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THE SMALLEST HYPERBOLIC 6-MANIFOLDS

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ABSTRACT. By gluing together copies of an all right-angled Coxeter polytope a number of open hyperbolic 6-manifolds with Euler characteristic -1 are constructed. They are the first known examples of hyperbolic 6-manifolds having the smallest possible volume.

1. INTRODUCTION

The last few decades have seen a surge of activity in the study of finite volume hyperbolic manifolds—that is, complete Riemannian n-manifolds of constant sectional curvature -1. Not surprisingly for geometrical objects, volume has been, and continues to be, the most important invariant for understanding their sociology. The possible volumes in a fixed dimension form a well-ordered subset of $\mathbb{R}$, indeed a discrete subset except in 3 dimensions (where the orientable manifolds at least have ordinal type ω^ω). Thus it is a natural problem with a long history to construct examples of manifolds with minimum volume in a given dimension.

In 2 dimensions the solution is classical, with the minimum volume in the compact orientable case achieved by a genus 2 surface, and in the noncompact orientable case by a once-punctured torus or thrice-punctured sphere (the identities of the manifolds are of course also known in the nonorientable case). In 3 dimensions the compact orientable case remains an open problem with the Matveev-Fomenko-Weeks manifold [16, 30] obtained via $(5, -2)$-Dehn surgery on the sister of the figure-eight knot complement conjecturally the smallest. Amongst the noncompact orientable 3-manifolds the figure-eight knot complement realizes the minimum volume [17], and the Gieseking manifold (obtained by identifying the sides of a regular hyperbolic tetrahedron as in [14, 20]) does so for the nonorientable ones [1]. One could also add "arithmetic" to our list of adjectives and so have eight optimization problems to play with (so that the Matveev-Fomenko-Weeks manifold is known to be the minimum volume orientable, arithmetic compact 3-manifold; see [5]).

When $n \geq 4$ the picture is murkier, although in even dimensions we have recourse to the Gauss-Bonnet Theorem, so that in particular the minimum volume a $2m$-dimensional hyperbolic manifold could possibly have, is when the Euler characteristic χ satisfies $|\chi| = 1$. The first examples of noncompact 4-manifolds with

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