LYAPUNOV TYPE INEQUALITIES FOR HAMMERSTEIN INTEGRAL EQUATIONS AND APPLICATIONS TO POPULATION DYNAMICS

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(Communicated by Michael Winkler)

Abstract. Lyapunov type inequalities for (linear or nonlinear) Hammerstein integral equations are established and applied to second order differential equations (ODEs) with general separated boundary conditions. These new inequalities provide necessary conditions for the Hammerstein integral equations and these boundary value problems to have nonzero nonnegative solutions. As applications of these inequalities for nonlinear ODEs, we obtain extinction criteria and optimal locations of favorable habitats for populations inhabiting one dimensional heterogeneous environments governed by reaction-diffusion equations with spatially varying growth rates and external forcing.

1. Introduction. The classical Lyapunov inequality is a definite integral inequality:

$$\int_0^1 \omega(s) \, ds > 4$$

of a continuous function $\omega$ defined on $[0,1]$, which is a necessary condition for a linear second order differential equation of the form

$$-z''(x) = m(x)z(x) \quad \text{for } x \in [0,1]$$

subject to the Dirichlet boundary condition (BC): $z(0) = z(1) = 0$ to have a solution satisfying $z(x) > 0$ for $x \in (0,1)$, see [3, p.68, (4)] or [16, Corollary 5.1, p.346]. It is known that Lyapunov inequality can be replaced by a better inequality

$$\int_0^1 m_+(s) \, ds > 4$$

or by Hartman type inequality

$$\int_0^1 s(1-s)m_+(s) \, ds > 1$$

where $m_+(s) = \{m(s), 0\}$ is the positive part of $m$.

The Lyapunov inequalities have been generalized to other linear boundary value problems (BVPs) such as multi-point BVPs, p-Laplacian BVPs and fractional BVPs ([2, 4, 5, 9, 13, 14, 15, 18, 19], [34–43]) and partial differential equations and monotone quasilinear operators [10, 11] and have applications in the study of qualitative

2010 Mathematics Subject Classification. Primary: 47H30; Secondary: 35P30, 45A05, 47H10, 92D25.

Key words and phrases. Hammerstein integral equations, Lyapunov type inequalities, population models, extinction criteria, Allee effects, optimal favorable habitats.

The first author was supported in part by the Natural Sciences and Engineering Research Council (NSERC) of Canada under grant no. 250187-2013 and 135752-2018, and the Shanghai Key Laboratory of Contemporary Applied Mathematics, and the second author was supported in part by the NNSF of China under grants no. 11322111 and no. 61773125.

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properties of solutions of differential equations such as oscillation, disconjugacy and eigenvalue problems [3]. However, some differential equations often arising in population dynamics are nonlinear and population extinction occurs. Hence, seeking sufficient conditions for these nonlinear equations to have no nonzero nonnegative solutions representing population extinction is of importance in ecology.

In this paper, we develop Lyapunov type inequalities for nonlinear integral and differential equations and provide the new connection between the Lyapunov type inequalities and population extinction in ecology. We establish four inequalities including Lyapunov and Hartman type inequalities for (linear or nonlinear) Hammerstein integral equations of the form

\[ z(x) = \int_0^1 k(x, s)f(s, z(s))\,ds \quad \text{for } x \in [0, 1]. \]  

These inequalities employ the spectral radius \( r(L) \) of a linear Hammerstein integral operator \( L \), the norm of \( L \), an integral inequality involving a positive upper bound \( g \) of \( f \), or the \( L^p \)-norm of \( g \) while Lyapunov and Hartman type inequalities for linear BVPs only employ integral inequalities and \( L^1 \)-norms.

We apply these inequalities to obtain corresponding inequalities for the boundary value problems (BVPs): \(-z''(x) = f(x, z(x))\) with general separated BCs including the Dirichlet BC. These new inequalities provide necessary conditions for the Hammerstein integral equations and the BVPs to have nonzero nonnegative solutions, and generalize the Lyapunov type inequalities mentioned above even when the BC is the Dirichlet BC.

As applications of the new inequalities for the nonlinear BVPs, we study extinction for a population inhabiting one dimensional heterogeneous environments governed by a reaction-diffusion equation with spatially varying growth rate and external forcing of the form

\[ z_t(x, t) - d z_{xx}(x, t) = z(x, t)G(x, z(x, t)) - Y(x, z(x, t)) \quad \text{on } [0, 1] \times \mathbb{R}_+ \]  

subject to the general separated BCs, where \( d > 0 \) represents the rate of diffusion, \( z(x, t) \) denotes the density of the population of one species at location \( x \) and at time \( t \), the density dependent term \( G(x, z(x, t)) \) denotes the local rate of change (or growth rate function) in the population density, and the term \( Y(x, z(x, t)) \) represents external forcing or external perturbation such as a harvesting rate or functional response. We derive the extinction criteria for the population by applying the newly established Lyapunov type inequalities to the steady-state solution equations of (1.3), which are nonlinear second order differential equations with changing-sign nonlinearities. These new extinction criteria use the spectral radius \( r(L) \) of a linear Hammerstein integral operator \( L \) arising from the steady-state solution equations, the norm of \( L \); an integral inequality involving a positive upper bound \( g \) of \( G \), and the \( L^p \)-norms of \( g \), as the measures of the unsuitability of an environment for a given population.

As illustrations of the extinction criteria, we consider the populations with four important growth rate functions including strong Allee effects, piecewise constants and non-constant functions and determine the optimal habitats for the populations.

In section 2 of this paper, we establish the Lyapunov type inequalities for Hammerstein integral equations. In section 3, we apply the results obtained in section 2 to derive the Lyapunov type inequalities for second order nonlinear or linear differential equations with general separated BCs. In section 4 we establish extinction
criteria for the populations and study the optimal locations of favorable habitats for populations inhabiting one dimensional heterogeneous environments.

2. Lyapunov type inequalities for Hammerstein integral equations. In this section, we study Lyapunov type inequality for Hammerstein integral equations of the form

\[ z(x) = \int_0^1 k(x, s)f(s, z(s)) \, ds := Az(x) \quad \text{for } x \in [0, 1]. \tag{2.1} \]

These inequalities are necessary conditions for the fixed point equation (2.1) to have nonzero nonnegative solutions.

We denote by \( C[0, 1] \) the Banach space of all continuous functions from \([0, 1]\) to \( \mathbb{R} \) with the maximum norm \( \| \cdot \| \), and by \( P \) the standard positive cone in \( C[0, 1] \), that is,

\[ P = \{ z \in C[0, 1] : z(x) \geq 0 \quad \text{for } x \in [0, 1] \}. \tag{2.2} \]

Recall that a function \( z : [0, 1] \to \mathbb{R} \) is said to be a nonnegative solution of (2.1) if \( z \) satisfies (2.1) and \( z \in P \). Hence, (2.1) has a nonzero nonnegative solution if and only of (2.1) has a solution in \( P \setminus \{ 0 \} \).

We always assume that \( p, q \in [1, \infty) \) is such that \( 1/p + 1/q = 1 \), where if \( p = 1 \), then \( q = \infty \), and if \( p = \infty \), then \( q = 1 \). We denote by \( L^p[0, 1] \) and \( L^q[0, 1] \) the Banach space of functions for which the \( p \)-th power of the absolute values are Lebesgue integrable, and its positive cone, respectively.

We list the following conditions.

\( (C_1) \) \quad \( k : [0, 1] \times [0, 1] \to \mathbb{R}_+ \) satisfies \( k(x, \cdot) : [0, 1] \to \mathbb{R}_+ \) is measurable for each \( x \in [0, 1] \), and \( k(\cdot, s) : [0, 1] \to \mathbb{R}_+ \) is continuous for almost every (a.e.) \( s \in [0, 1] \).

We denote by \( W \) the set of measurable functions \( g : [0, 1] \to \mathbb{R}_+ \) satisfying the following conditions:

\( (a) \) \quad \int_0^1 k(x, s)g(s) \, ds < \infty \text{ for } x \in [0, 1]. \)

\( (b) \) \quad For each \( \tau \in [0, 1] \), \( \lim_{x \to \tau} \int_0^1 [k(x, s) - k(\tau, s)]g(s) \, ds = 0. \)

\( (C_2) \) \quad \( f : [0, 1] \times \mathbb{R}_+ \to \mathbb{R} \) satisfies Carathéodory conditions, that is, \( f(\cdot, z) \) is measurable for \( z \in \mathbb{R}_+ \) and \( f(x, \cdot) \) is continuous for a.e. \( x \in [0, 1] \), and there exists a measurable function \( g : [0, 1] \to \mathbb{R}_+ \) and for each \( r > 0 \), there exists \( g_r \in W \) such that

\[ -g_r(x) \leq f(x, z) \leq g(x)z \quad \text{for a.e. } x \in [0, 1] \text{ and all } z \in [0, r]. \tag{2.3} \]

\( (C_3) \) \quad There exists a measurable function \( \Phi : [0, 1] \to \mathbb{R}_+ \) such that

\[ k(x, s) < \Phi(s) \quad \text{for } x \in [0, 1] \text{ and a.e. } s \in [0, 1]. \]

\( (C_3') \) \quad There exists a measurable function \( \Phi : [0, 1] \to \mathbb{R}_+ \) such that

\[ k(x, s) \leq \Phi(s) \quad \text{for } x \in [0, 1] \text{ and a.e. } s \in [0, 1]. \]

We remark that the first inequality of (2.3) depends on \( r \) but the second one is independent of \( r \), so it holds for \( z \in \mathbb{R}_+ \). With the measurable function \( g \) given in 

\( (C_2) \), we define a linear Hammerstein integral operator by

\[ Lz(x) = \int_0^1 k(x, s)g(s)z(s) \, ds \quad \text{for } x \in [0, 1]. \tag{2.4} \]

By Proposition 3.1 in [25, p.164] (or [31, Lemma 2.3]), we have the following result.

**Lemma 2.1.** If \( g \in W \), then the linear operator \( L \) defined in (2.4) maps \( C[0, 1] \) to \( C[0, 1] \) and is continuous such that \( L(P) \subset P \).
By Lemma 2.1, since \( L : C[0,1] \rightarrow C[0,1] \) is continuous, it follows that
\[
\|L\| = \max_{x \in [0,1]} \int_0^1 k(x,s)g(s) \, ds. \tag{2.5}
\]
The values \( \|L\| \) were widely used and estimated in the study of existence of nonzero nonnegative solutions for Hammerstein integral equations and BVPs, for example, see [20, 22, 23, 31].

Recall that the radius of the spectrum of \( L \) in \( C[0,1] \), denoted by \( r(L) \), is given by the well-known spectral radius formula
\[
r(L) = \lim_{m \to \infty} \sqrt[m]{\|L^m\|}. \tag{2.6}
\]
Now, we provide necessary conditions for (2.1) to have a nonzero nonnegative solution in \( C[0,1] \).

**Theorem 2.1.** Assume that \((C_1)\) and \((C_2)\) hold and (2.1) has a solution in \( P \setminus \{0\} \). Then the following assertions hold:

(i) (Spectral radius type inequality) If \( g \in W \), then \( r(L) \geq 1 \).

(ii) (Norm type inequality) If \( g \in W \), then \( \|L\| \geq 1 \).

(iii) (Hartman type inequality) Assume that \( g \Phi \in L^1[0,1] \). If \((C_3)\) holds, then
\[
\int_0^1 \Phi(s)g(s) \, ds > 1.
\]
If \((C_3)'\) holds, then
\[
\int_0^1 \Phi(s)g(s) \, ds \geq 1.
\]

(iv) (Lyapunov type inequality) Assume that \( \Phi \in L^1_+[0,1] \) and \( g \in L^p_+[0,1] \). If \((C_3)\) holds, then
\[
\|g\|_{L^p} > 1/\|\Phi\|_{L^p}.
\]
If \((C_3)'\) holds, then
\[
\|g\|_{L^p} \geq 1/\|\Phi\|_{L^p}.
\]

**Proof.** Note that under \((C_1)\), either \( g \Phi \in L^1[0,1] \) in the condition \((iii)\) or \( g \in L^p_+[0,1] \) in \((iv)\) implies \( g \in W \). By Lemma 2.1 and \((C_2)\), the operator \( A \) defined in (2.1) maps \( P \) into \( C[0,1] \). Assume that \( z \in P \setminus \{0\} \) is such that \( z = Az \). By (2.3), we have for \( x \in [0,1] \),
\[
z(x) = \int_0^1 \frac{1}{k(x,s)}f(s,z(s)) \, ds \leq \int_0^1 k(x,s)g(s)z(s) \, ds = Lz(x). \tag{2.7}
\]
Since \( z \in P \setminus \{0\} \), it follows from (2.7) that
\[
g \neq 0. \tag{2.8}
\]
(i) By (2.7), we have \( z \leq Lz \). Since \( L(P) \subset P \), we have \( z \leq Lz \leq L^2z \). Repeating the process implies \( z \leq L^mz \) for \( m \in \mathbb{N} \). Hence,
\[
\|z\| \leq \|L^mz\| \leq \|L^m\|\|z\| \quad \text{for } m \in \mathbb{N}
\]
and \( 1 \leq \|L^m\| \). Hence, \( 1 \leq \sqrt[m]{\|L^m\|} \) and taking limit implies \( r(L) \geq 1 \).

(ii) Since \( r(L) \leq \|L\| \), the result (ii) follows from the result (i).

(iii) For the first part of the result, we first prove that the following strict inequality holds:
\[
\|L\| < \int_0^1 \Phi(s)g(s) \, ds. \tag{2.9}
\]
In fact, if not, then \( \|L\| = \int_0^1 \Phi(s)g(s) \, ds \). By (2.5),

\[
\|L\| = \max_{x \in [0,1]} \int_0^1 k(x, s) g(s) \, ds.
\]

Hence, there exists \( x_0 \in [0,1] \) such that

\[
\int_0^1 k(x_0, s) g(s) \, ds = \int_0^1 \Phi(s) g(s) \, ds
\]

and \( \int_0^1 [\Phi(s) - k(x_0, s)] g(s) \, ds = 0 \). This implies that

\[
[\Phi(s) - k(x_0, s)] g(s) = 0 \quad \text{a.e. on } [0,1]. \tag{2.10}
\]

By (C3), \( \Phi(s) - k(x_0, s) > 0 \) for a.e. \( s \in [0,1] \). This, together with (2.10), implies \( g(s) = 0 \) a.e. on \( [0,1] \), which contradicts (2.8). The first part of the result follows from (2.8), (2.9) and the result (ii). The second part of the result follows from (C3'), (2.5) and the result (ii).

(iv) Since \( \Phi \in L^1_1[0,1] \) and \( g \in L^p[0,1] \), we have \( g \Phi \in L^1[0,1] \). By Hölder inequality, we have

\[
\int_0^1 \Phi(s) g(s) \, ds \leq \|\Phi\|_{L^p} \|g\|_{L^p}.
\]

This, together with the result (iii), implies (iv). \( \square \)

3. Lyapunov type inequalities for differential equations. In this section, we apply the results obtained in section 2 to derive Lyapunov type inequalities for a second order differential equation of the form

\[
-z''(x) = f(x, z(x)) \quad \text{a.e. on } [0,1] \tag{3.1}
\]

subject to the following BCs:

\((B_1)\): \( z(0) - \beta z'(0) = 0 \) and \( z(1) + \delta z'(1) = 0 \) with \( \beta, \delta \geq 0 \).

\((B_2)\): \( z(0) - \beta z'(0) = 0 \) and \( z'(1) = 0 \) with \( \beta \geq 0 \).

If \( \beta = \delta = 0 \), \((B_1)\) is the Dirichlet BC and if \( \beta, \delta > 0 \), \((B_1)\) is the BC \((B_3)\) in [20, page 697]. When \( \beta = \delta > 0 \), \((B_1)\) is the mixed or Robin BC considered in [7, page 335] and [24]. We note that the above BCs, together with \((B_2)'\): \( z'(0) = 0 \) and \( z(1) + \delta z'(1) = 0 \) with \( \delta \geq 0 \), are equivalent to the well-known general separated BCs (see for example, the BC (3.2) in [20] or the BC (5.2) in [31]). Also, it is easy to verify that if \( z \) is a solution of Eq. (3.1) with \((B_2)\) if and only if \( z \) is a solution of

\[
-z''(x) = g(x, z(x)) \quad \text{a.e. on } [0,1],
\]

with \( z'(0) = 0 \) and \( z(1) + \beta z'(1) = 0 \) with \( \beta \geq 0 \), where \( z(x) = z(1-x) \) for \( x \in [0,1] \) and \( g(x, z) = f(1-x, z) \) for \( (x, z) \in [0,1] \times \mathbb{R}_+ \). Hence, (4.4) with \((B_2)\) is equivalent to (4.4) with \((B_2)'\). When \( \beta = \delta = 0 \), the result was mentioned in [22, Remark 3.1].

As mentioned in the Introduction, the results obtained in section 2 can be applied to study Lyapunov type inequalities for other boundary value problems.

We always assume \( i \in \{1,2\} \). Let \( k_i : [0,1] \times [0,1] \to \mathbb{R}_+ \) be the Green’s function for the equation \(-z'' = 0\) subject to \((B_i)\). Then it is well known that

\[
k_1(x, s) = \frac{1}{1 + \beta + \delta} \begin{cases} (1 + \delta - x)(\beta + s), & \text{if } s \leq x, \\ (\beta + x)(1 + \delta - s), & \text{if } x < s; \end{cases} \tag{3.2}
\]

\[
k_2(x, s) = \begin{cases} \beta + s, & \text{if } s \leq x, \\ \beta + x, & \text{if } x < s. \end{cases} \tag{3.3}
\]
Let $\Phi_i(s) = k_i(s, s)$ for $s \in [0, 1]$. Then we have for $s \in [0, 1]$,

$$
\Phi_1(s) = \frac{(\beta + s)(1 + \delta - s)}{1 + \beta + \delta} \quad \text{and} \quad \Phi_2(s) = \beta + s. \tag{3.4}
$$

We always assume that the following condition holds.

\[ \Phi : [0, 1] \times \mathbb{R}_+ \to \mathbb{R} \text{ satisfies Carathéodory conditions and there exists a measurable function } g : [0, 1] \to \mathbb{R}_+ \text{ and for each } r > 0, \text{ there exists a measurable function } g_r : [0, 1] \to \mathbb{R}_+ \text{ such that } g_r \Phi_i \in L^1[0, 1] \text{ and (2.3) holds.} \]

Recall that a function $z : [0, 1] \to \mathbb{R}$ is said to be a nonnegative solution of (3.1)-(B_i) if $z' \in AC[0, 1]$, $z$ satisfies (3.1)-(B_i) and $z \in P$, where $AC[0, 1]$ denotes the set of all absolute continuous functions defined on $[0, 1]$. It is well known (see [31]) that $z : [0, 1] \to \mathbb{R}$ is a nonnegative solution of (3.1)-(B_i) if and only if $z$ is a nonnegative solution of the following Hammerstein integral equation

$$
z(x) = \int_0^1 k_i(x, s)f(s, z(s)) \, ds \quad \text{for } x \in [0, 1]. \tag{3.5}
$$

With the function $g$ given in the condition $(H)$, we define the linear Hammerstein integral operator

$$
L_i z(x) = \int_0^1 k_i(x, s)g(s)z(s) \, ds \quad \text{for } x \in [0, 1]. \tag{3.6}
$$

By applying Theorem 2.1, we obtain the following results.

**Theorem 3.1.** Assume that (3.1)-(B_i) has a solution in $P \setminus \{0\}$. Then the following assertions hold.

(i) If $g \in W$ with $k = k_i$, then $r(L_i) \geq 1$.

(ii) If $g \in W$ with $k = k_i$, then $\|L_i\| \geq 1$.

(iii) If $g \Phi_1 \in L^1[0, 1]$, then $\int_0^1 \Phi_1(s)g(s) \, ds \geq 1$ and if $g \Phi_2 \in L^1[0, 1]$, then $\int_0^1 \Phi_2(s)g(s) \, ds \geq 1$.

(iv) If $g \in L^r_0[0, 1]$, then $\|g\|_{L^r} > 1/\|\Phi_1\|_{L^r}$ and $\|g\|_{L^r} \geq 1/\|\Phi_2\|_{L^r}$.

**Proof.** Since $k_i$ is continuous, $(C_1)$ with $k = k_i$ holds. It is easy to verify $k_1$ and $\Phi_1$ satisfy $(C_2)$ and $k_2$ and $\Phi_2$ satisfy $(C_3)'$. It follows that $g_r \Phi_i \in W$ and $(C_2)$ holds. The results follow from Theorem 2.1.

**Remark 3.1.** For Theorem 3.1 (i), we refer to [31] for the values and estimations of $r(L_i)$ when $g$ is a constant function. We point out that it is not easy to estimate the values $r(L_i)$ if $g$ is not a constant function. But it is relatively easy to verify the conditions (ii), (iii) and (iv) in applications.

By Theorem 3.1 (iv), we see that the Lyapunov type inequality depends on the norm $\|\Phi_i\|_{L^r}$. The following proposition gives the formula of $\|\Phi_i\|_{L^r}$ which is useful when one uses the Lyapunov type inequality.

**Proposition 3.1.** (1)

$$
\|\Phi_1\| = \begin{cases} 
\frac{\beta(1+\delta)}{1+\beta+\delta} & \text{if } \delta - \beta < -1, \\
\frac{1+\delta}{1+\beta+\delta} & \text{if } |\delta - \beta| \leq 1, \quad \text{and } \|\Phi_2\| = 1 + \beta, \\
\frac{\beta(1+\delta)}{1+\beta+\delta} & \text{if } \delta - \beta > 1
\end{cases}
$$

(2) If $\beta = \delta = 0$, then $\|\Phi_1\|_{L^r} = [B(q + 1, q + 1)]^{1/q}$, where

$$
B(x, y) = \int_0^1 s^{x-1}(1-s)^{y-1} \, ds \quad \text{is the beta-function.}
$$
Proof. We only sketch the formula for \( \|\Phi_1\| \). The proofs for other formulas are straightforward and we omit them. Let \( s_0 = (1 + \delta - \beta)/2 \). Then
\[
\Phi_1'(s) = \frac{2(s_0 - s)}{1 + \beta + \delta} \quad \text{for } s \in [0, 1].
\]

If \( \delta - \beta < -1 \), then \( \Phi_1'(s) < 0 \) for \( s \in [0, 1] \) and \( \Phi_1(s) \leq \Phi_1(0) \) for \( s \in [0, 1] \); if \( |\delta - \beta| \leq 1 \), then \( s_0 \in [0, 1] \) and \( \Phi_1(s) \leq \Phi(s_0) \) for \( s \in [0, 1] \); and if \( \delta - \beta > 1 \), then \( \Phi_1'(s) > 0 \) for \( s \in [0, 1] \) and \( \Phi_1(s) \leq \Phi_1(1) \) for \( s \in [0, 1] \). The formula for \( \|\Phi_1\| \) follows.

As a special case of Theorem 3.1, we consider the following linear second order differential equation
\[
-z''(x) = m(x)z(x) \quad \text{a.e. on } [0, 1]
\]
subject to the BC \((B_1)\), where \( m : [0, 1] \to \mathbb{R} \) is a measurable function. Let
\[
g(x) = m_+(x) := \max\{m(x), 0\} \quad \text{for a.e. } x \in [0, 1].
\]

**Theorem 3.2.** Let \( L_i \) be defined by (3.6) with \( g \) in (3.8). Assume that (3.7)-(\( B_1 \)) has a solution in \( P \setminus \{0\} \). Then the following assertions hold:

(i) If \( m\Phi_1 \in L^1[0, 1] \), then \( r(L_i) \geq 1 \).

(ii) If \( m\Phi_1 \in L^1[0, 1] \), then \( \|L_i\| \geq 1 \).

(iii) If \( m\Phi_1 \in L^1[0, 1] \), then \( \int_0^1 \Phi_1(s)m_+(s)\,ds > 1 \) and if \( m\Phi_2 \in L^1[0, 1] \), then \( \int_0^1 \Phi_2(s)m_+(s)\,ds \geq 1 \).

(iv) If \( m \in L^p[0, 1] \), then \( \|m_+\|_{L^p} > 1/\|\Phi_1\|_{L^q} \) and \( \|m_+\|_{L^p} \geq 1/\|\Phi_2\|_{L^q} \).

**Proof.** We define a function \( f : [0, 1] \times \mathbb{R}_+ \to \mathbb{R} \) by
\[
f(x, z) = m_+(x)z \quad \text{for } (x, z) \in [0, 1] \times \mathbb{R}_+.
\]

Then for \( r > 0 \), we have
\[
-m(x)r \leq -m(x)z \leq f(x, z) \leq m_+(x)z \quad \text{for } (x, z) \in [0, 1] \times [0, r].
\]

Hence, (2.3) with \( g(x) = m_+(x) \) and \( g_r(x) = |m(x)|r \) holds. The result follows from Theorem 3.1.

**Remark 3.2.** Theorem 3.2 (iii) with the Dirichlet BC becomes the Hartman inequality ([16, Theorem 5.1, page 345]). By Proposition 3.1 (1) with \( \beta = \delta = 0 \) and Theorem 3.2 (iv) with \( p = 1 \) and the Dirichlet BC, we obtain the well-known Lyapunov inequality ([16, Corollary 5.1, page 346]): \( \int_0^1 m_+(x)\,dx > 4 \). Hence, Theorem 3.2 generalizes the Lyapunov and Hartman inequalities.

4. Extinction criteria for populations. In this section, as applications of our results obtained in section 3, we derive extinction criteria for a population inhabiting one dimensional heterogeneous environments governed by a reaction-diffusion equation with spatially varying growth rate and external forcing of the form
\[
z_t(x, t) - dzz_{xx}(x, t) = z(x, t)G(x, z(x, t)) - Y(x, z(x, t)) \quad \text{on } [0, 1] \times \mathbb{R}_+
\]
subject to one of the following BCs:
where \( d > 0 \) represents the rate of diffusion, \( z(x,t) \) denotes the density of the population of one species at location \( x \) and at time \( t \), the density dependent term \( G(x,z(x,t)) \) denotes the local rate of change in the population density, and the term \( Y(x,z(x,t)) \) represents external forcing or external perturbation such as a harvesting rate or functional response. The BC \( z(x,t) = 0 \) at \( x = 0 \) or \( x = 1 \) corresponds to a completely hostile exterior region, that is, the population that reaches the boundary \( x = 0 \) or \( x = 1 \) must die; the BC \( z_x(1,t) = 0 \) corresponds to the boundary acting as a perfect barrier to the population; and either \( z(0,t) - \beta z_x(0,t) = 0 \) with \( \beta > 0 \) or \( z(1,t) = -\delta z_x(1,t) \) with \( \delta > 0 \) corresponds to a situation where some members of the population that reach the boundary \( x = 0 \) or \( x = 1 \) would die and others would turn back. Following [7, page 335], 1 or \( \beta \) or \( 1/\beta \) measures the hostility of the exterior environment at \( x = 0 \) or \( x = 1 \), respectively. We refer to [6, 7, 8] for the biological interpretations for (4.1).

These extinction criteria will be obtained by studying nonexistence of nonzero nonnegative steady state solutions of (4.1) with (4.2) or (4.3), namely,

\[
-dz''(x) = z(x)G(x,z(x)) - Y(x,z(x)) \quad \text{for a.e. } x \in [0,1] \tag{4.4}
\]

subject to \((B_1)\) and \((B_2)\), respectively.

When the BC is the Dirichlet and Robin BCs, the models (4.1) with the logistic growth rate \( G(x,z) := G_1(x,z) = \lambda(1-z) \) or \( G(x,z) := G_2(x,z) = m(x)-\nu(x)z \) and \( Y(x,z) \equiv 0 \) or \( Y(x,z) = \delta h(x)z \) were studied in [24, sections 3 and 4], [6, 7, 8] and [26]. (4.1) with a quasi-constant-yield harvest rate \( Y(x,z) = Y_1(x,z) := \rho(x)p(z) \) was studied in [27] under Neumann BCs or under the assumption of these functions \( m, \nu, h \) being periodic functions and in [21] under the Dirichlet BCs and \( G(x,z) := G_1(x,z) \). The existence of nonzero nonnegative solutions of the model (4.4) with the logistic grow rate and the functional response \( Y(x,z) = Y_2(x,z) := a(x)z^2(x)/(1+z^2(x)) \) was studied by Yang and Lan [33] when the BC is the Dirichlet BC. Such equations arise from the reaction diffusion population models of spruce budworm [24, p.235, (6.1)].

Here, we establish the extinction criteria for (4.1) with (4.2) or (4.3).

We assume that the following conditions hold.

\((H_1)\) \( G : [0,1] \times \mathbb{R}_+ \to \mathbb{R} \) satisfies Carathéodory conditions, there exists a measurable function \( g : [0,1] \to \mathbb{R}_+ \) and for each \( r > 0 \), there exists a measurable function \( \eta_r : [0,1] \to \mathbb{R}_+ \) such that \( \eta_r \Phi_i \in L^1[0,1] \) and

\[
-dz''(x) \leq G(x,z) \leq g(x) \quad \text{for a.e. } x \in [0,1] \text{ and all } z \in [0,r]. \tag{4.5}
\]

\((H_2)\) \( Y : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+ \) satisfies Carathéodory conditions, \( Y(x,0) = 0 \) for \( x \in [0,1] \), and for \( r > 0 \), there exists a measurable function \( h_r : [0,1] \to \mathbb{R}_+ \) such that \( h_r \Phi_i \in L^1[0,1] \) and

\[
Y(x,z) \leq h_r(x) \quad \text{for a.e. } x \in [0,1] \text{ and all } z \in [0,r]. \tag{4.6}
\]

**Theorem 4.1 (Extinction criteria).** Let \( L_i \) be defined by (3.6) with \( g \) given in (4.5). Assume that one of the following conditions holds.

- (i) \( g \Phi_i \in L^1[0,1] \) and \( r(L_i) < d \).
(ii) \( g\Phi_i \in L^1[0,1] \) and \( \|L_i\| < d \).

(iii) \( g\Phi_i \in L^1[0,1], \int_0^1 \Phi_i(s)g(s)ds \leq d \) or \( g\Phi_i \in L^1[0,1], \int_0^1 \Phi_i(s)g(s)ds < d \).

(iv) \( g \in L^p[0,1] \), and \( \|g\|_{L^p} \leq d/\|\Phi_1\|_{L^p} \) or \( \|g\|_{L^p} < d/\|\Phi_2\|_{L^p} \).

Then (4.4)-(B_i) has no solutions in \( P \setminus \{0\} \).

**Proof.** We define a function \( f : [0,1] \times \mathbb{R}_+ \to \mathbb{R} \) by

\[
f(x,z) = \frac{1}{d}[zG(x,z) - Y(x,z)].
\]

For \( r > 0 \), we define a measurable function \( g_r : [0,1] \to \mathbb{R}_+ \) by

\[
g_r(x) = \frac{1}{d}[\eta_r(x) + h_r(x)].
\]

By (4.5) and (4.6), \( f, g, g_r \) satisfy (2.3). The results follow from Theorem 3.1. \( \square \)

Each criterion in Theorem 4.1 depends on both the positive upper bound \( g \) of \( G \) and the diffusion rate \( d \). As long as \( g \) and \( d \) satisfy one of these inequalities, the population becomes extinct due to the fact that (4.4)-(B_i) has no solutions in \( P \setminus \{0\} \).

**Remark 4.1.** The external force term \( Y \) in Theorem 4.1 is not critical for extinction of population since there are not restrictions on it except (B_2). But if these criteria given in Theorem 4.1 are not true, say \( r(L_i) \geq 1 \), then \( Y \) would play an important role for existence of positive steady states and would make a positive steady state disappear if it is large. The classical fixed point index theory \([1]\) cannot be applied in this case since the nonlinearity changes sign, but the newly established fixed point index by Yang and Lan \([33]\) would be a more appropriate tool to treat this question since this new index allows the nonlinearities to change sign.

As application of Theorem 4.1, we derive some extinction criteria by considering (4.4) with some special and important growth rate functions. We first consider the steady state solution equation of the form

\[-dz''(x) = k z(x)(1-z(x))(z(x) - \rho) - Y(x,z(x)) \]  \( \text{for a.e. } x \in [0,1] \)  (4.7)

subject to (B_1) and (B_2), respectively, where \( \rho \in (0,1) \) is a given constant.

The growth rate functions \( G(x,z) = k(1-z)(z-\rho) \) is of the strong Allee effects. Populations with (strong or weak) Allee effects have been widely studied, for example in \([29,30]\). In the following, we seek the intervals for the ratio \( d/k \) under which the population inhabiting such a habitat dies out.

**Theorem 4.2.** (1) If \( d/k > (1-\rho)^2/4\pi^2 \), then (4.7) with the Dirichlet BC: \( z(0) = z(1) = 0 \) has no solutions in \( P \setminus \{0\} \).

(2) If \( d/k > (1-\rho)^2/\pi^2 \), then (4.7) with the BC: \( z(0) = z'(1) = 0 \) has no solutions in \( P \setminus \{0\} \).

(3) If \( d/k > (1-\rho)^2(1+2\delta)[1+2\delta + 4\beta(1+\beta + \delta)]/32(1+\delta)^2 \), then (4.7)-(B_1) has no solutions in \( P \setminus \{0\} \).

(4) If \( d/k > (1-\rho)^2(1+2\beta)/8 \), then (4.7)-(B_2) has no solutions in \( P \setminus \{0\} \).

**Proof.** We define a function \( G : [0,1] \times \mathbb{R}_+ \to \mathbb{R} \) by

\[ G(x,z) = k(1-z)(z-\rho). \]
We show that $G$ satisfies (4.5). For $r > 0$, let $\eta_r(x) = -\min\{G(x, z) : z \in [0, r]\}$ for $x \in [0, 1]$. Then $\eta_r(x) > 0$ since $G(x, z) < 0$ for $(x, z) \in [0, 1] \times (0, \rho) \cup [1, \infty)$. Noting that $G(x, z) \geq 0$ for $(x, z) \in [0, 1] \times [\rho, 1]$, we have for each $r > 0$,

$$g(x, z) = \eta_r(x) \text{ for } (x, z) \in [0, 1] \times [0, r]$$

and $G$ satisfies the first inequality of (4.5). Let

$$g(x) = \frac{k(1 - \rho)^2}{4} \text{ for } x \in [0, 1]. \quad (4.8)$$

Then

$$g(x) - g(x, z) = \frac{k(1 - \rho)^2}{4} - k(1 - z)(z - \rho) = \frac{k}{4}[4z^2 - 4(1 + \rho)z + (1 + \rho)^2]$$

$$= \frac{k}{4}[2z - (1 + \rho)]^2 \geq 0.$$ 

It follows that $G(x, z) \leq g(x)$ for $(x, z) \in [0, 1] \times \mathbb{R}_+$ and the second inequality of (4.5) holds. Since $G$, $g$ and $\Phi_i$ are continuous functions, $G$ satisfies $(H_i)$.

With the function $g$ given in (4.8), we define for $i \in \{1, 2\}$ and $x \in [0, 1],

$$L_i z(x) = \int_0^1 k_i(x, s) g(s) z(s) ds = \int_0^1 k_i(x, s) z(s) ds. \quad (4.9)$$

(1) By (4.9) and [31, Theorem 5.1 (a)], $r(L_1) = k(1 - \rho)^2/4\pi^2$. The result follows from Theorem 4.1 (i).

(2) By (4.9) and [31, Theorem 5.1 (b)], $r(L_2) = k(1 - \rho)^2/\pi^2$. The result follows from Theorem 4.1 (i).

(3) By the proof of [21, Theorem 2.4], we have

$$\|L_1\| = \left(\frac{k(1 - \rho)^2}{4}\right) \max_{0 \leq x \leq 1} \int_0^1 k_1(x, s) ds$$

$$= \frac{k(1 - \rho)^2}{4} k(1 + 2\delta)[1 + 2\delta + 4\beta(1 + \beta + \delta)].$$

The result follows from Theorem 4.1 (ii).

(4) By computation, we have

$$\|L_2\| = \left(\frac{k(1 - \rho)^2}{4}\right) \max_{0 \leq x \leq 1} \int_0^1 k_2(x, s) ds$$

$$= \frac{k(1 - \rho)^2}{4} \frac{(1 + 2\delta)}{2}.$$ 

The result follows from Theorem 4.1 (ii).

\textbf{Remark 4.2.} Theorem 4.2 shows that the parameter $\rho$ in the strong Allee effects plays an important role since the possibility of extinction of the population is decreasing as $\rho$ goes to 0, and the harvest rate does not effect the extinction.

Next, we consider the steady state solutions governed by the BVP:

$$-dz''(x) = z(x)[m(x) - \nu(x)z(x)] - Y(x, z(x)) \text{ for a.e. } x \in [0, 1] \quad (4.10)$$

subject to $(B_1)$ and $(B_2)$, respectively, where $m(x)$ describes the intrinsic rate at which the population would grow or decline at $x$ in the absence of crowding or limitations on the availability of resources. The sign of $m(x)$ becomes positive on favorable habitats and negative on unfavorable ones. The term $-\nu(x)z$ represents the effects of crowding on the growth rate of the population; the size of $\nu(x)$ denotes the strength of the crowding effects at $x$. 

When the BC is the Dirichlet BC or Robin BC \( (i.e., \beta = \delta) \) and \( Y \equiv 0 \), the model (4.10) was studied by Cantrell and Cosner [7].

By Theorem 4.1, we obtain the following extinction criteria.

**Theorem 4.3.** Assume that \( m \Phi_1, v \Phi_1 \in L^1[0,1] \) and \( Y \) satisfies \( (H_2) \). Let \( L_i \) be defined by (3.6) with \( g = m_+ \). Assume that one of the following conditions holds.

\( r(L_i) < d \).

\( \|L_i\| < d \).

\( \int_1^0 \Phi_1(s)m_+(s)ds \leq d, \) and \( \int_1^1 \Phi_2(s)m_+(s)ds < d. \)

\( m \in L^p[0,1], \|m_+\|_{L^p} \leq d/\|\Phi_1\|_{L^q} \) or \( m \in L^p[0,1], \|m_+\|_{L^p} < d/\|\Phi_2\|_{L^q} \).

Then (4.10)-(B1) has no solutions in \( P \setminus \{0\} \).

As applications of Theorem 4.3, we study (4.10) with specific intrinsic rates including the piecewise constant growth rate function \( m(x) = m_{a,k} \) defined by

\[
m_{a,k}(x) = \begin{cases} 
-1, & \text{on } [0,a), \\
 k, & \text{on } [a, a+T], \\
 -1, & \text{on } (a + T, 1],
\end{cases}
\]

for a fixed \( T \in (0,1), a \in [0,1-T] \) and \( k > 0 \), which was initially studied by Cantrell and Cosner in [7] when the BC is the Dirichlet BC or Robin BC and \( Y \equiv 0 \).

**Notation:** Let \( \sigma = (1 + \beta + \delta)^{-1} \) and

\[
\eta_1(\beta, \delta, a, T) = \sigma T \left[ -a^2 + (1 - \beta + \delta - T)a + \beta(1 + \delta) \right] + \frac{T}{2} \left[ 1 - \beta + \delta \right] - \frac{T^2}{3},
\]

\[
\eta_2(\beta, \delta, a, T) = \beta T + a T + \frac{T^2}{2}.
\]

**Example 4.1.** (1) If \( d/k \geq \eta_1(\beta, \delta, a, T) \), then (4.10)-(B1) with \( m = m_{a,k} \) has no solutions in \( P \setminus \{0\} \).

(2) If

\[
d/k > \frac{(1 + 2\delta)\left[1 + 2\delta + 4\beta(1 + \beta + \delta)\right]}{8(1 + \delta)^2} := \xi(\beta, \delta),
\]

then (4.10)-(B1) with \( m(x) \equiv k \) has no solutions in \( P \setminus \{0\} \).

(3) If

\[
d/k \geq \sqrt[70]{\frac{(1 - e^{-1})^2}{170}} \approx 0.1787,
\]

then (4.10) with \( m(x) = ke^{-\sqrt[70]{(2x-1)^2}} \) and the Dirichlet BC has no solutions in \( P \setminus \{0\} \).
Proof. (1) Let \( \zeta = \int_0^1 \Phi_1(s)(m_{a,k})_+(s) \, ds \). Then by computation, we have

\[
\zeta = k\sigma \int_a^{a+T} (\beta + s) (1 + \delta - s) \, ds \\
= k\sigma \int_a^{a+T} \left[ \beta (1 + \delta) + (1 + \delta - \beta) s - s^2 \right] \, ds \\
= k\sigma \left\{ \beta (1 + \delta) T + \frac{1 + \delta - \beta}{2} [(a + T)^2 - a^2] - \frac{1}{3} [(a + T)^3 - a^3] \right\} \\
= k\sigma [\beta (1 + \delta) + \frac{(1 + \delta - \beta)(2a + T)}{2} - a^2 - aT - \frac{T^2}{3}] \\
= k\eta_1(\beta, \delta, a, T).
\]

The result with \((B_1)\) follows from Theorem 4.3 (3). Let

\[
h(x) = \int_0^x k_2(x, s)(m_{a,k})_+(s) \, ds \quad \text{for} \quad x \in [0, 1].
\]

Then by (3.3) and (4.11),

\[
h(x) = k \int_a^{a+T} k_2(x, s) \, ds \quad \text{for} \quad x \in [0, 1].
\]

We show that the following assertions hold.

(i) If \(0 \leq x \leq a\), then

\[
h(x) = k \int_a^{a+T} (\beta + x) \, ds = k(\beta + x)T
\]

and \(h(x) \leq k(\beta + a)T = h(a)\) for \(x \in [0, a]\).

(ii) If \(a \leq x \leq a + T\), then

\[
h(x) = k \int_a^x (\beta + s) \, ds + k \int_x^{a+T} (\beta + x) \, ds
\]

and \(h'(x) = k(a + T - x) \geq 0\) for \(x \in [a, a + T]\). Hence, \(h(x) \leq h(a + T)\) for \(x \in [a, a + T]\).

(iii) If \(a + T \leq x \leq 1\), then

\[
h(x) = k \int_a^{a+T} (\beta + s) \, ds = k(\beta T + aT + \frac{T^2}{2}).
\]

Hence, we have

\[
h(x) \leq k(\beta T + aT + \frac{T^2}{2}) \quad \text{for} \quad x \in [0, 1] \quad \text{and} \quad \|L_2\| = k(\beta T + aT + \frac{T^2}{2}).
\]

The result with \((B_2)\) follows from Theorem 4.3 (2).
(2) Let \( \xi(x) = \int_0^1 k_1(x, s) \, ds \) for \( x \in [0, 1] \). Then we have for \( x \in [0, 1] \),
\[
\xi(x) = \sigma \left[ \int_0^x (1 + \delta - x)(\beta + s) \, ds + \int_x^1 (\beta + x)(1 + \delta - s) \, ds \right]
\]
\[
= \sigma \left\{ (1 + \delta - x) \left( \beta x + \frac{x^2}{2} \right) + (\beta + x) \left[ \frac{1}{2} \delta - (1 + \delta) x + \frac{x^2}{2} \right] \right\}
\]
\[
= \sigma \left\{ -\frac{1}{2} (1 + \beta + \delta)x^2 + \frac{1}{2} (1 + \beta + \delta)x + \frac{1}{2} (1 + \beta) \beta \right\}
\]
\[
= \frac{(1 + 2\delta)[1 + 2\delta + 4\beta(1 + \beta + \delta)]}{8(1 + \delta)^2} - \frac{1}{2} \left[ x - \frac{1 + 2\delta}{2(1 + \beta + \delta)} \right]^2.
\]
This, together with \( L_1z(x) = k\xi(x) \), implies
\[
\|L_1\| = k \max_{0 \leq x \leq 1} \int_0^1 k_1(x, s) \, ds = \frac{k(1 + 2\delta)[1 + 2\delta + 4\beta(1 + \beta + \delta)]}{8(1 + \delta)^2}.
\]
The result follows from Theorem 4.3 (2).

(3) When the BC is the Dirichlet BC, \( \Phi_1(s) = s(1 - s) \) for \( s \in [0, 1] \). Since
\[
\|\Phi_1\|_{L^3} = \left( \int_0^1 s^3(1 - s)^3 \, ds \right)^{1/3} = \left[ B(4, 4) \right]^{1/4} = \left[ \frac{\Gamma(4)^2}{\Gamma(8)} \right]^{1/4}
\]
\[
= \frac{1}{\sqrt[3]{70}},
\]
where we have used the beta- and gamma-functions, and
\[
\int_0^1 |m_+(s)|^2 \, ds = k^2 \left[ \int_0^{1/2} e^{2s-1} \, ds + \int_{1/2}^1 e^{1-2s} \, ds \right] = k^2 (1 - e^{-1}),
\]
we have
\[
\|m_+\|_{L^2} = k(1 - e^{-1}) \leq \sqrt[3]{70}d = \frac{d}{\|\Phi_1\|_{L^3}}.
\]
The result follows from Theorem 4.3 (4).

**Remark 4.3.** From the proof of Example 4.1 (2) we see that \( \|L_1\| = \xi(\beta, \delta) \).
The value \( \xi(0, 0) = 1/8 \) was obtained in the proof of [22, Corollary 3.2] and the value \( \xi(0, \delta) = \frac{(1 + 2\delta)^2}{8(1 + \delta)} \) can be derived from Theorems 2, 4 and 6 in [23]. Hence, \( \|L_1\| = \xi(\beta, \delta) \) is new when \( \beta > 0 \).

Example 4.1 (1) depends heavily on \( T \) and \( a \). When the size \( T \) of a favorable habitat in the heterogeneous environment is fixed, the choice of the positions of the favorable habitat is of great importance in predicating extinction for the population. The larger the value of \( \eta_1(\beta, \delta, a, T) \), the smaller the possibility of extinction for the population becomes. Hence, we can obtain the optimal favorable habitats by finding the maximum value of \( \eta_1(\beta, \delta, a, T) \). The approach is different from that used in [7], where \( r(L_1) \) is used to find the optimal favorable habitats when \( \beta = \delta \). We refer to [12, 17, 28] for further study of minimization of the characteristic value for the linear boundary value problem \(-z''(x) = \lambda m(x)z(x)\) or elliptic BVPs with Neumann BCs.

As illustration, in the following, we find the the maximum values of \( \eta_1(\beta, \delta, a, T) \) to obtain the optimal arrangements for such heterogeneous environments.
Notation: Let
\[ a_1(T) = (1 - \beta + \delta - T)/2 \]  
(4.13)
and let
\[
\begin{align*}
\mathcal{D}_1 &= \{(\beta, \delta, T) : 1 + \delta \leq \beta \text{ and } T \in (0, 1)\}; \\
\mathcal{D}_2 &= \{(\beta, \delta, T) : \delta < \beta < 1 + \delta \text{ and } T \in [1 - \beta + \delta, 1)\}; \\
\mathcal{D}_3 &= \{(\beta, \delta, T) : \delta < \beta < 1 + \delta \text{ and } T \in (0, 1 - \beta + \delta)\}; \\
\mathcal{D}_4 &= \{(\beta, \delta, T) : \delta = \beta \text{ and } T \in (0, 1)\}; \\
\mathcal{D}_5 &= \{(\beta, \delta, T) : \delta < 1 + \beta \text{ and } T \in (0, 1 + \beta - \delta)\}; \\
\mathcal{D}_6 &= \{(\beta, \delta, T) : \beta < \delta < 1 + \beta \text{ and } T \in (1 + \beta - \delta, 1)\}; \\
\mathcal{D}_7 &= \{(\beta, \delta, T) : 1 + \beta \leq \delta \text{ and } T \in (0, 1)\}.
\end{align*}
\]

Example 4.2. Assume that one of the following conditions is satisfied.

\((h_1)\) \((\beta, \delta, T) \in \mathcal{D}_1 \cup \mathcal{D}_2\) and
\[ d/k \geq \eta_1(\beta, \delta, 0, T) = \sigma T \left[ \beta(1 + \delta) + \frac{T}{2}(1 - \beta + \delta) - \frac{T^2}{3} \right]. \]

\((h_2)\) \((\beta, \delta, T) \in \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}_5\) and
\[ d/k \geq \eta_1(\beta, \delta, a(T), T) = \frac{\sigma T}{12} \left[ 3(1 + \beta + \delta)^2 - T^2 \right]. \]

\((h_3)\) \((\beta, \delta, T) \in \mathcal{D}_6 \cup \mathcal{D}_7\) and
\[ d/k \geq \eta_1(\beta, \delta, 1 - T, T) = \sigma T \left[ (1 + \beta)\delta + \frac{T}{2}(1 + \beta - \delta) - \frac{T^2}{3} \right]. \]

Then (4.10)-(B1) with \(m = m_{a,k}\) has no solutions in \(P \setminus \{0\}\) for each \(a \in [0, 1 - T]\).

Proof. By (4.12), we have
\[ \frac{\partial \eta_1(\beta, \delta, a, T)}{\partial a} = 2\sigma \left[ \frac{1 - \beta + \delta - T}{2} - a \right] = 2\sigma T [a_1(T) - a]. \]
(4.14)
We consider the following three cases.

(1) If \((\beta, \delta, T) \in \mathcal{D}_1 \cup \mathcal{D}_2\), then \(a_1(T) \leq 0\). By (4.14), \(\frac{\partial \eta_1(\beta, \delta, a, T)}{\partial a} \leq 0\) for each \(a \in [0, 1 - T]\) and
\[ \eta_1(\beta, \delta, a, T) \leq \eta_1(\beta, \delta, 0, T) = \sigma T \left[ \beta(1 + \delta) + \frac{T}{2}(1 - \beta + \delta) - \frac{T^2}{3} \right]. \]

(2) If \((\beta, \delta, T) \in \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}_5\), then \(0 < a_1(T) < (1 - T)/2\) for \((\beta, \delta, T) \in \mathcal{D}_3\), \(a_1(T) = (1 - T)/2\) for \((\beta, \delta, T) \in \mathcal{D}_4\) and \((1 - T)/2 < a_1(T) \leq 1 - T\) for \((\beta, \delta, T) \in \mathcal{D}_5\). By (4.14), \(\frac{\partial \eta_1(\beta, \delta, a, T)}{\partial a} \geq 0\) for each \(a \in [0, a_1(T)]\) and
\[ \frac{\partial \eta_1(\beta, \delta, a, T)}{\partial a} \leq 0\) for each \(a \in (a_1(T), 0]\). Hence, we have for \(a \in [0, 1 - T]\),
\[ \eta_1(\beta, \delta, a, T) \leq \eta_1(\beta, \delta, a_1(T), T) = \frac{\sigma T}{12} \left[ 3(1 + \beta + \delta)^2 - T^2 \right]. \]

(3) If \((\beta, \delta, T) \in \mathcal{D}_6 \cup \mathcal{D}_7\), then we have \(a_1(T) > 1 - T\). By (4.14), \(\frac{\partial \eta_1(\beta, \delta, a, T)}{\partial a} > 0\) for each \(a \in [0, 1 - T]\). Hence, we have for \(a \in [0, 1 - T]\),
\[ \eta_1(\beta, \delta, a, T) \leq \eta_1(\beta, \delta, 1 - T, T) = \sigma T \left[ (1 + \beta)\delta + \frac{T}{2}(1 + \beta - \delta) - \frac{T^2}{3} \right]. \]
The results follow from the above inequalities and Example 4.1 (1) with (B1). □
By Example 4.2 and its proof, we see that the lower bound in each of \((h_1), (h_2)\) and \((h_3)\) is the maximum value of the function \(\eta_1(\beta, \delta, a, T)\) of \(a\) on \([0, 1 - T]\). The optimal location for the population is determined by the number \(a\) at which \(\eta_1(\beta, \delta, a, T)\) reaches its maximum value. The detailed interpretation and analysis for the optimal location are given below.

By Example 4.2 \((h_1)\) we see that if \((\beta, \delta, T) \in \mathcal{D}_1 \cup \mathcal{D}_2\), then \(\eta_1(\beta, \delta, a, T)\) reaches its maximum at \(a = 0\), so the favorable habitat of size \(T\) would be arranged to the left-hand side starting at the origin to get the optimal location for the heterogeneous environment, as shown in Figs. 1(a) and 1(b). If we move \(a\) from 0 to \(1 - T\), the habitat becomes worse and worse, as shown by the monotonically decreasing curves with respect to \(a \in [0, 1 - T] = [0, 0.2]\) for \((\beta, \delta, T)\) in different area. Here, the curve, plotted by squares, corresponds to the parameters in (a) and the curve, plotted by dots, to the parameters in (b).

By Example 4.2 \((h_2)\), we see that if \((\beta, \delta, T) \in \mathcal{D}_3\), then \(a_1(T) < (1 - T)/2\) and the optimal location for the single habitat of size \(T\) is the favorable habitat whose left endpoint would be arranged at \(a_1(T)\) which is smaller than \((1 - T)/2\), as shown in Fig. 2(a). If we move \(a\) from 0 to \(a_1(T)\), then the habitat becomes better and better and if we move \(a\) from \(a_1(T)\) to \(1 - T\), then the habitat becomes worse and worse. This is illustrated by the unimodal curve plotted by the dash line in Fig. 2(d).

By the proof of Example 4.2 \((h_2)\), we see that when \((\beta, \delta, T) \in \mathcal{D}_4\), that is, \(\beta = \delta\) and \(T \in (0, 1)\), we have \(a_1(T) = (1 - T)/2\). Hence, the optimal location is to put the favorable habitat at the center, as shown by Fig. 2(b) as well as by the symmetric and unimodal curve, plotted by the solid line, in Fig. 2(d). If \((\beta, \delta, T) \in \mathcal{D}_5\), then \(a_1(T) \in ((1 - T)/2, 1 - T]\). Hence, the optimal location for the single habitat of size
Figure 2. (a) The optimal position with $m_{a,k}$ when $(\beta, \delta, T) \in \mathcal{D}_3$, where $k = 3, a = 1 - T, \beta = 2.5, \delta = 2, T = 0.3$, and $a_1(T) = 0.1$. (b) The optimal position with $m_{a,k}$ when $(\beta, \delta, T) \in \mathcal{D}_4$, where $k = 3, a = 1 - T, \beta = \delta = 3, T = 0.3$, and $a_1(T) = \frac{1 - T}{2} = 0.35$. (c) The optimal position with $m_{a,k}$ when $(\beta, \delta, T) \in \mathcal{D}_5$, where $k = 3, a = 1 - T, \beta = 3, \delta = 3.5, T = 0.3$, and $a_1(T) = 0.6$. (d) The unimodal curve of $\eta_1$, as defined in (4.12), with respect to the variable $a \in [0, 1 - T] = [0, 0.7]$ for $(\beta, \delta, T)$ in different area. Here, the curve, plotted by the dash line, corresponds to the parameters in (a), the curve, plotted by the solid line, to the parameters in (b), and the curve, plotted by the dotted line, to the parameters in (c).

$T$ is the favorable habitat whose left endpoint would be arranged at $a_1(T)$ which is greater than $(1 - T)/2$, as shown in Fig. 2(c).

We end this section by mentioning the optimal location under $(B_2)$. Since

$$\eta_2(\beta, \delta, a, T) \leq \eta_2(\beta, \delta, 1 - T, T) \quad \text{for } a \in [0, 1 - T],$$

by Example 4.1 (1) with $(B_2)$, we see that the lower bound $\eta_2(\beta, \delta, 1 - T, T)$ is the maximum value of the function $\eta_2(\beta, \delta, a, T)$ of $a$ on $[0, 1 - T]$. The optimal location for the population which is determined by the number $a$ at which $\eta_2(\beta, \delta, a, T)$ reaches its maximum value. Hence, the favorable habitat of size $T$ would be arranged to the right-hand side to get the optimal location for the heterogeneous environment. Note that the BC $z_x(1, t) = 0$ corresponds to the boundary acting as a perfect barrier to the population; and $z(0, t) - \beta z_x(0, t) = 0$ with $\beta > 0$ corresponds to a situation where some members of the population would die and others would turn back. Our conclusion is that we would arrange the favorable habitat of size $T$ closer to the boundary $z_x(1, t) = 0$ to get more suitable favorable environment.

Acknowledgments. We would like to thank the referee very much for providing valuable comments.

References


Received February 2017; revised June 2018.

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