for $u \in U(\mathbb{A})$ and $h \in Spin_{11}(\mathbb{A})$.

The integral we consider is

$$
(5) \quad \int_{Spin_{11}(F) \backslash Spin_{11}(A)} \varphi(h) \left( \int_{U(F) \backslash U(A)} \theta^\psi_\phi(\ell(u)j(h))E(uh, s)\psi_U(u)du \right) dh,
$$

where $\varphi$ is a cusp form in the space of a generic irreducible cuspidal automorphic representation $\pi$ of $GSpin_{11}(\mathbb{A})$. This time, the integral over $U$ defines a Fourier coefficient obtained from the nilpotent orbit of $\epsilon_8$ associated to the weighted Dynkin
The result of the unfolding is similar. Let $w$ be the shortest element of $QwP$, which is of length 83, and $U_0 = U \cap w^{-1}Uw$, the product of all the groups $U_\alpha$ such that $\alpha > 0$ and $w\alpha < 0$.

Then for a certain codimension 5 subgroup $V''$ of $V$, defined similarly to $V'$ above, and a suitable element $\xi_0$ of $F_{16}$, we have

**Theorem.** For $\Re(s)$ large, the integral (5) is equal to

$$
\int_{V''(\mathbb{A})/Spin_{11}(\mathbb{A})} \omega_\psi(\ell(\mu)j(h))\phi(\xi_0)f(\mu h, s)\psi_U(u)du
$$

$$
W_\varphi(x_{-00000100}(x_3)x_{-00011100}(x_1)x_{-00001000}(x_2)w[645]h)dh.
$$

Once again, the factorization as a product of local integrals is now essentially immediate.

3. **Formal manipulations**

In this section, we describe how to obtain the $GL_5$ integral from the $GSpin_{11}$ integral via certain formal manipulations. We emphasize that every integral appearing at an intermediate stage is divergent, so this procedure is purely formal.

Let $\tau$ be an automorphic representation of $GL_5(\mathbb{A})$ and $E_{\tau}(g, s_1)$ an Eisenstein series defined on $Spin_{11}(\mathbb{A})$ using the section $f_{\tau}(g, s_1) \in Ind_{R(\mathbb{A})}^{GL_5(\mathbb{A})} \tau \otimes \delta_{R}^{s_1}$. Here $R$ is the standard maximal parabolic subgroup of $Spin_{11}$ having Levi part isomorphic to $GL_5$. We plug $E_{\tau}(g, s_1)$ into (5) for $\varphi$ and formally unfold the resulting divergent integral. We obtain

$$
\int_{R(F)\backslash Spin_{11}(\mathbb{A})} f_{\tau}(g, s_1) \left( \int_{U(F)\backslash U(\mathbb{A})} \theta_{\varphi}(\ell(u)j(g))E(ug, s_1)\psi_U(u)du \right) dg.
$$

Our next step is to replace integration over $R(F)\backslash Spin_{11}(\mathbb{A})$ by integration over $GL_5(F)\backslash GL_5(\mathbb{A}) \times U(R(F))\backslash U(\mathbb{A})$ where $GL_5$ is identified with the Levi part of $R$ and $U_R$ is its unipotent radical. At the same time, we replace $f_{\tau}(g, s_1)$ by a cusp from $\varphi_{\tau}$ in the space of $\tau$. This step may be thought of as plugging in the Iwasawa decomposition of $Spin_{11}$ with respect to the parabolic $R$ and passing to an “inner integral.”

We allow ourselves to plug in simplified, formal versions of the sorts of identities which commonly arise in the unfolding of Rankin-Selberg integrals and other similar computations.

For example, the next step is to replace integration over the whole group $U$ against a theta function by integration over a subgroup against a character. The group is obtained by deleting 16 of the subgroups $U_\alpha$ from the Heisenberg part, and the character is the one which is trivial on every $U_\alpha$ except 22343210 and $\psi(r)$ on $x_{22343210}(r)$. This may be regarded as a simplified formal version of the type of identities found on p. 5 of [11].
We also use a simplified formal version of the technique of “exchange of roots.” (Cf. [9], section 2.1, [12] pp. 748-750.) The idea is to replace integration over one unipotent group $U_2$ against a character $\psi_1$ by integration over a more advantageous group $U_1$ against a corresponding character $\psi_2$ by proving an identity of the form
\[
\int_{U_1(F)\backslash U_1(\mathbb{A})} f(u_1g)\psi_1(u_1)du_1 = \int_{Z(\mathbb{A})} \int_{U_2(F)\backslash U_2(\mathbb{A})} f(u_2zg)\psi_2(u_2)du_2dz.
\]
This is possible under suitable hypotheses on the groups $U_1, U_2$ and $Z$. For a more thorough discussion see the references cited above. In our simplified formal version, we omit the $z$ integral. This may be thought of as passing once again to an “inner integral,” corresponding to a fixed value of $z$.

Conjugation by $w[34254365428765431]$ (here we specify an element of the Weyl group as a word in the simple reflections) maps \{22343210,01011111,00111111\} to \{\alpha_1,01122221,12343210\}. It is then possible to exchange roots so as to obtain integration over the subgroup of $U$ generated by \{\(U_\alpha \subset U : \alpha \neq \alpha_8\). We then perform a Fourier expansion on $U_{\alpha_8}$.

We ignore the term corresponding to the trivial character. The remaining terms are permuted transitively by our (conjugated) $GL_5$, which acts by det on this one dimensional subgroup. The stabilizer is $SL_5$, embedded exactly as in section 1. In this manner, formally, we have obtained the $GL_5$ integral.

4. Unramified computations in the $GL_5$ case.

We consider the local analogue of integral (4), at a place where the data is unramified. (So, $W$ is now the normalized spherical vector in the Whittaker model of a local representation, etc.) We wish to show that $I(W,f,s) = L(s,\pi,Ad)/N(s)$ where $N(s)$ is the normalizing factor of the Eisenstein series, given by
\[
\zeta(11s)\zeta(11s-1)\zeta(11s-2)\zeta(11s-3)\zeta(11s-4)\zeta(22s-6)\zeta(22s-7)\zeta(22s-8)\zeta(22s-9)\zeta(22s-10)\zeta(33s-12)\zeta(33s-13)\zeta(33s-14)\zeta(33s-15)\zeta(44s-18)\zeta(44s-20)\zeta(55s-25).
\]
Performing a series of fairly standard steps, such as plugging in the Iwasawa decomposition and a suitable form of the Casselman-Shalika formula, we may express $I(W,f,s)$ as:
\[
\sum_{\lambda \in \Lambda^+_R} \chi_\lambda(t_\tau)x^{\ell(\lambda)}I(f,s;\lambda),
\]
where $\lambda$ is summed over dominant weights of $PGL_5(\mathbb{C})$ (i.e., dominant weights of $GL_5(\mathbb{C})$ which are in the span of the roots), $\chi_\lambda$ denotes the trace of the irreducible finite dimensional representation of $GL_5(\mathbb{C})$ having highest weight $\lambda$, and $\ell$ is an integer-valued linear function on the root lattice. Also, $x = q^{-11s+5}$, where $q$ is the absolute value of a uniformizer. The heart of the matter is the “inner integral” $I(f,s;\lambda)$. Let $\lambda = \sum_{i=1}^4 m_i\beta_i$, where $\beta_1,\ldots,\beta_4$ are the simple roots of $GL_5(\mathbb{C})$, and choose a uniformizer $\varpi$. Then $I(f,s;\lambda)$ can be written as
\[
\int_{F_R} \psi(\varpi^{m_1-m_3+m_4}r_1-\varpi^{-m_1+m_2}r_2)
\]
\[
\int_{U_1} \psi_{U_1}(u_1)f(\varpi_1x_{\alpha_7}(r_1)x_{\alpha_4}(r_2)x_00011100(r_3)\delta(m)u_0,s)du_0dr_1.
\]
with $U_1$ being a conjugate of $U_0$ above, $\psi_{U_1}$ a character, and

$$\delta(m) = x_{-\alpha_5}(\varpi_m^{m_2 + m_3})x_{-\alpha_5}(\varpi_m^{m_1 - m_2 + m_3})x_{-\alpha_5}(\varpi_m^{m_1 + m_4})x_{-\alpha_5}(\varpi_m^{m_2 - m_3}).$$

The delicate part of the dependence of $I(f, s; \lambda)$ on $\lambda$ comes from $\delta(m)$. As $m$ varies, the shape of the Iwasawa decomposition of this element varies and results in several different cases for the shape of the integral. The easiest is when $m_1 = m_4$ and $m_2 = m_3$. This case corresponds to weights $\lambda$ which are symmetric under the outer automorphism of $GL_5(\mathbb{C})$ that reverses the order of the simple roots, i.e., weights which, in terms of the fundamental weights, are of the form $\lambda = (m, k, k, m)$. For such weights, we find that $N(s)I(f, s; \lambda)$ equals

$$\frac{x^{k+2m}}{(1 - x)(1 - x^3)(1 - x^4)(1 - x^5)} \left[ \delta_{k,0} x^6 (1 - x^{m-1}) (1 - x^m) (1 - x^{m+1}) (1 - x^{m+2}) \right. $$

$$+ \frac{(1 - x^{k+1})(1 - x^{k+2})(1 - x^{k+3})}{(1 - x)(1 - x^2)(1 - x^3)} (1 - x^{m+1}) (1 - x^{m+2}) (1 - x^{m+3})(1 - x^{m+4})$$

$$+ x^3 \frac{(1 - x^{k+1})(1 - x^{k+2})(1 - x^{k+3})}{(1 - x)(1 - x^2)(1 - x^3)} (1 - x^m) (1 - x^{m+1}) (1 - x^{m+2})(1 - x^{m+3})$$

$$- x^7 \frac{(1 - x^{k-1})(1 - x^{k+1})(1 - x^{k+2})}{(1 - x)(1 - x^2)(1 - x^3)} (1 - x^{m-1})(1 - x^m)(1 - x^{m+1})(1 - x^{m+2})$$

$$- x^{12} \frac{(1 - x^{k-1})(1 - x^k)(1 - x^{k+1})}{(1 - x)(1 - x^2)(1 - x^3)} (1 - x^{m-2}) (1 - x^{m-1})(1 - x^m)(1 - x^{m+1}).$$

(Here $\delta_{k,0}$ is the Kronecker delta.)

Turning our attention to the other side of the desired identity, we need an analog of the identity appearing near the top of p.155 in [1]. The identity for our case is

$$\frac{(1 - x)}{(1 - x^5)} \sum_{k_1 = 0}^{\infty} x^{k_1 + 2k_2 + 3k_3 + 4k_4} \text{Trace}(t_\pi | V_{k_1, k_2, k_3, k_4} \otimes V_{k_4, k_3, k_2, k_1})$$

$$= \sum_{n=0}^{\infty} \text{Trace}(t_\pi | \text{sym}^n \text{Ad}),$$

where $V_{k_1, k_2, k_3, k_4}$ is the irreducible finite-dimensional representation of $GL_5(\mathbb{C})$ having highest weight $\sum_i k_i \varpi_i$. (Here $\varpi_1, \ldots, \varpi_4$ are the fundamental weights.) In fact, this is an identity of functions, which is then evaluated at $t_\pi$.

Fix a dominant weight $\lambda$. We may then consider the “coefficient” of $\chi_\lambda(t_\pi) = \text{Trace}(t_\pi | V_\lambda)$ – a power series in $x$ – on either side of (8). Let $J(\lambda)$ denote this coefficient, and let $\tilde{I}(f, s; \lambda) = N(s)I(f, s; \lambda)$ (and extend by zero to dominant weights which are not in the root lattice). Then our main assertion may be re-packaged as the assertion that $J(\lambda) = \tilde{I}(f, s; \lambda)$ for all $\lambda$.

**Proposition.** This holds for $\lambda = (m, k, k, m)$.

In fact, one may prove an equality of the “symmetric pieces”:

$$\sum_{k,m=0}^{\infty} \tilde{I}(f, s; (m, k, k, m)) x^{f(m,k,k,m)} \chi_{(m,k,k,m)}(t_\pi)$$

$$= \sum_{k,m=0}^{\infty} J(m,k,k,m) \chi_{(m,k,k,m)}(t_\pi),$$

where $f(m,k,k,m)$ is the irreducible finite-dimensional representation of $GL_5(\mathbb{C})$ having highest weight $\sum_i k_i \varpi_i$. (Here $\varpi_1, \ldots, \varpi_4$ are the fundamental weights.) In fact, this is an identity of functions, which is then evaluated at $t_\pi$.
by replacing $\chi_{(m,k,k,m)}(t_\pi)$ by $T_1T_2^n$, where $T_1$ and $T_2$ are indeterminates. The resulting power series in 3 variables may then be summed to obtain explicit rational functions, which turn out to be equal.

We remark that the desired equality can be verified by hand for small asymmetrical $\lambda$ as well. Also, the quantity $\tilde{I}(f,s;\lambda)$ which was introduced on an ad hoc basis above also appears naturally as an “inner integration” if we interpret part of the unipotent integration in $I(f,s;\lambda)$ above as an intertwining operator, and part as the Jacquet integral for the Whittaker function on the $GL_4 \times GL_5$ Levi part.

References