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HUREWICZ-LIKE TESTS FOR BOREL SUBSETS OF THE PLANE

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(Communicated by Alexander Kechris)

Abstract. Let $\xi \geq 1$ be a countable ordinal. We study the Borel subsets of the plane that can be made $\mathbf{\Pi}^0_\xi$ by refining the Polish topology on the real line. These sets are called potentially $\mathbf{\Pi}^0_\xi$. We give a Hurewicz-like test to recognize potentially $\mathbf{\Pi}^0_\xi$ sets.

1. Preliminaries in dimension one

Let us recall some results in dimension one before studying Borel subsets of the plane. In descriptive set theory, a standard way to see that a set is complicated is to note that it is more complicated than a well-known example. For instance, we have the following result (see [SR]):

Theorem 1 (Hurewicz)**.** *Let $P_f := \{\alpha \in 2^\mathbb{N} / \exists n \in \mathbb{N} \ \ \forall m \geq n \ \ \alpha(m) = 0\}$, X be a Polish space, and A a Borel subset of X. Then exactly one of the following holds:*

(a) *The set A is $\mathbf{\Pi}^0_2(X)$.*
(b) *There is $u : 2^\mathbb{N} \to X$ continuous and one-to-one with $P_f = u^{-1}(A)$.*

This result has been generalized to the other Baire classes (see [Lo-SR]). We state this generalization in two parts:

Theorem 2 (Louveau-Saint Raymond)**.** *Let $\xi < \aleph_1$, $A_{1+\xi} \in \mathbf{\Sigma}^0_{1+\xi}(2^\mathbb{N})$, X be a Polish space, and A, B disjoint analytic subsets of X. One of the following holds:*

(a) *The set A is separable from B by a $\mathbf{\Pi}^0_{1+\xi}(X)$ set.*
(b) *There is $u : 2^\mathbb{N} \to X$ continuous with $A_{1+\xi} \subseteq u^{-1}(A)$ and $2^\mathbb{N} \setminus A_{1+\xi} \subseteq u^{-1}(B)$.*

If we moreover assume that $A_{1+\xi} \notin \mathbf{\Pi}^0_{1+\xi}$, then this is a dichotomy (in this case, and if $\xi \geq 2$, we can have u one-to-one).

Theorem 3. *There is a concrete example of $A_{1+\xi} \in \mathbf{\Sigma}^0_{1+\xi}(2^\mathbb{N}) \setminus \mathbf{\Pi}^0_{1+\xi}(2^\mathbb{N})$, for $\xi < \aleph_1$.*

If we replace P_f (resp., $\mathbf{\Pi}^0_2$) with the set $A_{1+\xi}$ given by Theorem 3 (resp., $\mathbf{\Pi}^0_{1+\xi}$), we get the generalization of Theorem 1 for $\xi \geq 2$. We state this generalization in two parts for the following reasons:

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