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A COMBINATORIAL CURVATURE FLOW FOR COMPACT 3-MANIFOLDS WITH BOUNDARY

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Abstract. We introduce a combinatorial curvature flow for piecewise constant curvature metrics on compact triangulated 3-manifolds with boundary consisting of surfaces of negative Euler characteristic. The flow tends to find the complete hyperbolic metric with totally geodesic boundary on a manifold. Some of the basic properties of the combinatorial flow are established. The most important one is that the evolution of the combinatorial curvature satisfies a combinatorial heat equation. It implies that the total curvature decreases along the flow. The local convergence of the flow to the hyperbolic metric is also established if the triangulation is isotopic to a totally geodesic triangulation.

1. Introduction

1.1. The purpose of this paper is to construct a combinatorial curvature flow which is a 3-dimensional analog of the flows considered in [CL]. In [CL], we introduced a 2-dimensional combinatorial curvature flow for triangulated surfaces of non-positive Euler characteristic. It is shown that for any initial choice of PL metric of circle packing type, the flow exists for all times and converges exponentially fast to the Andreev-Koeb-Thurston's circle packing metrics. In the 3-dimensional case, it is shown that the curvature evolution equation of the combinatorial curvature flow satisfies a combinatorial heat equation. Furthermore, the flow tends to find the complete hyperbolic metric of totally geodesic boundary on the 3-manifold.

Following Hamilton [Ha], a curvature flow should deform the metrics in such a fashion that, as the metric evolves, the curvature evolves according to some heat type equation. It indicates infinitesimally, the metrics tend to improve themselves. In the case of combinatorial curvature flow, instead of using the space of all Riemannian metrics, we consider the space of all piecewise constant curvature metrics supported in a fixed triangulation. The goal is to produce a flow, or a vector field, in the space so that the combinatorial curvature evolves according to a combinatorial Laplace equation. The most general form of a combinatorial Laplacian is defined as follows. Given finite graph with vertices labelled by $\{1, 2, \ldots, n\}$, a *combinatorial Laplace operator* is a negative semidefinite $n \times n$ matrix $A = [a_{ij}]_{n \times n}$ so that it acts on the space of all functions defined on vertices by the linear transformation A. To be more precise, suppose $x = [x_1, \ldots, x_n]^t$ is a function (representated as a

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column vector) whose value at the ith vertex is x_i; then the combinatorial Laplace operator sends x to Ax. In the construction of the combinatorial curvature flow, we are guided by the principle that the curvature evolves according to some combinatorial Laplacian operator. Finding the correct combinatorial Laplacian, or more precisely, the negative semidefinite matrix, is the main ingredient of the paper. The combinatorial Laplacian operator used in this paper is obtained as the Hessian of the hyperbolic volume function with dihedral angles as variables. That the hyperbolic volume function for a hyperideal tetrahedron is strictly concave in terms of the dihedral angles was first observed by J. Schlenker [Sc]. The author thanks Igor Rivin for bringing him this earlier work of Schlenker. In fact, for strictly hyperideal polytopes of the same combinatorial type, the hyperbolic volume, viewed as a function of the dihedral angles, is smooth and strictly concave ([Sc]).

1.2. The basic building blocks for a 2-dimensional flow are hyperbolic and Euclidean triangles in [CL]. In the 3-dimensional case, the basic building blocks are the *strictly hyperideal tetrahedra* discovered by Bao and Bonahon [BB]. See also Frigerio and Petronio [FP].

Given an ideal triangulation of a compact 3-manifold with boundary consisting of surfaces of negative Euler characteristic, we replace each (truncated) tetrahedron with a strictly hyperideal tetrahedron by assigning the edge lengths. The isometric gluing of these tetrahedra gives a hyperbolic cone metric on the 3-manifold. The *combinatorial curvature* of the cone metric at an edge is 2π less the sum of dihedral angles at the edge. The combinatorial curvature flow that we propose is the following system of ordinary differential equations:

$$dx_i/dt = K_i, \tag{1.1}$$

where x_i is the length of the ith edge and K_i is the combinatorial curvature of the cone metric $(x_1, \ldots, x_n)$ at the ith edge. The equation (1.1) captures the essential features of the 2-dimensional combinatorial Ricci flow in [CL]. The most important of all is that the combinatorial curvature evolves according to a combinatorial heat equation. Thus the corresponding maximum principle applies. The flow has the tendency of finding the complete hyperbolic metric of totally geodesic boundary on the manifold. By analyzing the singularity formations in equation (1.1), it is conceivable that one could give a new proof of Thurston's geometrization theorem for these manifolds using (1.1). Furthermore, the flow (1.1) will be a useful tool to find algorithmically the complete hyperbolic metric.

1.3. Suppose M is a compact 3-manifold whose boundary is non-empty and is a union of surfaces of negative Euler characteristic. Let $C(M)$ be the compact 3-manifold obtained by coning off each boundary component of M to a point. In particular, if M has k boundary components, then there are exactly k cone points $\{v_1, \ldots, v_k\}$ in $C(M)$ so that $C(M) - \{v_1, \ldots, v_k\}$ is homeomorphic to $M - \partial M$. An *ideal triangulation* (or truncated triangulation) T of M is a triangulation $\mathcal{T}$ of $C(M)$ such that the vertices of the triangulation are exactly the cone points $\{v_1, \ldots, v_k\}$. By Moise [Mo], every compact 3-manifold can be ideally triangulated. We identify M with a subset of $C(M)$ as follows. Take $st(v_1, \ldots, v_k)$ to be the open star of the vertices $\{v_1, \ldots, v_k\}$ in the second barycentric subdivision of the triangulation. Then we take $M = C(M) - st(v_1, \ldots, v_k)$. By abuse of the language, the *edges, triangles* and *tetrahedra* in M in the ideal triangulation T are defined to be the intersection $a \cap M$, where a is an i-dimensional simplex in the triangulation

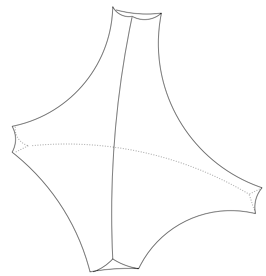

FIGURE 1. The six edges of a hyperideal tetrahedron are the intersections of its hexagonal faces.

$\mathcal{T}$ of $C(M)$ for $i = 1, 2, 3$ respectively. Note that the boundary ∂M of M has the induced triangulation from $\mathcal{T}$. The 1-simplex and 2-simplex in the boundary triangulation of ∂M are not called edges or triangles in the ideal triangulation.

Following Bao-Bonahon [BB], a *strictly hyperideal tetrahedron* in the 3-dimensional hyperbolic space is a compact convex polyhedron that is diffeomorphic to a truncated tetrahedron in the 3-dimensional Euclidean space and its four hexagonal faces are right-angled hyperbolic hexagons. (Note that two compact subsets of $\mathbf{R}^n$ are diffeomorphic if there is a diffeomorphism between two open neighborhoods of them sending one compact set to the other.) See Figure 1. In [BB], Bao and Bonahon give a complete characterization of hyperideal convex polyhedra. As a very special case of their work, they obtained a characterization of strictly hyperideal tetrahedra using dihedral angles. This characterization of a hyperideal simplex was also obtained by Frigerio and Petronio [FP]. Recall that an *edge* in a hyperideal tetrahedron is the intersection of two hexagonal faces. The works [BB] and [FP] show that the strictly hyperideal tetrahedra are completely characterized by their six dihedral angles at the six edges such that the sum of three dihedral angles associated to edges adjacent to each vertex is less than π. In particular, the space of all strictly hyperideal tetrahedra forms an open convex polytope in $\mathbf{R}^6$ when parametrized by the dihedral angles.

The main technical observation is the following. We were informed by I. Rivin that this was first known to J. Schlenker in [Sc].

Theorem 1 (Schlenker)**.** *The volume of a strictly hyperideal tetrahedron is a strictly concave function of its dihedral angles.*

The Hessian of the volume is a negative definite matrix by Theorem 1. This provides a basis for constructing the combinatorial Laplacian operator for the curvature evolution equation.

Given an ideal triangulated 3-manifold (M, T), let E be the set of edges in the ideal triangulation and let n be the number of edges in E. An assignment $x : E \to \mathbf{R}_{>0}$ is called a *hyperbolic cone metric associated to the ideal triangulation* T if for each tetrahedron t in T with edges $e_1, \ldots, e_6$, the six numbers $x_i = x(e_i)$ $(i = 1, \ldots, 6)$ are the edge lengths of a strictly hyperideal tetrahedron in $\mathbf{H}^3$. The set of all hyperbolic cone metrics associated to T is denoted by $L(M, T)$, which will be regarded as an open subset of $\mathbf{R}^n = \mathbf{R}^E$ by measuring the edge lengths. The combinatorial curvature of a cone metric $x \in L(M, T)$ is the map $K : E \to \mathbf{R}$

sending an edge e to the combinatorial curvature of x at the edge e. Again we identify the set of all combinatorial curvatures $\{K \mid x \in L(M,T)\}$ with a subset of $\mathbf{R}^n$. The combinatorial curvature flow is the vector field in $L(M,T)$ defined by equation (1.1), where $K_i = K_i(t)$ on the right-hand side is the combinatorial curvature of the metric $x = (x_1, \ldots, x_n)$ at time t at the ith edge.

Theorem 2. *For any ideal triangulated 3-manifold, under the combinatorial curvature flow* (1.1), *the combinatorial curvature* $K_i(t)$ *evolves according to a combinatorial heat equation,*

$$dK_i(t)/dt = \sum_{j=1}^{n} a_{ij} K_j(t), \tag{1.2}$$

where the matrix $[a_{ij}]_{n\times n}$ *is symmetric negative definite. Furthermore, the combinatorial curvature flow is the negative gradient flow of a locally strictly convex function.*

Corollary 3. *For any ideal triangulated 3-manifold* (M,T)*, under the combinatorial curvature flow* (1.1),

(a) *the total curvature* $\sum_{i=1}^n K_i^2(t)$ *is strictly decreasing along the flow unless* $K_i(t) = 0$ *for all* i*;*
(b) *the equilibrium points of the combinatorial curvature flow* (1.1) *are the complete hyperbolic metric with totally geodesic boundary;*
(c) *each equilibrium point is a local attractor of the flow.*

Another consequence of the convexity is the following local rigidity result for hyperbolic cone metrics without constrains on cone angles. Note that by [HK], hyperbolic cone metrics with cone angles at most 2π are locally rigid.

Theorem 4. *For any ideal triangulated 3-manifold* (M,T)*, the curvature map* $\Pi : L(M,T) \to \mathbf{R}^n$ *sending a metric* x *to its curvature* $K(x)$ *is a local diffeomorphism. In particular, a hyperbolic cone metric associated to an ideal triangulation is locally determined by its cone angles.*

1.4. In the rest of the paper, we prove the results. In the last section, we propose several questions related to the combinatorial curvature flow whose resolution will lead to a new proof of Thurston's geometrization theorem for this class of 3-manifolds.

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2. Proofs

In this section we prove theorems and corollaries stated in §1.

2.1. **Proof of Theorem 1.** Suppose $x_1, \ldots, x_6$ are the lengths of the six edges of a strictly hyperideal tetrahedron so that the corresponding dihedral angles are $a_1, \ldots, a_6$. Let $a = (a_1, \ldots, a_6)$, $x = (x_1, \ldots, x_6)$, and let V be the volume of the strictly hyperideal tetrahedron. By the results of [BB] and [FP], $x = x(a)$ is a function of a. Conversely, $a = a(x)$ is a function of x. This is due to the following results on convex polyhedron. First of all, by the Cauchy rigidity of a convex polytope, the isometry class of the convex polytope in the hyperbolic 3-space is determined by the intrinsic geometry of the boundary surface. In particular, the dihedral angles are determined by the induced metric on the boundary surface. On the other hand, the metric on the boundary surface is determined by the metrics on the four hexagonal faces since the other four faces are triangles. Finally, the metric on a right-angled hyperbolic hexagon is determined by the three lengths of its three pairwise non-adjacent edges. Thus $a = a(x)$. Obviously, these two functions $a = a(x)$ and $x = x(a)$ are inverses of each other. Next we claim that both functions $a = a(x)$ and $x = x(a)$ are smooth (in fact real analytic functions). Indeed, to express x_i in terms of a, we use the cosine law for the triangular faces of the strictly hyperideal tetrahedron. The cosine law expresses twelve edge lengths y_j of these four triangles in terms of a analytically. Next, we use the cosine law for the right-angled hexagon to express the length x_i analytically in terms of the y_j's. Thus we see that $x = x(a)$ is an analytic map. Conversely, we use the cosine law for hexgonal faces to express y_j analytically in terms of x_i's. Then we use the cosine law for triangles to express a_i analytically in terms of y_j's. Thus we see that both $a = a(x)$ and $x = x(a)$ are local diffeomorphisms. In particular, the Jacobi matrix $[\partial x_i/\partial a_j]_{6\times 6}$ is non-singular.

Now consider the volume $V = V(a)$ as a function of the dihedral angles a. By the Schlaefli formula, we have $\partial V/\partial a_i = -x_i/2$. This implies that $-\partial x_i/\partial a_j = -\partial x_j/\partial a_i$ for all i, j due to the symmetry of the Hessian matrix. Since the space of all strictly hyperideal tetrahedra parametrized by dihedral angles is connected and the matrix $[\partial x_i/\partial a_j]_{6\times 6}$ is non-singular, the signature of the symmetric matrix $[\partial x_i/\partial a_j]_{6\times 6}$ is independent of the choice of the strictly hyperideal tetrahedra. One checks directly that if the strictly hyperideal tetrahedron is regular (i.e., all x_i's are the same and all a_i's are the same), the Jacobian matrix is positive definite. Using the Schlaefli formula, this implies that the Hessian of the volume function V is locally strictly concave. Since the domain is convex by [BB], we see that the volume is a strictly concave function of the dihedral angles. □

We do not have a proof that the volume function defined on the space of all strictly hyperideal tetrahedra parametrized by the dihedral angles can be extended continuously to all hyperideal tetrahedra. This fact should be true and may follow from [Us]. See also the related work [Lu].

Since the inverse of a symmetric positive definite matrix is again symmetric and positive definite, we conclude:

Corollary 5. *For a strictly hyperideal tetrahedron with dihedral angle a_i and length x_i at the ith edge, the matrix $[\partial a_i/\partial x_j]_{6\times 6}$ is symmetric and positive definite. In particular, the function $F = 2V + \sum_{i=1}^6 a_i x_i$ is a locally strictly convex function of the length variables $(x_1, \ldots, x_6)$.*

Indeed, it suffices to verify that the Hessian matrix $[\partial^2 F/\partial x_i \partial x_j]$ is strictly positive definite. Now, by construction,

$$\begin{aligned}\partial F/\partial x_j &= 2\partial V/\partial x_j + a_j + \sum_{i=1}^6 x_i \partial a_i/\partial x_j \\ &= 2\sum_{i=1}^6 (\partial V/\partial a_i)(\partial a_i/\partial x_j) + a_j + \sum_{i=1}^6 x_i \partial a_i/\partial x_j.\end{aligned}$$

By the Schlaefli formula, $\partial V/\partial a_i = -x_i/2$. Thus

$$\partial F/\partial x_j = -\sum_{i=1}^{6} x_i \partial a_i/\partial x_j + a_j + \sum_{i=1}^{6} x_i \partial a_i/\partial x_j = a_j. \tag{2.1}$$

As a consequence, we see that the Hessian of F with respect to $(x_1, \ldots, x_6)$ is exactly $[\partial a_i/\partial x_j]_{6\times 6}$, which is known to be positive definite. This shows that the function F is a strictly locally convex function. Unfortunately, the space of all strictly hyperideal simplices parametrized by the edge lengths is not convex. Thus there is no global convexity for F.

2.2. **Proof of Theorem 2.** To prove Theorem 2, one uses equation (1.1) and Corollary 5. To begin with, we assume that there are n edges labelled by $1, 2, \ldots, n$. The length of the ith edge is x_i and the combinatorial curvature at the ith edge is K_i. If r is a tetrahedron in the ideal triangulation, let $I(r)$ be the set of all indices i so that the ith edge is a codimension-1 face of r. If $i \in I(r)$, we use $a_i^r = a_i^r(x)$ to denote the dihedral angle of the strictly hyperideal simplex r at the ith edge in the hyperbolic cone metric x. Define $a_i^r = 0$ if $i \notin I(r)$. Let $T^{(3)}$ be the set of all tetrahedra in the ideal triangulation. Then the combinatorial curvature K_i is given by $2\pi - \sum_{r\in T^{(3)}} a_i^r$ by definition. In particular,

$$dK_i/dt = -\sum_{r\in T^{(3)}} da_i^r/dt.$$

On the other hand, by the chain rule, we can express

$$da_i^r/dt = \sum_{j=1}^{n} (\partial a_i^r/\partial x_j)(dx_j/dt) = \sum_{j=1}^{n} \partial a_i^r/\partial x_j K_j.$$

Thus we have,

$$dK_i/dt = \sum_{r\in T^{(3)}} \sum_{j=1}^{n} (-\partial a_i^r/\partial x_j K_j).$$

In particular, if we express $dK_i/dt = \sum_{j=1}^{n} a_{ij} K_j$, then $a_{ij} = -\sum_{r\in T^{(3)}} \partial a_i^r/\partial x_j$. Since the Jacobian matrix $[\partial a_i/\partial x_j]_{6\times 6}$ is symmetric, we have $\partial a_i^r/\partial x_j = \partial a_j^r/\partial x_i$. (Indeed, by definition $\partial a_i^r/\partial x_j = 0$ unless $i, j \in I(r)$. In the latter case, it follows from Corollary 5.) Thus $a_{ij} = a_{ji}$. To finish the proof, we need to show that the matrix $[a_{ij}]_{n\times n}$ is negative definite. To this end, take a vector $(u_1, \ldots, u_n)$ in $\mathbf{R}^n$, and consider the quadratic expression

$$\sum_{i,j=1}^{n} u_i u_j a_{ij} = -\sum_{r\in T^{(3)}} \sum_{i,j=1}^{n} u_i u_j \partial a_i^r/\partial x_j. \tag{2.2}$$

For any fixed tetrahedron $r \in T^{(3)}$, let $x_i^r = x_i$ when $i \in I(r)$. Then

$$\sum_{i,j=1}^{n} u_i u_j \partial a_i^r/\partial x_j = \sum_{i,j\in I(r)} u_i u_j \partial a_i^r/\partial x_j^r.$$

By Corollary 5, the expression $-\sum_{i,j\in I(r)} u_i u_j \partial a_i^r/\partial x_j^r$ is less than or equal to zero, and is zero if and only if $u_i = 0$ for all indices $i \in I(r)$. This shows first of all that the expression (2.2) is less than or equal to zero. Furthermore, if it is zero, then all $u_i = 0$ since each edge is adjacent to some tetrahedron. We have thus established that fact that the matrix $[a_{ij}]_{n\times n}$ is symmetric and negative definite.

To see that the flow (1.1) is the gradient flow, recall that $L(M,T)$ denotes the open subset of $\mathbf{R}^n$ consisting of the hyperbolic cone metrics associated to the ideal triangulation. To be more precise, if $x = (x_1, \ldots, x_n) \in L(M,T)$, then x_i is the length of the ith edge in the cone metric. For $x \in L(M,T)$, let $V = V(x)$ be the volume of the cone metric which is the sum of all hyperbolic volumes of its strictly hyperideal tetrahedra. Now define a function $H : L(M,T) \to \mathbf{R}$ by $H(x) = 2V(x) - \sum_{i=1}^n x_i K_i$ where K_i is the curvature of the metric x at the ith edge. We can express the function H as the sum

$$H = \sum_{r \in T^{(3)}} \Big(2V_r(x) + \sum_{i \in I(r)} x_i a_i^r \Big) - 2\pi \sum_{i=1}^n x_i \tag{2.3}$$

where $V_r(x)$ is the volume of the tetrahedron r in the metric. By (2.1) in the proof of Corollary 5, we have $\partial(2V_r(x) + \sum_{i \in I(r)} x_i a_i^r)/\partial x_j = a_j^r$. This implies that $\partial H/\partial x_j = \sum_{r \in T^{(3)}} a_j^r - 2\pi = -K_j$. Furthermore, the Hessian matrix $[h_{ij}]$ of H is given by $h_{ij} = -\partial K_i/\partial x_j = \sum_{r \in T^{(3)}} \partial a_i^r/\partial x_j = -a_{ij}$ where a_{ij} is the quantity used in the above proof. By the argument above, we see that the Hessian of H is positive definite. Thus the function H is strictly locally convex in $L(M,T)$. □

2.3. **Proof of Corollary 3.** This corollary easily follows from Theorem 2 by standard tools of differential equations.

To see (a), that $f(t) = \sum_{i=1}^n K_i(t)^2$ is strictly decreasing in t, let us take the derivative $df(t)/dt$. We find $f'(t) = 2\sum_{i=1}^n K_i(t) dK_i/dt = 2\sum_{i,j=1}^n a_{ij} K_i K_j < 0$ unless $K_i(t) = 0$ for all i.

To see (b), first of all, observe that at the equilibrium points of (1.1), all curvatures $K_i(t) = 0$ for all i. Thus there are no singularities at the edges. This shows that the cone metric is a smooth complete hyperbolic metric with totally geodesic boundary. Since the matrix $[a_{ij}]$ in (1.2) is negative definite, the equilibrium points are always a local attractor. Thus (c) follows. □

We do not know if the equilibrium point is unique.

2.4. **Proof of Theorem 4.** Consider the smooth map $\Pi : L(M,T) \to \mathbf{R}^n$ sending a metric x to its curvature $\Pi(x) = (K_1, \ldots, K_n)$. By theorem 1, this map is the same as the gradient map $\nabla(-H) : L(M,T) \to \mathbf{R}^n$. Since the function $-H$ is shown to be strictly locally concave, the gradient map is a local diffeomorphism. This ends the proof. □

3. Some remarks and questions

This work and [CL] are motivated by the work of Richard Hamilton [Ha] on the Ricci flow. The strategy of [Ha] is to find a flow deforming metrics such that its curvature evolves according to a heat-type equation. The main focus of study then shifts from the evolution of the metrics to the evolution of its curvature using the maximum principle for heat equations. As long as one has control of the curvature evolution, one gets some control of the metric evolutions by studying either the singularity formation or the long time convergence. Theorem 2 above seems to indicate that the combinatorial flow (1.1) deforms the cone metrics in the "right" direction. There remains the task of understanding the singularity formations in (1.1) which corresponds to the degeneration of the strictly hyperideal tetrahedra.

This is being investigated. Below are some thoughts on this topic. The motivations come from [Th], [CV], [Ri2], [Le] and [CL].

3.1. To understand the singularity formation, we will focus our attention on the function H in (2.3). Following [CV] and [Ri2], our goal is to find the (linear) conditions on the ideal triangulation which will guarantee the existence of critical points for H. The existence of the ideal triangulation satisfying the (linear) condition will be related to the topology of the 3-manifold and will be resolved by topological arguments.

Suppose (M, T) is an ideal triangulated compact 3-manifold such that each boundary component of M has negative Euler characteristic. A pair (e, t), where e is an edge and t is a tetrahedron containing e, is called a *corner* in T. Following Rivin [Ri1], and Casson and Lackenby [La1], we say that the triangulated manifold (M, T) supports a *linear hyperbolic structure* if one can assign to each corner of T a positive number called the *dihedral angle* so that (1) the sum of dihedral angles of all corners adjacent to each fixed edge be 2π, and (2) the sum of dihedral angles of every triple of corners $(e_1, t), (e_2, t), (e_3, t)$, where e_1, e_2, e_3 are adjacent to a fixed vertex, be less than π. By [BB], given a linear hyperbolic structure on (M, T), we can realize each individual tetrahedron by a strictly hyperideal tetrahedron whose dihedral angles are the assinged numbers so that the sum of the dihedral angles at each edge be 2π. It can be shown that if (M, T) supports a linear hyperbolic structure, then the manifold M is irreducible without incompressible tori. One would ask if the converse is also true. The work of Lackenby [La2] gives some evidences that the following may have a positive answer. See also [KR].

Question 1. *Suppose M is a compact irreducible 3-manifold with incompressible boundary consisting of surfaces of negative Euler characteristic. If M contains no incompressible tori and annuli, is there any ideal triangulation of M which supports a linear hyperbolic structure?*

The next question is the 3-dimensional analog of the 2-dimensional singularity formation analysis presented in [CV], [Ri2] and [Le].

Question 2. *Suppose (M, T) is an ideal triangulated 3-manifold which supports a linear hyperbolic structure. Does H have a local minimal point in the space $L(M, T)$ of all cone metrics associated to (M, T)?*

Positive resolutions of these two questions will produce a new proof of Thurston's geometrization theorem for this class of 3-manifolds.

3.2. Suppose (M, T) supports a linear hyperbolic structure. We define the volume of a linear hyperbolic structure to be the sum of the volumes of its strictly hyperideal tetrahedra. Let $LH(M, T)$ be the space of all linear hyperbolic structures on (M, T). It can be shown, using Lagrangian multipliers, that the volume function is strictly concave on $LH(M, T)$ whose maximal point is exactly the complete hyperbolic metric on M. The situation is the same as for ideal triangulations of compact 3-manifolds whose boundary consists of tori. In this case, one realizes each tetrahedron by an ideal tetrahedron in the hyperbolic space. It can be shown that the complete hyperbolic metric of finite volume is exactly equal to the maximal point of the volume function defined on the space of all linear hyperbolic structures given in [Ri1]. This was also observed by Rivin [Ri3].

3.3. We remark that the moduli space of all strictly hyperideal tetrahedra parametrized by their edge lengths $x_1, \ldots, x_6$ is not a convex subset of $\mathbf{R}^6$. This is the main reason that we have only local convergence and local rigidity in Corollary 3 and Theorem 4. However, it is conceivable that Theorem 4 may still be true globally. We do not know yet if the space $L(M,T)$ of all cone metrics associated to the ideal triangulated manifold is homeomorphic to a Euclidean space. It is likely to be the case. In fact, one would hope that there is a diffeomorphism $h : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ so that if we parameterize the space of all strictly hyperideal tetrahedra by $(t_1, \ldots, t_6) = (h(x_1), \ldots, h(x_6))$, then the space becomes convex in t-coordinate. Evidently, if this holds, it implies that the space $L(M,T)$ is convex in the t-coordinate.

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