HERMITE SPECTRAL METHOD
FOR LONG-SHORT WAVE EQUATIONS

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Abstract. We are concerned with the initial boundary value problem of the Long-Short wave equations on the whole line. A fully discrete spectral approximation scheme is structured by means of Hermite functions in space and central difference in time. A priori estimates are established which are crucial to study the numerical stability and convergence of the fully discrete scheme. Then, unconditionally numerical stability is proved in a space of \( H^1(\mathbb{R}) \) for the envelope of the short wave and in a space of \( L^2(\mathbb{R}) \) for the amplitude of the long wave. Convergence of the fully discrete scheme is shown by the method of error estimates. Finally, numerical experiments are presented and numerical results are illustrated to agree well with the convergence order of the discrete scheme.

1. Introduction. The Long-Short (LS) wave equations are a kind of important nonlinear evolution equations in physics. One of pioneering works was tackled by Djordjevic and Redekopp [2], which described the motion of a two-dimensional packet of capillary gravity waves on water of finite depth, modeled by two partial differential equations: the nonlinear Schrödinger equation with a forcing term and a linear equation. Their analysis reveals the existence of a resonant interaction between a capillary gravity wave and a long gravity wave. The dispersion of the short wave is balanced by nonlinear interaction of the long wave, while the evolution of the long wave is driven by the self-interaction of the short wave. Nowadays, the LS wave equations have been seen to arise in various physical phenomena, such as in electron-plasma and ion-field interaction [10] and plasma physics [9] etc.

Because of their rich physical and mathematical properties and wide applications, the LS wave equations have drawn a lot of attention from a rather diverse group of scientists such as physicists and mathematicians in both theoretical and experimental manners. For the LS wave equations with periodic boundary conditions and initial conditions, Guo [3] studied well-posedness of solutions in the usual

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Sobolev spaces. Tsutsumi and Hatano [14, 15] investigated well-posedness of solutions for the Cauchy problem in fractional Sobolev spaces. By a numerical analysis with periodic boundary conditions, Chang et al [1] presented several finite-difference schemes and spectral schemes, and compared accuracy of each scheme with numerical experiments. Zhang and Xiang [19], and Rashid [12] developed the Fourier pseudo-spectral method to propose a three-level approximation scheme independently. Rashid [13] applied the spectral method to establish a two-level nonlinear discrete scheme. The convergence of the discrete scheme was also discussed in these literature and error estimates of approximate solutions under certain conditions were discussed.

However, very little has been undertaken on numerical results for the LS wave equations on unbounded domain, as far as our knowledge goes. This motivates us to start our study with considering the following LS wave equations:

\[
\begin{align*}
  is_t + s_{xx} - \alpha s l &= f, \quad x \in \mathbb{R}, \quad 0 < t \leq T, \\
l_t + \beta(|s|^2)_x &= g, \quad x \in \mathbb{R}, \quad 0 < t \leq T \\
s(x, 0) &= s_0(x), \quad l(x, 0) = l_0(x), \quad x \in \mathbb{R}, \\
\lim_{|x| \to \infty} s(x, t) &= \lim_{|x| \to \infty} s_x(x, t) = \lim_{|x| \to \infty} l(x, t) = 0, \quad 0 < t \leq T,
\end{align*}
\]

where the complex function \( s \) represents the envelope of the short wave, and the real function \( l \) represents the amplitude of the long wave, and \( \alpha \) and \( \beta \) are positive constants. The existence and uniqueness of the solution was investigated by Guo [3] in a particular case of \( f = g = 0 \).

As we know, the Hermite spectral method by using Hermite polynomials or functions usually can be applied to the domains on the whole line for analyzing nonlinear evolution systems. There are quite many developments for solving nonlinear differential equations by using the Hermite spectral method. We refer to [4, 17] for using Hermite polynomial methods and to [5, 8, 16, 6, 18, 7] for using Hermite function methods. Hermite polynomial methods with the weight \( \omega(x) = e^{-x^2} \) can destroy the crucial conservation properties of equations as well as symmetries and positive definiteness of bilinear operators and may lead to complication in analysis and implementation. And the weight is not natural and proper for some physical problems. Thus in some cases it is more appropriate to consider approximations by Hermite functions with the weight \( \omega(x) = 1 \).

In this paper, we apply the Hermite function method to study the problem (1.1)–(1.4). We first establish a three-level fully discrete spectral scheme, and use it to obtain a priori estimates. Then we analyze the numerical stability of the discrete scheme. Finally, we deal with the convergence of the discrete scheme by using the error estimate method.

An outline of this paper is as follows. We commence by reviewing some preliminaries and notations in Section 2. In Section 3, we construct a fully discrete spectral scheme and use it to study a priori estimates. In Section 4, we analyze unconditionally numerical stability of the derived discrete scheme. Section 5 is dedicated to the convergence of the discrete scheme and the error estimates of approximate solutions without any restrict conditions on discrete parameters of time-space variables. In Section 6, we illustrate numerical results which agree well our theoretical analysis on the convergence order of the discrete scheme.

2. Preliminaries and notations. Let \( H_l(x) \) be the Hermite polynomials of degree \( l \). The Hermite functions of degree \( l \) are defined by
\[ \hat{H}_l(x) = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{2^l l!}} e^{-x^2/2} H_l(x). \]

The functions \( \hat{H}_l(x) \) are mutually-orthogonal in \( L^2(\mathbb{R}) \), i.e.,

\[ \int_{\mathbb{R}} \hat{H}_l(x) \hat{H}_m(x) \, dx = \delta_{l,m}. \] (2.1)

Moreover, we have

\[ \int_{\mathbb{R}} \hat{H}_l(x) \hat{H}_m(x) \, dx = \begin{cases} 
- \sqrt{\frac{l(l-1)}{2}}, & m = l - 2, \\
\frac{l + 1}{2}, & m = l, \\
- \sqrt{\frac{(l+1)(l+2)}{2}}, & m = l + 2, \\
0, & \text{otherwise}.
\end{cases} \] (2.2)

For any given positive integer \( N \), let

\[ \mathcal{H}_N = \text{span} \left\{ \hat{H}_0(x), \hat{H}_1(x), \cdots, \hat{H}_N(x) \right\}, \]

and denote by \( P_N : L^2(\mathbb{R}) \to \mathcal{H}_N \) the orthogonal projection. For any \( v \in L^2(\mathbb{R}) \), it satisfies

\[ (P_N v - v, \phi) = 0, \quad \forall \phi \in \mathcal{H}_N. \]

Let \( L^2(\mathbb{R}), L^\infty(\mathbb{R}), \) and \( H^m(\mathbb{R}) \) denote the usual Sobolev spaces equipped with norms \( \| \cdot \|, \| \cdot \|_\infty \) and \( \| \cdot \|_m \), respectively. The inner products in \( L^2(\mathbb{R}) \) and \( H^m(\mathbb{R}) \) are denoted by \((\cdot, \cdot)\) and \((\cdot, \cdot)_m\), respectively. Let \( | \cdot |_m \) denote the semi-norm of \( H^m(\mathbb{R}) \).

Denote

\[ A v(x) = -e^{\frac{1}{2} x^2} \partial_x \left[ e^{-x^2} \partial_x \left( e^{\frac{1}{2} x^2} v(x) \right) \right]. \]

It is easy to verify that \( A \) is a nonnegative self-adjoint operator. For any integer \( r \geq 0 \), we define the normed space as follows:

\[ H^r_A(\mathbb{R}) = \{ u : \| u \|_{r,A} < \infty \}, \]

where

\[ \| u \|_{r,A} = \left( \sum_{k=0}^{r} \left\| (x^2 + 1)^{\frac{k}{2}} \partial_x^k u \right\|^2 \right)^{\frac{1}{2}}. \]

For any real \( r > 0 \), we define the space and its norm by function space interpolation.

As the end of this section, we give two lemmas which will be frequently used in the next two sections.

**Lemma 2.1.** [16] For any \( v \in H^r_A(\mathbb{R}) \) and \( 0 \leq \mu \leq r \), we have

\[ |P_N v - v|_{\mu} \leq cN^{\frac{r}{2} - \frac{\mu}{2}} \| v \|_{r,A}. \]

**Lemma 2.2** ([11]). Assume that \( g_1 \geq 0, h_n \) and \( \varphi_n \) are non-negative sequences for \( n \geq 1 \), and \( \varphi_n \) satisfies

\[ \left\{ \begin{array}{l}
\varphi_1 \leq g_1, \\
\varphi_n \leq g_1 + \tau \sum_{j=1}^{n-1} h_j \varphi_j, \quad n \geq 2.
\end{array} \right. \]
Then there holds
\[ \varphi_n \leq g_1 \exp \left( \tau \sum_{j=1}^{n-1} h_j \right), \quad n \geq 2. \]

3. A priori estimates. In this section, we first construct a fully discrete scheme using Hermite functions in space and central difference in time for the problem (1.1)-(1.4). Then we explore a priori estimates of discrete solutions which play an important role in the proofs of stability and convergence.

Let \( \tau \) be the step-size in variable \( t \), \( t_k = k \tau \) \( (k = 0, 1, \cdots, M; M = [T/\tau]) \), \( u^k = u(x, t_k) \), and
\[ u_t^k = \frac{1}{\tau} (u^{k+1} - u^{k-1}), \quad u^k = \frac{1}{2} (u^{k+1} + u^{k-1}). \]

We see that the fully discrete Hermite spectral scheme for the problem (1.1)-(1.4) is to find \( s^k_N(x, t), l^k_N(x, t) \in H_N \) such that
\[
\begin{aligned}
&\left\{ \begin{array}{l}
isk_N + PN s^k_{Nxx} - \alpha PN (s^k_N) = PNF^k, \quad 1 \leq k \leq M - 1, \\
l^k_N + \beta PN |s^k_N|^2 = PN G^k, \quad 1 \leq k \leq M - 1, \\
s^0_N = PNs_0, \quad l^0_N = PNl_0, \\
s^1_N = s^0_N + PN(\tau(s_{0xx} - \alpha s_0l_0) - f^0), \quad l^1_N = l^0_N - PN(\tau(\beta |s_0|^2 - g^0)).
\end{array} \right.
\end{aligned}
\] (3.1, 3.2, 3.3, 3.4)

This is a linear iteration scheme. It only needs to solve \( l^k_N \) through (3.2) explicitly, and solve linear algebraic equations (3.1) for \( s^k_N \) in each iteration.

In what follows, we shall make a priori estimates for the discrete scheme (3.1)-(3.4). To this end, we introduce two technical lemmas.

Lemma 3.1. For any two discrete functions \( u^k \) and \( v^k \), \( (k = 0, 1, \cdots, n) \), we have
\[
2\tau \sum_{k=1}^{n-1} (u^k, v_t^k) = -2\tau \sum_{k=2}^{n-2} (u^k, v^k) + (u^{n-2}, v^{n-1}) + (u^{n-1}, v^n) - (u^1, v^0) - (u^2, v^1).
\]

The proof is straightforward based on the equality:
\[
2\tau \sum_{k=1}^{n-1} (u^k, v_t^k) = \sum_{k=2}^{n} (u^{k-1}, v^k) - \sum_{k=0}^{n-2} (u^{k+1}, v^k)
= -2\tau \sum_{k=2}^{n-2} (u^k, v^k) + (u^{n-2}, v^{n-1}) + (u^{n-1}, v^n) - (u^1, v^0) - (u^2, v^1).
\]

Lemma 3.2. Assume the complex function \( u \in C^1[0, T] \). Then we have
\[
\tau \sum_{k=1}^{n} |u^k|^2 \leq \tau \int_{0}^{t_n} |u_t|^2 dt + (1 + \tau) \int_{0}^{t_n} |u|^2 dt.
\]

Proof. By integration by parts, we get
\[
\int_{t_k}^{t_{k+1}} |u|^2 dt = \tau |u^{k+1}|^2 - 2Re \int_{t_k}^{t_{k+1}} (t - t_k) u_u dt.
\]

Using Young’s inequality, we have
\[
\tau |u^{k+1}|^2 = \int_{t_k}^{t_{k+1}} |u|^2 dt + 2Re \int_{t_k}^{t_{k+1}} (t - t_k) u_u u_d dt
\]
Evaluating the sum of $\tau|s_{k+1}|^2$ by letting the index $k$ vary from 0 to $n - 1$, we obtain the desired result.

Proposition 3.1. If $s_0 \in H^2(\mathbb{R})$ and $l_0$, $f$, $f_t \in L^2(\mathbb{R})$, then we have

$$\|s_N^n\| \leq E_{0n}, \quad n = 0, 1, 2, \cdots, M,$$

where $E_{0n}$ is a constant depending on $s_0$ and $l_0$.

Proof. In view of (3.1), taking the inner product with $s_N^k$ in $L^2$ and looking at the imaginary part, we have

$$\|s_N^k\| \leq \|f^k\|, \quad k = 1, 2, \cdots, M - 1.$$

Using Lemma 3.2, we get

$$\|s_N^n\| + \|s_N^{n-1}\| \leq \|s_N^n\| + \|s_N^{n-1}\| + 2 \left( \int_0^T \|f_n\|^2 dt + 2 \int_0^T \|f\|^2 dt \right).$$

In view of (3.3) and the definition of $P_N$, we deduce that

$$\|s_N^0\| = \|P_Ns_0\| \leq \|s_0\|$$

and

$$\|s_N^1\| = \|P_N(s_0 + ir(s_{0xx} - \alpha s_0l_0 - f^0))\| \leq \|s_0\| + \tau \left( \|s_{0xx}\| + \alpha \|s_0\| \|l_0\| + \|f^0\| \right).$$

Thus, we obtain

$$\|s_N^n\| \leq 2\|s_0\| + \|s_{0xx}\| + \alpha \|s_0\| \|l_0\| + \|f^0\| + 2 \left( \int_0^T \|f_n\|^2 dt + 2 \int_0^T \|f\|^2 dt \right) \leq E_{0n}, \quad n = 0, 1, \cdots, M.$$
Using Hölder's inequality, Young's inequality, as well as the Sobolev inequality and
\[ M_{s} - N x_{2} + 1 \leq 5 \leq 1, \]
we get
\[ b_{1}. \]
and
\[ \|l_{N}\| \leq \|l_{0} - \beta(\tau|s_{0}|_{x}^{2} - g^{0})\| \]
\[ \leq \|l_{0}\| + 2\beta\tau\|s_{0}\|_{\infty}\|s_{0}\|_{1, A} + \tau\|g^{0}\| \]
\[ \leq b_{1}. \]

Hence, one can see that inequality (3.7) holds for \( n = 0, 1 \).

We now prove that inequality (3.7) holds for \( M \geq n \geq 2 \) and inequality (3.8) holds \( M - 1 \geq n \geq 1 \).

To estimate \( \|s_{N,Nx}\| \), we take the inner product of (3.1) with \( s_{Nt}^{k} \) in \( L^{2} \) and consider the real part. It has
\[ \frac{1}{2} \|s_{N,Nx}^{k}\|^{2} \]
\[ = -\alpha \text{Re} \left( s_{Nt}^{k} l_{Nt}^{k} + l_{Nt}^{k} s_{Nt}^{k}\right) + \text{Re} \left( f^{k}, s_{Nt}^{k}\right) \]
\[ = -\frac{\alpha}{2} \left( l_{Nt}^{k}, s_{Nt}^{k}\right) + \text{Re} \left( f^{k}, s_{Nt}^{k}\right) \].

Using Hölder's inequality, Young's inequality, as well as the Sobolev inequality
\[ \|s_{N}\|_{\infty} \leq \|s_{N}\|_{\frac{3}{2}} \|s_{N,Nx}\|_{\frac{1}{2}}, \]
we deduce that
\[ \|s_{N,Nx}^{k}\|^{2} + \|s_{N,Nx}^{n-1}\|^{2} \]
\[ = \|s_{N,Nx}^{0}\|^{2} + \|s_{N,Nx}^{1}\|^{2} + 2\tau \sum_{k=2}^{n-2} \left( \alpha(l_{Nt}^{k}, s_{Nt}^{k}) + 2(f_{Nt}^{k}, s_{Nt}^{k}) \right) \]
\[ - \alpha \|l_{Nt}^{n-2}, s_{Nt}^{n-1}\|^{2} + \|l_{Nt}^{n-1}, s_{Nt}^{n}\|^{2} \]
\[ = 2 \left( f^{n-2}, s_{Nt}^{n-1}\right) + \left( f^{n-1}, s_{Nt}^{n}\right) - \left( f^{1}, s_{Nt}^{0}\right) \]
\[ \leq \frac{5}{4} \left( \|s_{N,Nx}^{0}\|_{1}^{2} + \|s_{N,Nx}^{1}\|_{1}^{2} + b_{2}\tau \sum_{k=2}^{n-2} \left( \|l_{Nt}^{k}\|_{\frac{3}{2}} + \|s_{Nt}^{k}\|_{\frac{1}{2}} + \|f_{Nt}^{k}\|_{2}\right) \]
\[ + \frac{1}{4} \left( \|s_{N,Nx}^{0}\|^{2} + \|s_{N,Nx}^{n-1}\|^{2} \right) \]
\[ + \frac{b_{2}}{2} \left( \|l_{Nt}^{n-1}\|_{\frac{3}{2}} + \|l_{Nt}^{n-2}\|_{\frac{3}{2}} + \|l_{Nt}^{n-1}\|_{\frac{1}{2}} + \|f_{Nt}^{n}\|_{\frac{3}{2}} + \|f_{Nt}^{n-1}\|_{\frac{3}{2}} \right) \]
\[ \frac{5}{3} \left( \|s_{N,Nx}^{0}\|_{1}^{2} + \|s_{N,Nx}^{1}\|_{1}^{2} \right) + \frac{4b_{2}}{3} \tau \sum_{k=2}^{n-2} \left( \|l_{Nt}^{k}\|_{\frac{3}{2}} + \|s_{Nt}^{k}\|_{\frac{1}{2}} + \|f_{Nt}^{k}\|_{2}\right) \]
\[ + \frac{2b_{2}}{3} \left( \|l_{Nt}^{n-1}\|_{\frac{3}{2}} + \|l_{Nt}^{n-2}\|_{\frac{3}{2}} + \|l_{Nt}^{n-1}\|_{\frac{1}{2}} + \|f_{Nt}^{n}\|_{\frac{3}{2}} + \|f_{Nt}^{n-1}\|_{\frac{3}{2}} \right) \].

where \( b_{2} = \max \left\{ \frac{3}{2} \alpha^{2} E_{0}, 2 \right\} \).

After simplification, we get
\[ \|s_{N,Nx}^{n}\|^{2} + \|s_{N,Nx}^{n-1}\|^{2} \]
\[ \leq \frac{5}{3} \left( \|s_{N,Nx}^{0}\|_{1}^{2} + \|s_{N,Nx}^{1}\|_{1}^{2} \right) + \frac{4b_{2}}{3} \tau \sum_{k=2}^{n-2} \left( \|l_{Nt}^{k}\|_{\frac{3}{2}} + \|s_{Nt}^{k}\|_{\frac{1}{2}} + \|f_{Nt}^{k}\|_{2}\right) \]
\[ + \frac{2b_{2}}{3} \left( \|l_{Nt}^{n-1}\|_{\frac{3}{2}} + \|l_{Nt}^{n-2}\|_{\frac{3}{2}} + \|l_{Nt}^{n-1}\|_{\frac{1}{2}} + \|f_{Nt}^{n}\|_{\frac{3}{2}} + \|f_{Nt}^{n-1}\|_{\frac{3}{2}} \right). \]
To estimate norms of $||l_N^k||$ and $||l_N^{-1}||$, we take the inner product of (3.2) with $l_N^k$ in $L^2$. Making use of proposition 3.1 and the Sobolev inequality, we derive that
\[
\frac{1}{2} ||l_N^k||^2 \leq 2\beta ||s_N^k||^{\frac{2}{3}} ||s_N^{k+1}||^\frac{2}{3} ||l_N^k|| + ||g^k|| ||l_N^k|| \leq (2\beta E_{0s}^\frac{1}{3} ||s_N^{k+1}||^\frac{2}{3} + ||g^k||) ||l_N^k||.
\]

It further gives
\[
||l_N^{k+1}|| - ||l_N^{-1}|| \leq 4\beta E_{0s}^\frac{1}{3} \tau ||s_N^{k+1}||^\frac{2}{3} + 2\tau ||g^k||.
\]

Finding the sum on both sides for $k$ from 1 to $n - 1$ and using (3.10)-(3.12) leads to
\[
||l_N^n|| + ||l_N^{-1}|| \leq ||l_0|| + b_2 + 2\tau \sum_{k=1}^{n-1} \left(2\beta E_{0s}^\frac{1}{3} ||s_N^{k+1}||^\frac{2}{3} + ||g^k||\right), \quad n = 2, 3, \cdots, M. \quad (3.14)
\]

Using Hölder’s inequality again, we deduce that
\[
||l_N^n||^\frac{2}{3} + ||l_N^{-1}||^\frac{4}{3} \leq \left(||l_0||^\frac{2}{3} + b_2^4 + 2\tau \sum_{k=1}^{n-1} (2\beta E_{0s}^\frac{1}{3} ||s_N^{k+1}||^\frac{2}{3} + ||g^k||) \right)^{\frac{2}{3}} \leq \left(2\beta E_{0s}^\frac{1}{3} T + 4 \int_0^T (||g|| + ||g_t||)dt + 2\right)^{\frac{2}{3}} \cdot \left(||l_0||^\frac{2}{3} + b_1^4 + 4 \int_0^T (||g|| + ||g_t||)dt + 4\beta E_{0s}^\frac{1}{3} \tau \sum_{k=1}^{n-1} ||s_N^{k+1}||^2\right). \quad (3.15)
\]

By (3.2) and Proposition 3.1, we find
\[
||l_N^n|| \leq 2\beta ||s_N^1||^{\frac{2}{3}} ||s_N^{k+1}||^\frac{2}{3} \leq 2\beta E_{0s}^\frac{1}{3} ||s_N^{k+1}||^\frac{2}{3} + ||g^k||, \quad k = 1, 2, \cdots, M - 1. \quad (3.16)
\]

Substituting (3.15) and (3.16) into (3.13) yields
\[
||s_N^n||^2 \leq g_1 + h\tau \sum_{j=1}^{n-1} ||s_N^j||^2, \quad n = 2, 3, \cdots, M. \quad (3.17)
\]

where
\[
g_1 = \frac{5}{3} (c||s_0||_{L^2} + b_0^2) + \left(\frac{5}{3} + \frac{2b_2}{3} + \frac{4b_2 T}{3}\right) E_{0s}^2 + \frac{16b_2}{3} \int_0^T ||f_t||^2 dt + \frac{4b_2}{3} \left[\left(2\beta E_{0s}^\frac{1}{3} T + 4 \int_0^T (||g|| + ||g_t||)dt + 2\right)^{\frac{2}{3}} \right.
\]
\[
\left. \left(||l_0||^\frac{2}{3} + b_1^4 + 4 \int_0^T (||g|| + ||g_t||)dt\right)\right]
\]
\[
\text{and}
\]
\[
h = \frac{8}{3} b_2 \beta \left[\left(2\beta E_{0s}^\frac{1}{3} T + 2 \int_0^T (||g|| + ||g_t||)dt\right)^{\frac{2}{3}}\right]
\]
Set $\varphi_n = \|s_{N,x}^n\|^2$. Clearly, $\varphi_1 \leq g_1$. By Lemma 2.2 and (3.17), we infer that
\[
\|s_{N,x}^n\|^2 \leq g_1 \exp(hT) \equiv E_{1s}^2, \quad n = 2, 3, \cdots, M. \tag{3.18}
\]
Substituting (3.18) into (3.14) and (3.16) respectively, we obtain
\[
\|l_{N}^n\| \leq \|l_0\| + b_2 + 4\beta E_{0s}\frac{T}{2}E_{1s}^2 + 4 \int_0^T (\|g\| + \|g_t\|) dt \\
\equiv E_{0t}, \quad n = 2, 3, \cdots, M, \tag{3.19}
\]
and
\[
\|l_{N}^n\| \leq 2\beta E_{0s}\frac{T}{2}E_{1s}^2 + \|g\|_{L^\infty((0,T);L^2(\mathbb{R}))} \\
\equiv E_{0t}, \quad n = 1, 2, \cdots, M - 1. \tag{3.20}
\]
Consequently, the proof of Proposition 3.2 is completed. \hfill \Box

**Proposition 3.3.** If $s_0$, $s_{0xx} \in H_A^2(\mathbb{R})$, $l_0 \in H^2(\mathbb{R})$, $f_t$, $g_t \in L^2((0,T);L^2(\mathbb{R}))$ and $f, g \in L^\infty((0,T);L^2(\mathbb{R}))$, then there holds
\[
\|s_{N,t}\|^2 \leq E_{0t}, \quad n = 1, 2, \cdots, M - 1. \tag{3.21}
\]
where $E_{0t}$ is a constant depending on $s_0$ and $l_0$.

**Proof.** Set $w^k_N = s^k_{N,t}$. Making the central difference quotient for (3.1), we get
\[
iw^k_{N,t} + P_Niw^k_{N,xx} = \frac{\alpha}{2T} P_N(s^k_{N,t} s^k_{N,xx} - s^{k-1}_{N,xx}, s^{k-1}_{N,t}) + f^k_t, \quad k = 2, 3, \cdots, M - 2. \tag{3.22}
\]
We take the inner product of (3.21) with $w^k_N$ in $L^2$ and consider the imaginary part. By using the Sobolev inequality and Propositions 3.1 and 3.2, we have
\[
\frac{1}{2} \|w^k_N\|^2 = \text{Im} \left( \sum_{k=0}^{M-1} (\int_0^T f^k_s, w^k_N) \right) \leq \alpha E_{0s}^2 E_{1s}^2 E_{0t} + f^k_t \|w^k_N\|,
\]
and
\[
\|w^{k+1}_N\| - \|w^{k-1}_N\| \leq 2\tau (\alpha E_{0s}^2 E_{1s}^2 E_{0t} + f^k_t). \tag{3.23}
\]
Evaluating the sum on both sides for $k$ from 2 to $n - 2$ yields
\[
\|w^{n-1}_N\| + \|w^{n-2}_N\| \leq \|w^1_N\| + \|w^2_N\| + 2\alpha T E_{0s}^2 E_{1s}^2 E_{0t} + 2\sqrt{2} \left( \int_0^T \|f_t\|^2 dt \right)^{\frac{1}{2}}. \tag{3.24}
\]
We now estimate $\|w^1_N\|$ and $\|w^2_N\|$, in other words, $\|s^1_{N,t}\|$ and $\|s^2_{N,t}\|$. Let $k = 1$ in (3.1). We take the inner product of (3.1) with $s^1_{N,t}$ in $L^2$ and consider the imaginary part. It gives
\[
\|s^1_{N,t}\|^2 = \text{Im} (s^{0}_{N,xx}, s^1_{N,t}) + \alpha \text{Im} (l^1_N s^{0}_{N,t}, s^1_{N,t}) + (f^1, s^1_{N,t}) \leq (\|s^{0}_{N,xx}\| + \alpha \|s^1_{N,t}\|^2)^{\frac{1}{2}} \|l^1_N\| + 2\alpha T E_{0s}^2 E_{1s}^2 E_{0t} + 2\sqrt{2} \left( \int_0^T \|f_t\|^2 dt \right)^{\frac{1}{2}}.
\]
That is,
\[
\|s^1_{N,t}\| \leq \|s^{0}_{N,xx}\| + \alpha \|s^1_{N,t}\|^2 \|s^{0}_{N,xx}\|^2 \|l^1_N\| + 2\alpha T E_{0s}^2 E_{1s}^2 E_{0t} + 2\sqrt{2} \left( \int_0^T \|f_t\|^2 dt \right)^{\frac{1}{2}}. \tag{3.25}
\]
For $\|s^{0}_{N,xx}\|$, using Lemma 2.1 yields
\[
\|s^{0}_{N,xx}\| \leq \|(P_N s_0 - s_0)_{xx}\| + \|s_{0xx}\| \leq c \|s_0\|_{2,A}. \tag{3.26}
\]
Substituting (3.5), (3.9), (3.12) and (3.24) into (3.23), we obtain
\[ \|s_N^1\| \leq c\|s_0\|_{2,A} + \alpha c\|s_0\| \frac{\alpha}{2} \|s\|_{1,A} E_0t + \|f\|_{L^\infty((0,T);L^2(\mathbb{R}))}. \] (3.25)

Similarly, letting \( k = 2 \) in (3.1) we have
\[ \|s_N^2\| \leq \|s_N^{1,xx}\| + \alpha \|s_N^1\|^\frac{\alpha}{2} \|s_N^{1,x}\| \frac{\alpha}{2} \|P_N^{1}\| + \|f^2\|. \] (3.26)

For \( \|s_{N,xx}^1\| \), using Lemma 2.1 again gives
\[ \|s_{N,xx}^1\| \leq c\|s_0\|_{2,A} + \alpha c\|s_0\| \frac{\alpha}{2} \|s_0\|_{2,A} + \alpha c\|s_0\| \frac{\alpha}{2} \|s_0\|_{\infty} ||0||_x + \alpha c\|s_0\| \frac{\alpha}{2} \|l_{0xx}\| \]
\[ \equiv b_3. \] (3.27)

Substituting (3.27), (3.6), (3.11) and (3.19) into (3.26) leads to
\[ \|s_{N,xx}^2\| \leq b_3 + \alpha b_3^2 b_1^2 E_0t + \|f\|_{L^\infty((0,T);L^2(\mathbb{R}))}. \] (3.28)

Furthermore, substituting (3.25) and (3.28) into (3.22), we obtain
\[ \|u_N^{n-1}\| \leq E_0t, \]
where
\[ E_0t = b_3 + \|s_0\|_{2,A} + \|s_0\|_{2,A} + \alpha \|s_0\| \frac{\alpha}{2} \|s_0\|_{\infty} ||0||_x + \alpha \|s_0\| \frac{\alpha}{2} \|l_{0xx}\| \]
\[ + 2\|f\|_{L^\infty((0,T);L^2(\mathbb{R}))} + 2\sqrt{2} \left( \int_0^T \|f_0\|^2 dt \right)^\frac{\alpha}{2}. \]

Consequently, we arrive at (3.20).

4. Numerical stability. In this section our goal is to discuss the unconditional stability for the discrete scheme (3.1)-(3.4).

Suppose that \( s_{N,j}^k \) and \( l_{N,j}^k \) are two solutions of the discrete scheme (3.1)-(3.4) with the initial values \( s_{N,j}^0, s_{N,j}^1, l_{N,j}^0, l_{N,j}^1 \) and the source terms \( f_j \) and \( g_j \) (\( j = 1, 2 \)). According to Propositions 3.1-3.3 in the preceding section, we know that
\[ \|s_{N,j}^k\| \leq E_{0,j}, \|s_{N,j}^k\| \leq E_{1,j}, \|s_{N,j}^k\| \leq E_{0,j}', \quad j = 1, 2 \] (4.1)

and
\[ \|l_{N,j}^k\| \leq E_{0,j}, \|l_{N,j}^k\| \leq E_{0,j}', \quad j = 1, 2, \] (4.2)

where \( E_{0,j}, E_{1,j} \) and \( E_{0,j}' \) depend on \( s_{N,j}^0, s_{N,j}^1, l_{N,j}^0 \) and \( l_{N,j}^1 \) (\( j = 1, 2 \)).

Set
\[ u_N^k = s_{N,j}^k - s_{N,j}^2, \quad v_N^k = l_{N,j}^k - l_{N,j}^2, \quad s_N^1 = f_{N,j}^k - f_{N,j}^k, \quad g_N^k = g_{N,j}^k - g_{N,j}^k. \]

We find that \( u_N^k \) and \( v_N^k \) satisfy
\[ \begin{cases}
  u_{N,j}^k + P_N u_{N,xx}^k = \alpha P_N(s_{N,j}^k t_{N,j}^k) - \alpha P_N(s_{N,j}^k t_{N,j}^k) + P_N f_{N,j}^k, & 1 \leq k \leq M - 1 \quad (4.3) \\
  u_{N,j}^k + \beta P_N(s_{N,j}^k t_{N,j}^k) - \beta P_N(s_{N,j}^k t_{N,j}^k) \|x\| = P_N g_{N,j}^k, & 1 \leq k \leq M - 1. \quad (4.4)
\end{cases} \]

The following result is regarding the stability of the fully discrete scheme (3.1)-(3.4).
Theorem 4.1. The fully discrete scheme (3.1)-(3.4) is unconditionally stable, and
\[ \|u_N^k\|^2 + \|v_N^k\|^2 \leq C \left[ \|u_N^0\|^2 + \|v_N^0\|^2 + \|u_N^1\|^2 + \|v_N^1\|^2 + \int_0^T (\|g_N\|^2 + \|f_N\|^2 + \|g_{Nt}\|^2) \, dt + \|f_N\|_{L^\infty(0,T)} \right], \quad n = 2, 3, \ldots, M, \]
where the constant C depends on T, s_{Nj}^0, s_{Nj}^1, t_{Nj}^0 and \beta_{Nj} (j = 1, 2).

Proof. We take the inner product of (4.3) with \( u_N^k \) in \( L^2 \) and consider the imaginary part. Using the Sobolev inequality and (4.1), we have
\[ \frac{1}{2} \|u_N^k\|^2 = \alpha \text{Im} \left( s_{N1}^k v_N^k, u_N^k \right) + \text{Im} \left( f_N^k, u_N^k \right) \leq \left( \alpha E_{N1}^k E_{N1}^k \right) \|v_N^k\| \||u_N^k||, \]
namely,
\[ \|u_N^k\| \leq 2\alpha E_{N1}^k E_{N1}^k \|v_N^k\| + \|f_N^k\|. \tag{4.5} \]
Finding the sum on both sides for \( k \) from 1 to \( n - 1 \) gives
\[ \|u_N^0\| + \|u_N^{n-1}\| \leq \|u_N^0\| + \|u_N^n\| + 2\tau \sum_{k=1}^{n-1} \|f_N^k\| + 2\alpha b_4 \tau \sum_{k=1}^{n-1} \|v_N^k\|, \tag{4.6} \]
where
\[ b_4 = E_{N1}^2 E_{N1}^2. \]
In order to estimate \( \|v_N^k\| \), we take the inner product of (4.4) with \( v_N^k \) in \( L^2 \). By using the Sobolev inequality and Young’s inequality, it follows from (4.1) that
\[ \frac{1}{2} \|v_N^k\|^2 = -2\beta \text{Re} \left( s_{N1}^k u_N^k + s_{N2}^k u_N^k, v_N^k \right) + (g_N^k, v_N^k) \leq 2\beta \left( \|s_{N1}^k\| \|u_N^k\| + \|s_{N2}^k\| \|u_N^k\|^2 \right) \|v_N^k\| + \|g_N^k\| \|v_N^k\| \leq \beta \left( 2b_4 + E_{1x2} \right) \|u_N^k\|^2 + E_{1x2} \|u_N^k\| \|v_N^k\| + \|g_N^k\| \|v_N^k\| \]
That is,
\[ \|v_N^k\| \leq \beta b_5 (\|u_N^k\|^2 + \|u_N^k\|) + \|g_N^k\|, \]
where
\[ b_5 = 2b_4 + E_{1x2}. \]
Evaluating the sum on both sides for \( k \) from 1 to \( n - 1 \) leads to
\[ \|v_N^n\| + \|v_N^{n-1}\| \leq \|v_N^0\| + \|v_N^1\| + 2\tau \sum_{k=1}^{n-1} \|g_N^k\| + 2\beta \tau \sum_{k=1}^{n-1} (\|u_N^k\|^2 + \|u_N^k\|). \tag{4.7} \]
To estimate \( \|u_N^k\| \) on the right side of the above inequality, we take the inner product of (4.3) with \( u_N^k \) in \( L^2 \) and consider the real part. It gives
\[ \frac{1}{2} \|u_N^k\|^2 = -\alpha \text{Re} \left( s_{N1}^k v_N^k, u_N^k \right) - \frac{\alpha}{2} \left( t_{N2}^k, u_N^k \right)^2 - (f_N^k, u_N^k). \]
Considering the sum on both sides for \( k \) from 1 to \( n - 1 \) yields
\[ \|u_N^n\|^2 + \|u_N^{n-1}\|^2 \]
\[ = -4\alpha \tau \text{Re} \sum_{k=1}^{n-1} \left( s_{N1}^k v_N^k, u_N^k \right) - 2\alpha \tau \sum_{k=1}^{n-1} (t_{N2}^k, u_N^k)^2 - 4\tau \sum_{k=1}^{n-1} (f_N^k, u_N^k). \tag{4.8} \]
For the first term of the right hand of (4.8), it follows from Lemma 3.1 and Hölder’s inequality that

\[-2\tau \sum_{k=1}^{n-1} \left( s_{N1}^k v_N^k, u_{N1}^k \right) \]

\[= 2\tau \sum_{k=2}^{n-2} \left( \left( s_{N1}^k v_N^k \right)_i, u_N^k \right) - \left( s_{N1}^{n-1} v_N^{n-1}, u_N^n \right) - \left( s_{N1}^{n-2} v_N^{n-2}, u_N^{n-1} \right) + \left( s_{N1}^2 v_N^2, u_N^0 \right) \]

\[\leq 2\tau \sum_{k=2}^{n-2} \left( \| s_{N1}^k \| \| v_N^{k+1} \| \| u_N^k \|^2 + \| s_{N1}^{n-1} \| \| v_N^{n-1} \| \| u_N^n \| \right) + \| s_{N1}^{n-2} \| \| v_N^{n-2} \| \| u_N^{n-1} \| + \| s_{N1}^1 \| \| v_N^1 \| \| u_N^0 \| \| . \]

(4.9)

From (4.4), by using the Sobolev inequality and (4.1) we find

\[\| v_N^k \| \leq \beta b_5 (\| u_N^x \| + \| u_N^k \|) . \]

(4.10)

Substituting (4.10) into (4.9) and using (4.1) and (4.2), we deduce that

\[-2\tau \sum_{k=1}^{n-1} \left( s_{N1}^k v_N^k, u_{N1}^k \right) \]

\[\leq 2\tau \sum_{k=2}^{n-2} \left( \| s_{N1}^k \| \| v_N^{k+1} \| \| u_N^k \|^2 + \| s_{N1}^{n-1} \| \| v_N^{n-1} \| \| u_N^n \| \right) \]

\[+ 2\tau \sum_{k=2}^{n-2} \left( \| s_{N1}^{n-1} \| \| v_N^{n-1} \| \| u_N^k \| \right) + \| s_{N1}^{n-1} \| \| v_N^{n-1} \| \| u_N^n \| \]

\[+ s_{N1}^{n-1} \| v_N^{n-1} \| \| u_N^k \| + \| s_{N1}^1 \| \| v_N^1 \| \| u_N^0 \| \]

\[\leq b_0 \tau \sum_{k=2}^{n-2} \left( (\| v_N^{k+1} \|^2 + \| u_N^k \|^2) + \frac{b_4}{2} (\| v_N^{n-1} \|^2 + \| v_N^{n-2} \|^2 + \| v_N^2 \|^2 + \| v_N^1 \|^2) \right) \]

\[+ \frac{b_4}{2} (\| u_N^{n-1} \|^2 + \| u_N^2 \|^2 + \| u_N^1 \|^2 + \| u_N^0 \|^2) , \]

(4.11)

where

\[b_0 = E_{0a1}^t + 3\beta b_4 b_5 . \]

For the second term on the right hand of (4.8), it follows from Hölder’s inequality that

\[-2\alpha \tau \sum_{k=1}^{n-1} \left( l_{N2}^k, |u_N^k|^2 \right) \]

\[= \alpha \left[ 2\tau \sum_{k=2}^{n-2} \left( l_{N2}^k, |u_N|^2 \right) \right. \]

\[\left. - \left( \left( l_{N2}^{n-1}, |u_N^{n-1}|^2 \right) - \left( l_{N2}^{n-2}, |u_N^{n-2}|^2 \right) + \left( l_{N2}^2, |u_N^1|^2 \right) + \left( l_{N2}^1, |u_N^0|^2 \right) \right) \right] \]

\[\leq 2\alpha \tau \sum_{k=2}^{n-2} \left( \| l_{N2}^k \| \| u_N^k \|^2 + \| u_N^{n-1} \|^2 + \alpha \| l_{N2}^{n-1} \| \| u_N^{n-1} \|^2 + \| u_N^0 \|^2 \right) \]

\[+ \frac{\alpha}{2} \| u_N^2 \|^2 + \| u_N^1 \|^2 + \| u_N^0 \|^2 , \]
By substituting (4.11), (4.12) and (4.13) into (4.8), using the Sobolev inequality and Young’s inequality, it follows from (4.6) and (4.7) that

\[ \|u_N^n\|^2 + \|u_N^0\|^2 + \|u_N^1\|^2 + \|u_N^2\|^2 \]

\[ + \frac{1}{4} (\|u_N^n\|^2 + \|u_N^0\|^2 + \|u_N^1\|^2 + \|u_N^2\|^2). \quad (4.12) \]

Similarly, we have

\[ 4\tau \sum_{k=1}^{n-1} (f_N^k, u_N^k) \]

\[ = 2\tau \sum_{k=2}^{n-2} (\|f_N^k\|^2 + \|u_N^k\|^2) + \|f_N^{n-2}\|^2 + \|f_N^{n-1}\|^2 + \|f_N^2\|^2 + \|f_N^1\|^2 \]

\[ + \|u_N^n\|^2 + \|u_N^0\|^2 + \|u_N^1\|^2 + \|u_N^2\|^2. \quad (4.13) \]

By substituting (4.11), (4.12) and (4.13) into (4.8), using the Sobolev inequality and Young’s inequality, it follows from (4.6) and (4.7) that

\[ \|u_N^0\|^2 + \|u_N^1\|^2 \]

\[ \leq \tau \sum_{k=1}^{n-1} \left( \|v_N^k\|^2 + \|u_N^n\|_1^2 \right) + b_8 \left( \|u_N^0\|_1^2 + \|u_N^1\|_1^2 + \|v_N^0\|^2 + \|v_N^1\|^2 \right) \]

\[ + b_8 \left( \tau \sum_{k=1}^{n-1} \|f_N^k\|^2 + \|f_N\|_{L^\infty([0,T]; L^2(\mathbb{R}))}^2 \right), \quad (4.14) \]

where

\[ b_7 = \frac{8}{3} b_6 + \max \left\{ 2\alpha E_0^2 + \frac{128}{3} b_4 T \beta^2, 12\alpha^2 b_4 T \left( \alpha^4 E_0^2 + \frac{4}{3} b_4 + \frac{4}{3} \right) \right\} \]

and

\[ b_8 = \max \left\{ 4\alpha^4 E_0^2 + \frac{16\alpha}{3} b_4, \frac{32}{3} b_4, \frac{16}{3} \right\}. \]

Combining (4.6), (4.7) and (4.14) and applying Lemma 3.2, we get

\[ \|u_N^n\|^2 + \|v_N^n\|^2 \leq g_1 + b_7 \sum_{k=1}^{n-1} \left( \|v_N^k\|^2 + \|u_N^n\|_1^2 \right), \quad n = 2, 3, \ldots, M, \quad (4.15) \]

where

\[ g_1 = (b_8 + 5) \left( \|u_N^0\|_1^2 + \|u_N^1\|_1^2 + \|v_N^0\|^2 + \|v_N^1\|^2 \right) \]

\[ + (b_8 + 80T) \int_0^T \left( \|g_N\|^2 + \|f_N\|^2 + \|g_N\|^2 \right) dt + \|f_N\|_{L^\infty([0,T]; L^2(\mathbb{R}))}^2 \]

and

\[ b = \max \left\{ b_7 + 20\beta^2 b_5 T, b_7 + 16\alpha^2 b_4 T \right\}. \]

Set \( \varphi_n = \|u_N^n\|^2 + \|v_N^n\|^2 \). Clearly, \( \varphi_1 \leq g_1 \). By virtue of Lemma 2.2 and inequality (4.15), we obtain

\[ \varphi_n \leq g_1 \exp(bT), \quad n = 2, 3, \ldots, M. \]

Therefore, the desired result is attained.
5. Convergence of the fully discrete scheme. In this section, we analyze the convergence of the fully discrete scheme (3.1)-(3.4) by using the error estimate method and explore the order of convergence $O\left(\tau^2 + N^{1-\frac{1}{2}}\right)$.

**Theorem 5.1.** Assume that

(i) $s \in L^\infty((0,T]; H_{\Delta}^{-1}(\mathbb{R})) \cap L^2((0,T]; H_{\Delta}^{-2}(\mathbb{R}))$, $l \in L^\infty((0,T]; H_{\Delta}^{-2}(\mathbb{R}))$, $s_t \in L^2((0,T]; H_{\Delta}^{-1}(\mathbb{R}))$ and $l_t \in L^2((0,T]; H_{\Delta}^{-2}(\mathbb{R}))$ for $r \geq 2$;
(ii) $s_{tt} \in L^\infty((0,T]; H_{\Delta}^{-2}(\mathbb{R}))$, $l_{tt}$ and $s_{ttt} \in L^\infty((0,T]; L^2(\mathbb{R}))$, $l_{ttt}$, $s_{xttt}$, $s_{xttt}$ and $s_{tttt} \in L^2((0,T]; L^2(\mathbb{R}))$.

Then we have

$$
\| s^n - s_N^n \|_1 + \| l^n - l_N^n \| \leq C (\tau^2 + N^{1-\frac{1}{2}}), \quad n = 0, 1, 2, \ldots, M,
$$

where $C$ is independent of $N$ and $\tau$.

**Proof.** Let

$$
e^k = s_N^k - s^k = (s_N^k - P_N s^k) + (P_N s^k - s^k) \triangleq e_{1}^k + e_{2}^k,
$$

$$
\eta^k = l_N^k - l^k = (l_N^k - P_N l^k) + (P_N l^k - l^k) \triangleq \eta_{1}^k + \eta_{2}^k.
$$

Using (1.1)-(1.4) and (3.1)-(3.4), for any $v \in \mathcal{H}_N$ we have

$$
i \left( e_{1t}^k, v \right) - \left( e_{1x}^k, v_x \right) = \alpha \left( s_N^k - s^k, v \right) + i \left( \eta_{1}^k - s_{tt}^k, v \right),
$$

$$
\left( \eta_{1t}^k, v \right) + \beta \left( |s_N^k|^2 |v|^2 - |s^k|^2 |v|^2 \right) = \left( l_N^k - l^k, v \right), \quad k = 1, 2, \ldots, M - 1,
$$

$$
e_{1}^0 = \eta_{1}^0 = 0,
$$

$$
e_{1}^k = \int_0^\tau (t - \tau) P_N s_{tt} dt \text{ and } \eta_{1}^k = \int_0^\tau (t - \tau) P_N l_{tt} dt.
$$

It follows Lemma 2.1 and Hölder’s inequality that

$$
\| e_{1t}^k \|^2 \leq \frac{\tau^4}{3} \| s_{tt} \|_{L^\infty((0,T]; L^2(\mathbb{R}))}, \quad \| e_{1x}^k \|^2 \leq \frac{\tau^4}{3} \| s_{tt} \|_{L^\infty((0,T]; L^2(\mathbb{R}))}\ (5.5)
$$

and

$$
\| \eta_{1}^k \|^2 \leq \frac{\tau^4}{3} \| l_{tt} \|_{L^\infty((0,T]; L^2(\mathbb{R}))}.\ (5.6)
$$

To estimate $\| e_{1}^n \|$ and $\| \eta_{1}^n \|$ ($n = 2, 3, \ldots, M$), we take $v = e_{1x}^k$ in (5.1) and consider the imaginary part. It gives

$$
\frac{1}{2} \| e_{1x}^k \|^2 = \alpha \text{Im} \left( s_N^k l_N^k - s^k l^k, e_{1x}^k \right) + \text{Re} \left( s_{1x}^k - s_{tt}^k, e_{1x}^k \right) - \text{Im} \left( s_{x}^k - \left( P_N s^k \right)_x, e_{1x}^k \right).\ (5.7)
$$

In view of the right-hand side of (5.7), it follows from Hölder’s inequality that

$$
\text{Re} \left( s_{1x}^k - s_{tt}^k, e_{1x}^k \right) \leq \| s_{1x}^k - s_{tt}^k \| \| e_{1x}^k \|,
$$

$$
\alpha \text{Im} \left( s_N^k l_N^k - s^k l^k, e_{1x}^k \right) = \alpha \text{Im} \left( s_N^k (\eta_{1}^k + \eta_{2}^k) + l^k e_{1x}^k + l^k (s^k - s^k), e_{1x}^k \right)
$$

$$
\leq \alpha \left( \| s_N^k \| \| \eta_{1}^k \| + \| s_N^k \| \| \eta_{2}^k \| + \| l^k \| \| e_{1x}^k \| + \| l^k \| \| s^k \| \| s^k \| \right) \| e_{1x}^k \|,
$$

and
and
\[ \text{Im} \left( s_x^k - (P_N s_N^k)_x, e_{1x}^k \right) \leq \left( \| s_{xx}^k - s_{xx}^k \| + \| e_{2xx}^k \| \right) \| e_{1x}^k \|. \]

Let
\[ C_1 = \alpha E_0^2 E_{1s}^2 \quad \text{and} \quad C_2 = \max \left\{ \alpha \left( \| l \|_{L^\infty((0,T);H^1(\Omega))} + E_0^2 E_{1s}^2 \right), 1 \right\}. \]

Substituting the above three inequalities into (5.7), using Propositions 3.1 and 3.2 we deduce that
\[ \| e_1^k \| \leq C_1 \| \eta_1^k \| + C_2 \left( \| e_2^k \| + \| e_{2xx}^k \| + \| s^k - s^k \| + \| s_{xx}^k - s_{xx}^k \| + \| s_t^k - s_t^k \| \right). \]

Evaluating the sum on both sides for \( k \) from 1 to \( n - 1 \), we have
\[ \| e_1^n \| + \| e_1^{n-1} \| \leq 2C_1 \tau \sum_{k=1}^{n-1} \| \eta_1^k \| + \| e_1^k \|
+ 2C_2 \tau \sum_{k=1}^{n-1} \left( \| e_2^k \| + \| e_{2xx}^k \| + \| s^k - s^k \|_2 + \| s_t^k - s_t^k \| \right), \]

Using Hölder’s inequality leads to
\[ \| e_1^n \| + \| e_1^{n-1} \|^2 \leq 24 \left( C_1^2 T \tau \sum_{k=1}^{n-1} \| \eta_1^k \|^2 + C_2^2 T \tau \sum_{k=0}^{n-1} \| e_2^k \|^2 \right)
+ 24C_2^2 T \tau \sum_{k=1}^{n-1} \left( \| s^k - s^k \|_2^2 + \| s_t^k - s_t^k \|_2^2 \right) + 6 \| e_1^n \|^2. \]

To estimate \( \| \eta_1^k \| \), we let \( v = \eta_1^k \) in (5.2). It follows from Propositions 3.1 and 3.2 that
\[ \| \eta_1^k \| \leq \beta \left( \| s_N^k \|_x^2 + \| s_{xx}^k \|_x^2 \| + \| l_t^k - l_t^k \| \right)
\leq 2\beta \left( \| s_N^k \|_x^2 + \| s_{xx}^k \|_x^2 \| + \| l_t^k - l_t^k \| \right)
\leq C_3 \left( \| e_1^k \| + \| e_{1x}^k \| + \| e_{2x}^k \| + \| e_{2xx}^k \| \right) + \| l_t^k - l_t^k \|, \]
where
\[ C_3 = 2\beta \left( \| s \|_{L^\infty((0,T);H^1(\Omega))} + E_0^2 E_{1s}^2 \right). \]

Evaluating the sum on both sides of (5.9) for \( k \) from 1 to \( n - 1 \) leads to
\[ \| \eta_1^n \| + \| \eta_1^{n-1} \|
\leq 2C_3 \tau \sum_{k=1}^{n-1} \left( \| e_1^k \| + \| e_{1x}^k \| + \| e_{2x}^k \| + \| e_{2xx}^k \| \right) + 2\tau \sum_{k=1}^{n-1} \| l_t^k - l_t^k \| + \| \eta_1^k \|. \]

Using Hölder’s inequality again, we find
\[ \| \eta_1^n \|^2 \| + \| \eta_1^{n-1} \|^2 \|
\leq 24T \left[ C_3^2 \tau \sum_{k=1}^{n-1} \| e_1^k \|_1^2 + \tau \sum_{k=1}^{n-1} \left( C_3^2 \| e_2^k \|_1^2 + \| l_t^k - l_t^k \|_2^2 \right) \right] + 6 \| \eta_1^1 \|^2. \]

To estimate \( \| e_{1x}^k \| \), we let \( v = e_{1x}^k \) in (5.1) and consider the real part. It gives
\[ \frac{1}{2} \| e_{1x}^k \|^2 = -\alpha \text{Re}(s_N^k t_N^k - s_t^k t^k, e_{1x}^k) + \text{Im}(s_t^k - s_t^k, e_{1x}^k) + \text{Re}(s_t^k - (P_N s_N^k)_x, e_{1x}^k). \]
Considering the sum on both sides for \( k \) from 1 to \( n - 1 \) yields
\[
||e_{nx}^k||^2 + ||e_{nx}^{n-1}||^2 - ||e_{nx}^1||^2
\]
\[
= 4\tau \sum_{k=1}^{n-1} \text{Im} (s_{tx}^k - s_{tx}^{k-1}, e_1^k) + 4\tau \sum_{k=1}^{n-1} \text{Re} \left( s_x^k - (P_N s^k) x, e_{1tx}^k \right)
\]
\[
-4\alpha \tau \sum_{k=1}^{n-1} \text{Re} \left( s_{tx}^k l_N x - s_x^{k-1} l_{tx}^k, e_1^k \right)
\]
\[
\triangleq \text{Im} I_1 + \text{Re} I_2 - \alpha \text{Re} I_3.
\] (5.11)

In view of each term on the right-side hand of (5.11), according to Lemma 3.1 and (5.8), it follows from Hölder’s inequality and Young’s inequality that
\[
I_1 = -4\tau \sum_{k=1}^{n-2} (s_{tx}^k - s_{tx}^{k-1}, e_1^k) + 2(s_{n-1}^k - s_{n-2}^k, e_1^{n-1})
\]
\[
+2(s_{n-1}^k - s_{n-1}^k, e_1^k) - 2(s_1^2 - s_1^2, e_1^k)
\]
\[
\leq 2\tau \sum_{k=1}^{n-1} (||e_1^k||^2 + 12C^2T||\eta^k||^2) + 24C^2T\tau \sum_{k=0}^{n-1} (||e_2||^2 + 7||e_1||^2)
\]
\[
+24C^2T\tau \sum_{k=1}^{n-1} (||s^k - s^k||^2 + ||s^k - s^k||^2) + 2\tau \sum_{k=2}^{n-2} (||s_{tx}^k - s_{tx}^{k-1}||^2)
\]
\[
+||s_{n-2}^k - s_{n-2}^k||^2 + ||s_{n-1}^k - s_{n-1}^k||^2 + ||s_1^2 - s_1^2||^2,
\] (5.12)

and
\[
I_2 \leq 2\tau \sum_{k=2}^{n-2} (||e_{nx}^k||^2 + 1/2 (||e_{nx}^n||^2 + ||e_{nx}^{n-1}||^2) + 4\tau \sum_{k=2}^{n-2} (||s_{tx}^k - s_{tx}^{k-1}||^2 + ||e_{tx}^k||^2)
\]
\[
+1/2||e_{nx}^1||^2 + 4\left( ||s_{n-2}^k - s_{n-2}^{n-2}||^2 + ||s_{n-1}^k - s_{n-1}^{n-1}||^2 + ||s_1^2 - s_1^2||^2 \right)
\]
\[
+4\left( ||e_{2x}^k||^2 + ||e_{nx}^n||^2 + ||e_{2x}^k||^2 \right).
\] (5.13)

Rewrite \( \text{Re} I_3 \) as follows:
\[
\text{Re} I_3 = 4\text{Re} \sum_{k=1}^{n-1} \left( s_{n}^k (\eta_1^k + \eta_2^k) + l^k (e_1^k + e_2^k) + l^k (\hat{s}_1^k - s^k), e_1^k \right)
\]
\[
\triangleq I_{31} + I_{32} + I_{33}.
\]

We are now left to estimate \( I_{31}, I_{32} \) and \( I_{33} \), respectively. Using Lemma 3.2 and Hölder’s inequality, we deduce that
\[
I_{31} \leq 4\tau \sum_{k=2}^{n-2} \left| \left( \frac{s_{n}^k (\eta_1^k + \eta_2^k) + l^k (e_1^k + e_2^k) + l^k (\hat{s}_1^k - s^k), e_1^k} \right) \right|
\]
\[
+2\left( s_{n}^{n-1} (\eta_1^{n-1} + \eta_2^{n-1}) + s_{n}^{n-2} (\eta_1^{n-2} + \eta_2^{n-2}), e_1^{n} \right) + 2\left( \frac{s_{n}^{n-1} + s_{n}^{n-2} (\eta_1^{n-1} + \eta_2^{n-2}), e_1^{n} \right)
\]
\[
\leq 4\tau \sum_{k=2}^{n-2} \left( ||s_{n}^k||^2 + ||\eta_1^k||^2 + ||\eta_2^k||^2 + ||e_1^k||^2 + ||e_2^k||^2 + ||s_{n}^{k-1}||^2 + ||\eta_1^{k-1}||^2 + ||\eta_2^{k-1}||^2 \right)
\]
\[
+2 ||s_{n}^k||^2 (||\eta_1^k|| + ||\eta_2^k||^2) ||e_1^k|| + 2 ||s_{n}^{n-2}||^2 (||\eta_1^{n-2}|| + ||\eta_2^{n-2}||) ||e_1^{n-1}||
\]
For the estimate of $\|\eta_{k}^n\|$, the proof is similar to that of $\eta_{k}^n$. So we have

$$\|\eta_{k}^n\| \leq C_{3}(\|e_{1}^n\| + \|e_{2}^n\|) + \|t_{k}^n - \tilde{t}_{k}^n\|. \quad (5.15)$$

Substituting (5.15) into (5.14) and using the Sobolev inequality and Young’s inequality, we have

$$I_{31} \leq C_{4} \tau \sum_{k=2}^{n-2} (\|e_{k}^n\|^2 + \|\eta_{k}^{n-1}\|^2) + C_{4}(\|\eta_{n-2}^n\|^2 + \|\eta_{k}^{n-1}\|^2 + \|\eta_{2}^{n-1}\|^2)$$

$$+ C_{4}(\|e_{1}^1\|^2 + \|e_{2}^1\|^2) + C_{4} \tau \sum_{k=2}^{n-2} (\|e_{2}^n\|^2 + \|\eta_{2}^{n-1}\|^2 + \|\eta_{2}^{n-1}\|^2 + \|t_{k}^n - \tilde{t}_{k}^n\|^2)$$

$$+ C_{4}(\|e_{1}^1\|^2 + \|\eta_{2}^{n-2}\|^2 + \|\eta_{2}^{n-1}\|^2 + \|\eta_{2}^{n-1}\|^2), \quad (5.16)$$

where

$$C_{4} = 10E_{0_{s}}^{1}E_{1_{s}}^{1}C_{3} + 4E_{0_{s}}^{1}E_{1_{s}}^{1} + 2E_{0_{s}}^{0_{s}}.$$ 

For the estimate of $I_{32}$, it follows from Lemma 3.1, Hölder’s inequality and Young’s inequality that

$$I_{32} = 2\tau \sum_{k=2}^{n-2} (t_{k}^n, e_{1}^1) + (t_{k}^{n-2}, e_{1}^{n-1}) - (t_{k}^{n}, e_{1}^{n})$$

$$+ 2\left(\|e_{2}^n\|^2 + \|e_{2}^{n-1}\|^2 + \|e_{2}^{n-1}\|^2 + \|e_{2}^{n-1}\|^2\right)$$

$$+ 4\tau \sum_{k=2}^{n-2} (t_{k}^n, e_{2}^1) + t_{k}^{n-1}, e_{1}^{n})$$

$$\leq 2C_{5} \tau \sum_{k=2}^{n-2} (\|e_{1}^n\|^2 + \|e_{1}^{n-1}\|^2) + 2C_{5} \tau \sum_{k=2}^{n-2} (\|e_{2}^{n-1}\|^2 + \|e_{2}^{n-1}\|^2)$$

$$+ C_{5}(\|e_{1}^1\|^2 + \|e_{2}^1\|^2) + \|\eta_{2}^{n-2}\|^2 + \|\eta_{2}^{n-1}\|^2), \quad (5.17)$$

where

$$C_{5} = 2\|l\|_{L_{\infty}((0,T);L_{2}^{1}(R))} \|l_{\tau}\|_{L_{\infty}((0,T);L_{2}^{1}(R))} + \|l\|_{L_{\infty}((0,T);L_{2}^{1}(R))}.$$ 

For the term $I_{33}$, using Lemma 3.1, Hölder’s inequality and Young’s inequality, we derive that

$$I_{33} \leq 4\tau \sum_{k=2}^{n-2} (t_{k}^{n+1}(s_{k}^{1} - s_{k}^{1}) + t_{k}^{n}(s_{k}^{n-1} - s_{k}^{n-1}), e_{1}^1) + 2\|l_{2}^2 - s_{2}^2, e_{1}^1\|$$

$$+ 2\|l_{2}^1(s_{n}^{n-1} - s_{n}^{n-1}), e_{1}^1\| + 2\|l_{2}^1(s_{n}^{n-1} - s_{n}^{n-1}), e_{1}^1\|$$

$$\leq 2C_{5} \tau \sum_{k=2}^{n-2} \|e_{k}^1\|^2 + C_{5}(\|e_{1}^{n-1}\|^2 + \|e_{1}^1\|^2)$$

$$+ 2C_{5} \sum_{k=2}^{n-2} (\|s_{k}^{1} - s_{k}^{1}\|^2 + \|s_{k}^{n-1} - s_{k}^{n-1}\|^2)$$

$$+ C_{5}(\|s_{k}^{1}\|^2 + C_{5}(\|s_{k}^{n-1} - s_{k}^{n-1}\|^2 + \|s_{k}^{n-1} - s_{k}^{n-1}\|^2. \quad (5.18)$$

By using (5.8) and (5.10) and (5.16)-(5.18), we have
\[
I_3 \leq C_6 \tau \sum_{k=1}^{n-1} \left( \|e_1^k\|^2 + \|\eta_1^k\|^2 \right) + C_7 \tau \sum_{k=0}^{n} \|e_2^k\|^2 + 2C_5 \tau \sum_{k=2}^{n-2} \left( \|s_{1i}^k - s_{1i}^k\|^2 + \|e_{2i}^k\|^2 \right)
+ C_7 \tau \sum_{k=1}^{n-1} \left( \|\eta_2^k\|^2 + \|\eta_2^k\|^2 + \|\eta_1^k - t_1^k\|^2 + \|s_1^k - s_1^k\|^2 + \|s_1^k - s_1^k\|^2 \right)
+ C_5 \left( \|e_2^{n-1}\|^2 + \|e_2^0\|^2 + \|e_2^1\|^2 \right) + C_4 \left( \|\eta_2^{n-2}\|^2 + \|\eta_2^{n-1}\|^2 + \|\eta_2^2\|^2 \right)
+ C_5 \left( \|s_1^{n-2} - s_1^{n-1}\|^2 + \|s_1^{n-1} - s_1^2\|^2 + \|s_1^2 - s_1^2\|^2 \right)
+ (12C_4 + 14C_5) \left( \|e_1^1\|^2 + \|\eta_1^1\|^2 \right),
\]

(5.19)

where

\[C_6 = \max \left\{ C_4 + 2C_5 + 48C_3^2 C_4 T, C_4 + 24C_1^2 T(C_4 + 2C_5) \right\}\]

and

\[C_7 = C_4 + 2C_5 + 24C_1^2 T(C_4 + 2C_5) + 48C_4 T \max \{1, C_3^2 \} \]

Substituting (5.12), (5.13) and (5.19) into (5.11), we infer that

\[
\|e_{1x}\|^2 + \|e_{n-1}\|^2 \leq C_8 \tau \sum_{k=1}^{n-1} \left( \|e_{1i}^k\|^2 + \|\eta_1^k\|^2 \right) + C_9 \left( \|e_1^1\|^2 + \|\eta_1^1\|^2 \right) + 2(C_7 + 24C_1^2 T) \tau \sum_{k=0}^{n} \|e_2^k\|^2
+ (2C_5 + 4) \tau \sum_{k=2}^{n-2} \left( \|s_{1i}^k - s_{1i}^k\|^2 + \|s_1^k - s_1^k\|^2 + \|e_{2i}^k\|^2 \right)
+ C_10 \tau \sum_{k=1}^{n-1} \left( \|\eta_2^k\|^2 + \|\eta_2^k\|^2 + \|t_1^k - t_1^k\|^2 + \|s_1^k - s_1^k\|^2 + \|s_1^k - s_1^k\|^2 \right)
+ C_11 \left( \|e_2^{n-1}\|^2 + \|e_2^0\|^2 + \|e_2^1\|^2 \right) + C_4 \left( \|\eta_2^{n-2}\|^2 + \|\eta_2^{n-1}\|^2 + \|\eta_2^2\|^2 \right)
+ \|s_1^{n-2} - s_1^{n-1}\|^2 + \|s_1^{n-1} - s_1^2\|^2 + \|s_1^2 - s_1^2\|^2
+ C_11 \left( \|s_1^{n-2} - s_1^{n-1}\|^2 + \|s_1^{n-1} - s_1^2\|^2 + \|s_1^2 - s_1^2\|^2 \right),
\]

(5.20)

where

\[C_8 = 2 \max \{ C_6 + 2, C_6 + 12C_1^2 T \}, \quad C_9 = 2(12C_4 + 14C_5) + 7, \]
\[C_{10} = 2(C_7 + 24C_1^2 T), \quad C_{11} = 2 \max \{ C_5, 4 \} \]

Combining (5.8), (5.10) and (5.20) leads to

\[
\|e_{1i}\|^2 + \|\eta_{1i}\|^2 \leq C_{12} \tau \sum_{k=1}^{n-1} \left( \|e_{1i}^k\|^2 + \|\eta_1^k\|^2 \right) + C_{13} \left( \|e_1^1\|^2 + \|\eta_1^1\|^2 \right) + C_{14} \tau \sum_{k=0}^{n} \|e_2^k\|^2
+ C_{14} \tau \sum_{k=1}^{n-1} \left( \|\eta_2^k\|^2 + \|\eta_2^k\|^2 + \|t_1^k - t_1^k\|^2 + \|s_1^k - s_1^k\|^2 + \|s_1^k - s_1^k\|^2 \right)
+ C_{15} \tau \sum_{k=2}^{n-2} \left( \|s_{1i}^k - s_{1i}^k\|^2 + \|s_1^k - s_1^k\|^2 + \|e_{2i}^k\|^2 \right)
+ C_4 \left( \|\eta_2^{n-2}\|^2 + \|\eta_2^{n-1}\|^2 + \|\eta_2^2\|^2 \right) + C_4 \left( \|e_2^{n-1}\|^2 + \|e_2^0\|^2 + \|e_2^1\|^2 \right)
+ C_{11} \left( \|s_1^{n-2} - s_1^{n-1}\|^2 + \|s_1^{n-1} - s_1^2\|^2 + \|s_1^2 - s_1^2\|^2 \right)
+ \|s_1^{n-2} - s_1^{n-1}\|^2 + \|s_1^{n-1} - s_1^2\|^2 + \|s_1^2 - s_1^2\|^2,
\]

(5.21)
for $n = 2, 3, \cdots, M$, where
\[
C_{12} = C_8 + \max \{24C^2_7, 24TC^2_3\}, \quad C_{13} = C_9 + 6, \\
C_{14} = C_{10} + 24C^2_7 + 24T, \quad C_{15} = 2C_5 + 4.
\]

Consider all terms on the right hand of (5.21) except $C_{12} \tau \sum_{k=1}^{n-1} (\|e^k\|^2 + \|\eta^k\|^2)$ and the initial values. It follows from Lemma 2.1 and 3.2 that
\[
\tau \sum_{k=0}^{n} \|e^k\|^2_2 \leq cN^{2-r} \tau \sum_{k=0}^{n} \|\hat{s}^k\|^2_{r,A} \leq cN^{2-r} \int_{0}^{T} (\|s_t\|^2_{r,A} + \|s\|^2_{r,A}) dt.
\]

By an analogous argument, we can derive that
\[
\tau \sum_{k=1}^{n} \|e^k\|^2_{1,1} \leq cN^{2-r} \tau \sum_{k=1}^{n} \left[ \int_{t_{k-1}}^{t_{k+1}} \|s_t\|^2_{r-1,A} dt \leq cN^{2-r} \int_{0}^{T} \|s_t\|^2_{r-1,A} dt,
\]
\[
\tau \sum_{k=1}^{n} \|e^k\|^2_{2,1} \leq cN^{2-r} \int_{0}^{T} \|s_t\|^2_{r-1,A} dt,
\]
\[
\tau \sum_{k=1}^{n} \|\eta^k\|^2 \leq cN^{2-r} \int_{0}^{T} \|s_t\|^2_{r-1,A} dt,
\]
\[
\tau \sum_{k=1}^{n} \|\hat{e}^k\|^2 \leq cN^{2-r} \|s\|^2_{r-1,A} \leq cN^{2-r} \|s\|^2_{L^\infty(0,T;H^{-1}_A)} \quad k = 2, n - 1, n - 2
\]

and
\[
\|\hat{\eta}^k\|^2 \leq cN^{2-r} \|\hat{l}\|^2_{L^\infty(0,T;H^{-2}_A)} \quad k = 2, n - 1, n - 2.
\]

Applying Taylor’s expansion, we have
\[
s^k_{ti} - s^k_{ti} = \frac{1}{4T} \left( \int_{t_k}^{t_{k+1}} (t_{k+1} - t)^2 s_{tttt} dt + \int_{t_{k-1}}^{t_k} (t_{k-1} - t)^2 s_{tttt} dt \right)
\]+\frac{1}{24T^2} \left( \int_{t_k}^{t_{k+2}} (t_{k+2} - t)^3 s_{tttt} dt + \int_{t_{k-2}}^{t_k} (t - t_{k-2})^3 s_{tttt} dt \right).
\]

Using Hölder’s inequality yields
\[
\tau \sum_{k=2}^{n-2} \|s^k_{ti} - s^k_{ti}\|^2 \leq \tau^4 \sum_{k=2}^{n-2} \left( \frac{1}{20} \int_{t_{k-1}}^{t_{k+1}} \|s_{tttt}\|^2 dt + \frac{8}{63} \int_{t_{k-2}}^{t_{k+2}} \|s_{tttt}\|^2 dt \right)
\]
\[
\leq \frac{383\tau^4}{630} \int_{0}^{T} \|s_{tttt}\|^2 dt.
\]

Similarly, we can deduce that
\[
\tau \sum_{k=2}^{n-2} \|s^k_t - s^k_t\|^2_1 \leq \tau^4 \sum_{k=2}^{n-2} \left( \frac{1}{5} \int_{t_{k-1}}^{t_{k+1}} \|s_{tttt}\|^2 dt + \frac{1}{20} \int_{t_{k-2}}^{t_{k+2}} \|s^k_{tt}\|^2 dt \right)
\]
\[
\leq \frac{17\tau^4}{10} \int_{0}^{T} \|s_{tttt}\|^2 dt,
\]
\[
\tau \sum_{k=1}^{n-1} \|s^k_t - s^k_t\|^2 \leq \tau^4 \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_{k+1}} \|s_{tttt}\|^2 dt \leq \frac{\tau^4}{40} \int_{0}^{T} \|s_{tttt}\|^2 dt,
\]
Therefore, we have completed the proof of Theorem 5.1.

From (5.5) and (5.6), it is easy to see that

\[ \tau \sum_{k=1}^{n-1} \| l^k_t - l^k_s \|^2 \leq \frac{\tau^4}{40} \sum_{k=1}^{n-1} \int_{t_k-1}^{t_{k+1}} \| l_{ttt} \|^2 \, dt \leq \frac{\tau^4}{20} \int_0^T \| l_{ttt} \|^2 \, dt, \]

\[ \tau \sum_{k=1}^{n-1} \| s^k - s^k_s \|^2 \leq \frac{\tau^4}{6} \sum_{k=1}^{n-1} \int_{t_k-1}^{t_{k+1}} \| s_{tt} \|^2 \, dt \leq \frac{\tau^4}{3} \int_0^T \| s_{tt} \|^2 \, dt, \]

\[ \| s^k - s^k_s \|^2 \leq \frac{\tau^3}{40} \int_{t_k-1}^{t_{k+1}} \| s_{ttt} \|^2 \, dt \leq \frac{\tau^4}{20} \| s_{ttt} \|^2_{L_2((0,T]; L^2(\mathbb{R}))}, \quad k = 2, n - 2, n - 1 \]

and

\[ \| s^k - s^k_s \|^2 \leq \frac{\tau^4}{3} \| s_{ttt} \|^2_{L_2((0,T]; H^1(\mathbb{R}))}, \quad k = 2, n - 2, n - 1. \]

Substituting (5.5), (5.6) and all the above estimates to (5.21), we see that

\[ \| e^k_1 \|^2 + \| \eta^k_1 \|^2 \leq C_{16} \tau^4 + C_{17} N^{2-r} + C_{12} \tau \sum_{k=1}^{n-1} \left( \| e^k_1 \|^2 + \| \eta^k_1 \|^2 \right) \]

(5.22)

holds for \( n = 2, 3, \ldots, M \), where

\[ C_{16} = \left( \frac{1}{20} + \frac{C_{14}}{20} \right) \| s_{ttt} \|^2_{L_\infty((0,T]; L^2(\mathbb{R}))} + \left( \frac{C_{13}}{3} + C_{11} \right) \| s_{tt} \|^2_{L_\infty((0,T]; H^1(\mathbb{R}))} \]

\[ + \left( \frac{C_{14}}{3} + \frac{17C_{15}}{10} \right) \int_0^T (\| l_{ttt} \|^2 + \| s_{tt} \|^2 + \| s_{ttt} \|^2 + \| s_{tttt} \|^2) \, dt \]

\[ + \frac{C_{13}}{3} \| l_t \|^2_{L_\infty((0,T]; L^2(\mathbb{R}))} \]

and

\[ C_{17} = c(C_{11} + C_4) \left( \| s \|^2_{L_\infty((0,T]; H^{-1}(\mathbb{R}))} + \| l \|^2_{L_\infty((0,T]; H^{-2}(\mathbb{R}))} \right) \]

\[ + cC_{14} \int_0^T (\| s \|^2_{L^2} + \| s_t \|^2_{L^2} + \| l_t \|^2_{L^2 - 2, A}) \, dt. \]

Set

\[ g_1 = C_{18}(\tau^4 + N^{2-r}), \quad \varphi_n = \| e^k_1 \|^2 + \| \eta^k_1 \|^2, \quad h_j = C_{12}, \]

where

\[ C_{18} = \max\{C_{16}, C_{17}\}. \]

From (5.5) and (5.6), it is easy to see that \( \varphi_1 \leq g_1 \). By virtue of Lemma 2.2, we deduce that

\[ \varphi_n \leq C_{18}(\tau^4 + N^{2-r}) \exp(C_{12}T), \quad n = 2, 3, \ldots, M, \]

By the triangle inequality and Lemma 2.1, we further obtain

\[ \| s^n - s_N^n \| + \| l^n - l_N^n \| \leq \| e^n_2 \| + \| \eta^n_2 \| + \| e^n_1 \| + \| \eta^n_1 \| \leq C(N^{\frac{2-r}{2}} + \tau^2), \quad n = 0, 1, \ldots, M, \]

where

\[ C = \sqrt{2C_{18}} \exp(C_{12}T/2). \]

Therefore, we have completed the proof of Theorem 5.1.
6. Numerical results. In this section, we give an example to demonstrate numerical implementations and present numerical results for the fully discrete scheme (3.1)-(3.4).

We consider the LS wave equations (1.1)-(1.4) with $\alpha = \beta = 1$ and the following source terms:

$$ f(x, t) = -\frac{(\cosh^2(x + 2t) + 3)e^{i(t-x)}}{\cosh^2(x + 2t)} \quad \text{and} \quad g(x, t) = -\frac{6\sinh^2(x + 2t)}{\cosh^2(x + 2t)}. $$

Exact solutions of the LS wave equations (1.1)-(1.4) are:

$$ s(x, t) = \text{sech}(x + 2t)e^{i(t-x)} \quad \text{and} \quad l(x, t) = \text{sech}^2(x + 2t). $$

We choose Hermite functions as the basis functions, and then rewrite numerical solutions as

$$ s_{N+1}^{k+1}(x) = \sum_{m=0}^{N} s_{m}^{k+1} \hat{H}_{m}(x) \quad \text{and} \quad l_{N+1}^{k+1}(x) = \sum_{n=0}^{N} l_{n}^{k+1} \hat{H}_{n}(x), \quad (6.1) $$

where $s_{m}^{k+1}$ and $l_{n}^{k+1}$ are Hermite coefficients.

Using (6.1) and taking the inner product of (3.1) and (3.2) ($\alpha = \beta = 1$) with $\hat{H}_q(x)$ in $L^2$, respectively, we obtain

$$ i \sum_{m=0}^{N} s_{m}^{k+1}(\hat{H}_m(x), \hat{H}_q(x)) - \tau \sum_{m=0}^{N} s_{m}^{k+1}(\hat{H}_{mx}(x), \hat{H}_{qx}(x)) $$

$$ -\tau \sum_{m=0}^{N} s_{m}^{k+1}([l_{m}^{k}(x)\hat{H}_{n}(x), \hat{H}_{q}(x)) = J_{1q}, \quad (6.2) $$

and

$$ \sum_{n=0}^{N} l_{n}^{k+1}(\hat{H}_{n}(x), \hat{H}_{q}(x)) = J_{2q}, \quad (6.3) $$

for $q = 0, 1, \ldots, N$, where

$$ J_{1q} = i(s_{N}^{k}(x), \hat{H}_q(x)) + \tau(s_{N,x}^{k-1}, \hat{H}_{qx}(x)) $$

$$ +\tau(s_{N}^{k-1}(x)l_{N}^{k}(x), \hat{H}_{q}(x)) + 2\tau(f^{k}(x), \hat{H}_{q}(x)) $$

and

$$ J_{2q} = (l_{N}^{k-1}(x), \hat{H}_q(x)) - 2\tau(|s_{N}^{k}|^{2} x, \hat{H}_{q}(x)) + 2\tau(g^{k}(x), \hat{H}_{q}(x)). $$

Note that it is difficult to calculate the nonlinear terms and the source terms in a straightforward manner. Thus we use the Gauss integral formula to calculate the terms described above. In other words, we select Hermite-Gauss nodes $\{x_j\}_{j=0}^{N}$, let $\{w_{j}\}_{j=0}^{N}$ be the corresponding weights, and use

$$ \tau \sum_{j=0}^{N} \sum_{m=0}^{N} s_{m}^{k+1} l_{j}^{k} \hat{H}_{m}(x_{j}) \hat{H}_{q}(x_{j}) w_{j}, \quad \tau \sum_{j=0}^{N} (|s_{N}^{k}|^{2} x_{j}) \hat{H}_{q}(x_{j}) w_{j}, $$

$$ 2\tau \sum_{j=0}^{N} f^{k}(x_{j}) \hat{H}_{q}(x_{j}) w_{j} \quad \text{and} \quad 2\tau \sum_{j=0}^{N} g^{k}(x_{j}) \hat{H}_{q}(x_{j}) w_{j}, $$

instead of the nonlinear terms and the source terms appearing in (6.2) and (6.3). Then we obtain the following system of linear algebraic equations:

$$ (I - \tau A - \tau B)s^{k+1} = (I + \tau A + \tau B)s^{k-1} + 2\tau \phi^{k}, $$

where

$$ A = \sum_{j=0}^{N} \sum_{m=0}^{N} s_{m}^{k+1} l_{j}^{k} \hat{H}_{m}(x_{j}) \hat{H}_{q}(x_{j}) w_{j}, \quad B = \sum_{j=0}^{N} (|s_{N}^{k}|^{2} x_{j}) \hat{H}_{q}(x_{j}) w_{j}. $$

The final numerical results are presented in Table 1.
\[ II^{k+1} = I^{k-1} - 4\tau ReCs^k + 2\tau \psi^k, \]

where \( s^{k+1} = (s_0^{k+1}, s_1^{k+1}, \ldots, s_N^{k+1})^T \) and \( l^{k+1} = (l_0^{k+1}, l_1^{k+1}, \ldots, l_N^{k+1})^T \). According to (2.1) and (2.2), we know that \( I \) is an identity matrix and \( A \) is a pentadiagonal matrix defined as follows:

\[ A = \begin{pmatrix}
\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & \cdots & 0 \\
0 & \frac{3}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 & \cdots & 0 \\
-\frac{\sqrt{3}}{2} & 0 & \frac{5}{2} & 0 & -\sqrt{3} & \cdots & 0 \\
0 & -\frac{\sqrt{3}}{2} & 0 & \frac{7}{2} & 0 & \cdots & \vdots \\
\vdots & \cdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & \frac{2N-1}{2} & 0 & \cdots \\
0 & \cdots & \cdots & 0 & -\sqrt{N(N-1)} & 0 & \frac{2N+1}{2}
\end{pmatrix}. \]

It is also not difficult to see that \( B = DE_1D^T \) is a symmetrical matrix, and

\[ C = DE_2D^{(1)}^T, \quad \phi^k = DE_3 \quad \text{and} \quad \psi^k = DE_4, \]

where

\[
D = (d_{ij}) = (\bar{H}_i(x_j)), \quad D^{(1)} = (d_{ij}^{(1)}) = (\bar{H}_j(x_i)), \quad i, j = 0, 1, \ldots, N,
\]

\[
E_1 = \text{diag}(l_N^k(x_0)w_0, l_N^k(x_1)w_1, \ldots, l_N^k(x_N)w_N),
\]

\[
E_2 = \text{diag}(s_N^k(x_0)w_0, s_N^k(x_1)w_1, \ldots, s_N^k(x_N)w_N),
\]

\[
E_3 = (f^k(x_0)w_0, f^k(x_1)w_1, \ldots, f^k(x_N)w_N)^T,
\]

and

\[
E_4 = (g^k(x_0)w_0, g^k(x_1)w_1, \ldots, g^k(x_N)w_N)^T.
\]

The convergence rate of time in the \( L^2 \)-norm sense is defined as follows:

\[
\text{rate} = \frac{\log(||e(t_1, N)||)}{\log(t_1/t_2)}.
\]

The convergence rate in \( L^\infty \)-norm sense can be defined in a similar way. To see the order of accuracy, we present two tables of \( L^2 \)-error and \( L^\infty \)-error at \( t = 1 \) of the scheme (3.1)-(3.4) for the solutions of the problem (1.1)-(1.4) with different values of \( \tau \) and \( N \).

**Table 1.** Errors and convergence rates for the Short wave

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( L^2 )-error ( \tau = 1/10 )</th>
<th>( \text{rate} )</th>
<th>( L^\infty )-error ( \tau = 1/10 )</th>
<th>( \text{rate} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/100</td>
<td>1.9011e-05</td>
<td>*</td>
<td>7.605e-06</td>
<td>2.1346</td>
</tr>
<tr>
<td>1/500</td>
<td>1.8917e-05</td>
<td>2.0033</td>
<td>7.605e-06</td>
<td>2.0049</td>
</tr>
<tr>
<td>1/1000</td>
<td>4.7371e-04</td>
<td>*</td>
<td>1.9027e-04</td>
<td>2.1346</td>
</tr>
<tr>
<td>1/5000</td>
<td>1.8917e-05</td>
<td>2.0033</td>
<td>7.605e-06</td>
<td>2.0049</td>
</tr>
<tr>
<td>1/10000</td>
<td>4.7371e-04</td>
<td>*</td>
<td>1.9027e-04</td>
<td>2.1346</td>
</tr>
</tbody>
</table>

Table 1 and Table 2 show the errors with different values of \( \tau \) for the given \( N \). One can see that both \( L^2 \)-error and \( L^\infty \)-error indicate a second-order accuracy in time for \( N = 128 \).

From Fig.1 and Fig.2, we can see that both \( L^2 \)-error and \( L^\infty \)-error decay exponentially which is the so-called exponential convergence.
Table 2. Errors and convergence rates for the Long wave

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$L^2$-error</th>
<th>rate</th>
<th>$L^\infty$-error</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/10</td>
<td>3.7789e-02</td>
<td>*</td>
<td>1.5177e-02</td>
<td>*</td>
</tr>
<tr>
<td>1/50</td>
<td>1.1696e-03</td>
<td>2.1596</td>
<td>4.7509e-04</td>
<td>2.1532</td>
</tr>
<tr>
<td>1/100</td>
<td>2.8829e-04</td>
<td>2.0204</td>
<td>1.1614e-04</td>
<td>2.0323</td>
</tr>
<tr>
<td>1/500</td>
<td>1.1430e-05</td>
<td>2.0055</td>
<td>4.5963e-06</td>
<td>2.0066</td>
</tr>
<tr>
<td>1/1000</td>
<td>2.8605e-06</td>
<td>1.9985</td>
<td>1.1781e-06</td>
<td>1.9640</td>
</tr>
</tbody>
</table>

Fig. 1. Errors for the Short wave

Fig. 2. Errors for the Long wave

Table 3 illustrates the errors with different values of $N$ for the given $\tau$. We see that the accuracy reaches e-06 at $N = 128$ when $\tau = 10^{-3}$, while it does not achieve the same accuracy when $N$ is no more than 64. This is because that $CN^{1-\frac{1}{2}}$ in the convergence analysis result $C (\tau^2 + N^{1-\frac{1}{2}})$ plays the critical role. We have to point out that the coefficient $C$ which includes $\| \cdot \|_{r,A}$ really affects the accuracy, as we know that the norm $\| \cdot \|_{r,A}$ becomes larger as $r$ increases. We take $N = 64$, for example: if $N^{1-\frac{1}{2}}$ has the same accuracy as e-06, we need $r > 8$ while
Table 3. Errors for both the Short wave and Long wave

<table>
<thead>
<tr>
<th>N</th>
<th>$L^2$-error</th>
<th>$L^\infty$-error</th>
<th>$L^2$-error</th>
<th>$L^\infty$-error</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>7.0116e-03</td>
<td>3.8134e-03</td>
<td>1.3843e-02</td>
<td>1.796e-03</td>
</tr>
<tr>
<td>32</td>
<td>1.2206e-03</td>
<td>1.0903e-03</td>
<td>1.1102e-03</td>
<td>5.4417e-04</td>
</tr>
<tr>
<td>64</td>
<td>5.3158e-05</td>
<td>4.8213e-05</td>
<td>3.2689e-05</td>
<td>1.4387e-05</td>
</tr>
<tr>
<td>128</td>
<td>1.2742e-06</td>
<td>4.7993e-07</td>
<td>7.3697e-07</td>
<td>3.2600e-07</td>
</tr>
</tbody>
</table>

$\|s\|_{s,A} = 1.5314e + 05$. Then $\|s\|_{s,A}N^{-3}$ has the accuracy e-01 which is far larger than the accuracy e-05 in Table 3. Hence, as we have observed, the accuracy in Table 3 agrees well with our theoretical analysis.

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