SPECIFIED HOMOGENIZATION OF A DISCRETE TRAFFIC MODEL LEADING TO AN EFFECTIVE JUNCTION CONDITION

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ABSTRACT. In this paper, we focus on deriving traffic flow macroscopic models from microscopic models containing a local perturbation such as a traffic light. At the microscopic scale, we consider a first order model of the form “follow the leader” i.e. the velocity of each vehicle depends on the distance to the vehicle in front of it. We consider a local perturbation located at the origin that slows down the vehicles. At the macroscopic scale, we obtain an explicit Hamilton-Jacobi equation left and right of the origin and a junction condition at the origin (in the sense of [25]) which keeps the memory of the local perturbation. As it turns out, the macroscopic model is equivalent to a LWR model, with a flux limiting condition at the junction. Finally, we also present qualitative properties concerning the flux limiter at the junction.

1. Introduction. The goal of this paper is to derive a macroscopic model for traffic flow problems from a microscopic model. The idea is to rescale the microscopic model, which describes the dynamics of each vehicle individually, in order to get a macroscopic model which describes the dynamics of density of vehicles. The main motivation for deriving macroscopic models from microscopic models comes from the fact that macroscopic models are more adapted to simulate traffic at large scales. Moreover, microscopic models are based on assumptions that are easier to verify and therefore to derive a macroscopic model allows to rigorously verify it.

Several techniques were proposed for the connections micro-macro and the derivation of a macroscopic model from a microscopic one was studied by several authors. In 1970, Payne [35] used the method of expansion of the gradient to derive a LWR model from a Newel model [34]. The authors of [22] established a micro-macro connection supposing that the macroscopic variables in a point $x$ of the space can be defined by the microscopic variables if a vehicle is present at $x$. Otherwise, the macroscopic variables $x$ are defined by linear interpolation. In [5], the authors derived an Aw-Rascale model from a second order model of the type “follow the leader”. The reader can refer also to [11, 37, 12, 29] to enrich their knowledge.

The originality of our work is that we assume that there is a local perturbation that slows down the vehicles and we want to understand how this local perturbation influences the macroscopic dynamics. To our knowledge, this result is the first in

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the presence of a local perturbation. Note that this result has also been extended by the authors to second order microscopic models ([19]). This local perturbation can be constant in time and represent a slowdown near a school or due to a car crash near the road. It can also depend (periodically) in time and represent for example a traffic light. The schematic representation of the microscopic model is given in Figure 1.

\[ \dot{U}_j = V(U_{j+1} - U_j) \]

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\[ \text{Perturbation: radius} = r \]

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**Figure 1.** Schematic representation of the microscopic model.

We denote by \( U_j(t) \) the position of the \( j \)th vehicle and we assume that the velocity of each vehicle is given by the function \( V \). In order to obtain our homogenization result, we proceed as in [15, 18, 17, 16] and rescale the microscopic model which describes the dynamics of each vehicle, to obtain a macroscopic model that describes the density of vehicles. If the local perturbation is located around zero, at the macroscopic scale it is natural to get an Hamilton-Jacobi equation with a junction condition at the origin (see Figure 2, \( u_0^\ast \) is the primitive of the density of vehicles and the effective Hamiltonian \( H \) is defined later in the paper), since the size of the perturbation goes to zero when we do the rescaling. This junction condition keeps the memory of the presence of the local perturbation.

\[ u_0^\ast + H(u_0^\ast) = 0 \]

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**Figure 2.** Schematic representation of the macroscopic model.

Recently, the theory of Hamilton-Jacobi equations with junction or more generally on networks has known important developments in particular since the works of Achdou, Camilli, Cutri, and Tchou [2] and the work of Achdou and Imbert, Monneau, and Zidani [26].

In [2], an optimal control problem was studied in \( \mathbb{R}^2 \) supposing that the trajectories of the controlled system stay in network. In [10], the authors studied the vanishing viscosity method for Hamilton-Jacobi equations on junctions. In [6, 7, 3], the authors studied a regional control problem with regular dynamics and costs on each side of the hyperplane and a condition on the interfaces separating the regions where the dynamic and running cost are different.
In this article, we will use the results obtained by Imbert and Monneau in [25, 24] where the authors considered a discontinuous Hamilton-Jacobi equation and they introduced the notion of “flux-limited solutions” (see also [33] for a simpler approach). The link between the theory developed in [6, 7] and “flux-limited solutions” in [25, 24] is explored in [8]. In particular, [8] contains a simpler proof of the comparison principle than in [25]. Finally, let us mention in this context the lectures of Lions at the “College de France” [31].

In this paper, we will use the ideas developed in [18] in order to pass from microscopic models to macroscopic ones. In particular, we will show that this problem can be seen as an homogenization result. The difficulty here is that, due to the local perturbation, we are not in a periodic setting and so the construction of suitable correctors is more complicated. In particular, we will use the idea developed by Achdou and Tchou in [1], by Galise, Imbert, and Monneau in [20], and in the lectures of Lions at the “College de France” [32], which consists in constructing correctors on truncated domains.

Finally, we would like to mention the work [38] where the second author provided a numerical scheme for the computation of an approximation of the flux limiter $\Lambda$.

2. Main results.

2.1. The microscopic model. In this paper, we are interested in a first order microscopic model of the form

$$\dot{U}_i(t) = V(U_{i+1}(t) - U_i(t)) \cdot \phi(t, U_i(t)),$$

where $U_i : [0, +\infty) \to \mathbb{R}$ denotes the position of the $i$-th vehicle and $\dot{U}_i$ is its velocity. The function $\phi : \mathbb{R} \times \mathbb{R} \to [0, 1]$ simulates the presence of a local perturbation around the origin. We denote by $r$ the radius of influence of the perturbation.

The function $V$ is called the optimal velocity function and we make the following assumptions on $V$ and $\phi$:

Assumption (A).

(A1) $V : \mathbb{R} \to \mathbb{R}^+$ is Lipschitz continuous, non-negative.

(A2) $V$ is non-decreasing on $\mathbb{R}$.

(A3) There exists a $h_0 \in (0, +\infty)$ such that for all $h \leq h_0$, $V(h) = 0$.

(A4) There exists $h_{\text{max}} \in (h_0, +\infty)$ such that for all $h \geq h_{\text{max}}$, $V(h) = V(h_{\text{max}}) =: V_{\text{max}}$.

(A5) There exists a real $p_0 \in [-1/h_0, 0)$ such that the function $p \mapsto pV(-1/p)$ is decreasing on $[-1/h_0, p_0)$ and increasing on $(p_0, 0)$.

(A6) The function $\phi : \mathbb{R} \times \mathbb{R} \to [0, 1]$ is Lipschitz continuous and there exists $r > 0$ such that $\phi(t, x) = 1$ for $|x| \geq r$. We assume also that $\phi$ is $\mathbb{Z}$-periodic in time.

Remark 1. Assumptions (A1)-(A2)-(A3)-(A5) are satisfied by several classical optimal velocity functions. To be more precise, since $V$ gives the velocity of a vehicle it is normal to assume that the function should be regular, continuous and non-negative (the vehicles only go forward) which explains assumption (A1). Moreover, a vehicle should go faster if he has more space in front of it, which explains assumption (A2). Assumption (A3) comes from the fact that we want to avoid any collisions and we added a safety distance $h_0$ to our model: if a vehicles has less than $h_0$ in front of it the vehicles should not advance. Assumption (A5) plays a crucial role at the macroscopic scale. In fact, the reader will see later that the macroscopic model is
the Hamilton-Jacobi equation with junction condition at the point zero introduced by Imbert and Monneau in [25]. Assumption (A5) provides an essential assumption used by the authors in [25] which is the quasi-convexity of the Hamiltonian. We have added assumption (A4) to work with $V'$ with a bounded support. But by modifying slightly the classical optimal velocity functions, we obtain a function that satisfies all the assumptions. For instance, in the case of the Greenshields based models [21](see also [9]):

$$V(h) = \begin{cases} 
0 & \text{for } h \leq h_0, \\
V_{\text{max}} \left( 1 - \left( \frac{h_0}{h} \right)^n \right) & \text{for } h_0 < h \leq h_{\text{max}}, \\
V_{\text{max}} \left( 1 - \left( \frac{h_{\text{max}}}{h} \right)^n \right) & \text{for } h > h_{\text{max}},
\end{cases}$$

with $n \in \mathbb{N}\setminus\{0\}$. Another optimal velocity function, based on the Newell model [34](see also [13]), is given by:

$$V(h) = \begin{cases} 
0 & \text{for } h \leq h_0, \\
V_{\text{max}} \left( 1 - \exp \left( - \left( \frac{h - h_0}{b} \right)^n \right) \right) & \text{for } h_0 < h \leq h_{\text{max}}, \\
V_{\text{max}} \left( 1 - \exp \left( - \left( \frac{h_{\text{max}} - h_0}{b} \right)^n \right) \right) & \text{for } h > h_{\text{max}},
\end{cases}$$

with $n \in \mathbb{N}\setminus\{0\}$ and $b \in [0, +\infty)$. See Figure 3 for a schematic representation of an optimal velocity function satisfying assumption (A).

**Remark 2.** We will give an example of the function $\phi$. We will define $\phi$ on the interval $[0, 1]$ since it’s a $\mathbb{Z}$-periodic function. For $t \in [0, 1]$,

$$\phi(t, x) = \begin{cases} 
1 & \text{if } |x| > r \\
\frac{(\phi_0(t) - 1)}{r} x + \phi_0(t) & \text{if } x \in [-r, 0] \\
\frac{(1 - \phi_0(t))}{r} x + \phi_0(t) & \text{if } x \in (0, r].
\end{cases}$$

where $\phi_0$ is defined in the following form:

$$\phi_0(t) = \begin{cases} 
4t & \text{if } 0 < t < \frac{1}{4}, \text{ The end of the red light time} \\
1 & \text{if } \frac{1}{4} < t < \frac{1}{2}, \text{ Green light time} \\
-4t + 3 & \text{if } \frac{1}{2} < t < \frac{3}{4}, \text{ Orange light time} \\
0 & \text{if } \frac{3}{4} < t < 1, \text{ Red light time.}
\end{cases}$$

2.2. **The macroscopic model.** We recall that $k_0 = 1/h_0$ and we define $H : \mathbb{R} \to \mathbb{R}$, by

$$H(p) = \begin{cases} 
-p - k_0 & \text{for } p < -k_0, \\
-V \left( \frac{-1}{p} \right) |p| & \text{for } -k_0 \leq p \leq 0, \\
p & \text{for } p > 0.
\end{cases}$$

(2)

Note that such a $H$ is continuous, coercive $\lim_{|p| \to +\infty} H(p) = +\infty$ and because of (A5), there exists a unique point $p_0 \in [-k_0, 0]$ such that

$$\begin{cases} 
H \text{ is decreasing on } (-\infty, p_0), \\
H \text{ is increasing on } (p_0, +\infty).
\end{cases}$$

(3)
We denote by
\[ H_0 = \min_{p \in \mathbb{R}} H(p) = \overline{H}(p_0) \] (4)
and we refer to Figure 4 for a schematic representation of \( \overline{H} \).

The macroscopic model of this paper is the Hamilton-Jacobi equation with flux limiting condition at the junction point introduced by Imbert and Monneau in [25] and is given by
\[
\begin{align*}
&w_0^0 + \overline{H}(u_0^0) = 0 \quad \text{for } (t, x) \in (0, +\infty) \times (-\infty, 0) \\
&w_0^0 + \overline{H}(u_0^0) = 0 \quad \text{for } (t, x) \in (0, +\infty) \times (0, +\infty) \\
&w_0^0 + F_{\overline{A}}(u_0^0(t, 0^-), u_0^0(t, 0^+)) = 0 \quad \text{for } (t, x) \in (0, +\infty) \times \{0\} \\
&w_0^0(0, x) = u_0(x) \quad \text{for } x \in \mathbb{R},
\end{align*}
\] (5)
where \( \overline{A} \) has to be determined and \( F_{\overline{A}} \) is defined by
\[
F_{\overline{A}}(p_-, p_+) = \max \left( \overline{A}, \overline{H}^+(p_-), \overline{H}^-(p_+) \right),
\] (6)
with
\[ \mathcal{H}^-(p) = \begin{cases} \mathcal{P}(p) & \text{if } p \leq p_0, \\ \mathcal{P}(p_0) & \text{if } p \geq p_0, \end{cases} \quad \text{and} \quad \mathcal{H}^+(p) = \begin{cases} \mathcal{P}(p_0) & \text{if } p \leq p_0, \\ \mathcal{P}(p) & \text{if } p \geq p_0. \end{cases} \] (7)

**Remark 3.** We notice that in the case of traffic flow, (5) is equivalent (deriving in space) to a LWR model (see [30, 36, 26, 28]) with a flux limiting condition at the origin. In fact, the fundamental diagram of the model is \( pV(1/p) \) and \( u^0_x \) corresponds to the density of vehicles.

2.3. **Main result: transition from micro to macro.** In this paper, we will study the traffic flow when the number of vehicles per unit length tends to infinity by introducing the rescaled “cumulative distribution function” of vehicles, \( \rho^\varepsilon \), defined by
\[ \rho^\varepsilon(t, y) = \varepsilon \rho \left( \frac{t}{\varepsilon}, \frac{y}{\varepsilon} \right) \] (8)
where
\[ \rho(t, y) = -\left( \sum_{i \geq 0} H(y - U_i(t)) + \sum_{i < 0} (-1 + H(y - U_i(t))) \right), \] (9)
with
\[ H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \] (10)

The following theorem is the main result of this paper

**Theorem 2.1** (Junction condition by homogenization: application to traffic flow). Assume (A) and that at the initial time, we have, for all \( i \in \mathbb{Z} \),
\[ U_i(0) \leq U_{i+1}(0) - h_0. \] (11)
We also assume that there exists a constant $R > 0$ such that, for all $i \in \mathbb{Z}$,
\[
\begin{cases}
U_{i+1}(0) - U_i(0) = h_1 & \text{if } U_i(0) \leq -R, \\
U_{i+1}(0) - U_i(0) = h_2 & \text{if } U_i(0) \geq R,
\end{cases}
\tag{12}
\]
with $h_1, h_2 \geq h_0$. We define the function $u_0$ by $u_0(x) = -\frac{x}{h_1}1_{\{x \leq 0\}} - \frac{x}{h_2}1_{\{x > 0\}}$ for all $x \in \mathbb{R}$. Then there exists $\bar{A} \in [H_0, 0]$ such that the function $\rho^\varepsilon$ defined by (9) converges towards the unique solution $u^0$ of (5).

**Remark 4.** As in [19], we can prove that there exists a constant $C > 0$ such that
\[
0 \leq U_{i+1}(t) - U_i(t) \leq C.
\tag{13}
\]
We deduce that
\[
0 \leq \varepsilon U_{i+1}(t) - \varepsilon U_i(t) \leq \varepsilon C.
\tag{14}
\]
On the other hand, we remark that the function $\rho^\varepsilon$ gives the position $\varepsilon U_i(\frac{t}{\varepsilon})$ a label $-\varepsilon (i + 1)$. Equation (14) implies that the distance between two consecutive vehicles tends to zero as $\varepsilon$ tend to zero, which can be seen as a passage from micro to macro. This fact also implies that $\rho^\varepsilon$ should go to the function giving the vehicles labels at the macroscopic scale.

**Remark 5** (Extension to second order microscopic model). The results presented in this paper have been extended by the authors to the case of second order microscopic models in [19].

2.4. **Strategy of the proof of the main result.** We will show now in three steps the path to obtain the convergence result.

1) **Injecting the system of ODEs into a single PDE:** as in [17, 18], we inject the system of ordinary differential equations in a partial differential equation. We look to construct a non-local operator $M$ which satisfies
\[
M[\rho(t, \cdot)](U_i(t)) = -V(U_{i+1} - U_i).
\tag{15}
\]
In fact, if we construct $M$ satisfying (15), we obtain that $\rho$ is a discontinuous viscosity solution of

$$u_t + M[u(t, \cdot)](x) \cdot \phi(t, x) \cdot |u_x| = 0 \quad \text{on } (0, +\infty) \times \mathbb{R}.$$  

The “suitable” non-local operator which satisfies (15) is given by

$$M[U](x) = \int_{-\infty}^{+\infty} \frac{J(z)}{\varepsilon} E(U(x + z) - U(x)) \, dz - \frac{3}{2} V_{\max}^\varepsilon$$  

with

$$E(z) = \begin{cases} 
0 & \text{if } z \geq 0 \\
1/2 & \text{if } -1 \leq z < 0 \\
3/2 & \text{if } z < -1,
\end{cases} \quad \text{and } J = V' \text{ on } \mathbb{R}.$$  

We deduce that $\rho^\varepsilon$ is a discontinuous viscosity solution of

$$u_\varepsilon^t + M^\varepsilon\left[\frac{u^\varepsilon(t, \cdot)}{\varepsilon}\right](x) \cdot \phi \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \cdot |u_\varepsilon^x| = 0 \quad \text{on } (0, +\infty) \times \mathbb{R},$$  

with

$$M^\varepsilon[U](x) = \int_{-\infty}^{+\infty} \frac{J(z)}{\varepsilon} E(U(x + \varepsilon z) - U(x)) \, dz - \frac{3}{2} V_{\max}^\varepsilon$$  

Theorem 2.2. The cumulative distribution function $\rho$ defined by (9) is a discontinuous viscosity solution of

$$\rho_t + M[\rho(t, \cdot)](x) \cdot \phi(t, x) \cdot |\rho_x| = 0 \quad \text{for } (t, x) \in [0, +\infty) \times \mathbb{R}. \quad \text{(19)}$$  

Conversely, if $u$ is a bounded and continuous viscosity solution of (19) satisfying for some time $T > 0$, and for all $t \in (0, T)$

$$u(t, x) \text{ is decreasing in } x,$$

then the points $U_i(t)$, defined by $u(t, U_i(t)) = -(i + 1)$ for $i \in \mathbb{Z}$, satisfy the system (1) on $(0, T)$.

2) Convergence of the continuous solution: we couple equation (17) with the following initial condition

$$u^\varepsilon(0, x) = u_0(x) \quad \text{on } \mathbb{R}.$$  

We also assume that the initial condition satisfies the following assumption:

(A0) (Gradient bound). The function $u_0$ is Lipschitz continuous and satisfies

$$-k_0 := -1/h_0 \leq (u_0)_x \leq 0 \quad \text{for all } x \in \mathbb{R}.$$  

The second step is to prove that the unique solution $u^\varepsilon$ of the equation

$$\begin{cases} 
u^\varepsilon_t + M^\varepsilon\left[\frac{u^\varepsilon(t, \cdot)}{\varepsilon}\right](x) \cdot \phi \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \cdot |u_\varepsilon^x| = 0 \\
u(0, x) = u_0(x),
\end{cases}$$  

converges locally uniformly towards the unique solution of (5).

Theorem 2.3 (Junction condition by homogenization). Assume (A) and (A0). For $\varepsilon > 0$, let $u^\varepsilon$ be the solution of (22). Then there exists $\overline{\mathbf{A}} \in [H_0, 0]$ such that $u^\varepsilon$ converges locally uniformly to the unique viscosity solution $u_0^\varepsilon$ of (5) (in the sense of Definition 3.3).
The proof of convergence relies on the construction of good correctors [17, 27] in order to use them in the perturbed test function method introduced by evans [14]. The particular form of the limit equation (5) requires to construct two correctors, one for \( x \neq 0 \) and one for \( x = 0 \).

\(-\) If \( x \neq 0 \): we use the classical Ansatz
\[
\left. u^\varepsilon (t, x) = u^0 (t, x) + \varepsilon v \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right). \tag{23}
\]

In fact, using the fact that \( u^\varepsilon \) is a solution of (22) and denoting by \( \lambda = -u^0(t, x) \) and \( p = u^0(t, x) \), the expression (23) implies that for \( x \neq 0 \), a corrector \( v \) must verify: for all \( p \in [-k_0, 0] \), there exists a unique \( \lambda \in \mathbb{R} \), such that there exists a bounded solution \( v \) of
\[
\begin{cases}
M_p[v](x) \cdot |v_x + p| = \lambda, \\
v \text{ is } \mathbb{Z}^-\text{periodique},
\end{cases}
\tag{24}
\]
with
\[
M_p[U](x) = \int_{-\infty}^{+\infty} J(z) E (U(x + z) - U(x) + p \cdot z) \, dz - \frac{3}{2} V_{\text{max}}. \tag{25}
\]

**Proposition 1** (Homogenization left and right of the perturbation). Assume (A). Then for every \( p \in [-k_0, 0] \), there exists a unique \( \lambda \in \mathbb{R} \), such that there exists a bounded solution \( v \) of (24). Moreover, for \( p \in [-k_0, 0] \), we have \( \lambda = \overline{H}(p) \).

\(-\) If \( x = 0 \): in this case, formally, using Theorem 3.7 we assume that near zero,
\[
u^0(t, x) = u^0(t, 0) + \overline{p}_+ 1_{\{x > 0\}} + \overline{p}_--1_{\{x < 0\}}
\]
where \( \overline{p}_+ \) and \( \overline{p}_- \) are the two constants satisfying
\[
\begin{cases}
\overline{H}(\overline{p}_+) = \overline{H}^+ (\overline{p}_+) = \overline{A} \\
\overline{H}(\overline{p}_-) = \overline{H}^- (\overline{p}_-) = \overline{A}.
\end{cases}
\]

We then use the following Ansatz
\[
u^\varepsilon(t, x) = u^0(t, 0) + \varepsilon w \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right). \tag{26}
\]

Using the fact that \( u^\varepsilon \) and \( u^0 \) are viscosity solutions respectively of (22) and (5), the expression (26) implies that a corrector \( w \) must verify the following theorem: there exists a unique constant \( \lambda = \overline{A} \) such that \( w \) is solution of
\[
\begin{cases}
\left. w_t + M[w(t, \cdot)](x) \cdot \phi(t, x) \cdot |w_x| = \lambda \right\} \\
\text{for } (t, x) \in \mathbb{R} \times \mathbb{R}.
\end{cases}
\tag{27}
\]

and such that \( w^\varepsilon(t, x) = \varepsilon w \left( \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right) \) satisfies locally uniformly:
\[
\left. w^\varepsilon(t, x) \right\} \xrightarrow{\varepsilon \to 0} W(x) = \overline{p}_+ 1_{\{x > 0\}} + \overline{p}_- 1_{\{x < 0\}}
\]

**Theorem 2.4** (Existence of a global corrector for the junction). Assume (A).

\( i \) (General properties) There exists a constant \( \overline{A} \in [H_0, 0] \) such that there exists a solution \( w \) of (27) with \( \lambda = \overline{A} \) and such that there exists a constant \( C \) and a globally Lipschitz continuous function \( m \) such that for all \( x \in \mathbb{R} \),
\[
|w(t, x) - m(x)| \leq C. \tag{28}
\]
ii) (Bound from below at infinity) If $\bar{A} > H_0$, then there exists a $\gamma_0$ such that for every $\gamma \in (0, \gamma_0)$, we have

$$w(t, x + h) - w(t, x) \geq (\bar{p}_+ - \gamma)h - C \quad \text{for } x \geq r \text{ and } h \geq 0$$

and

$$w(t, x - h) - w(t, x) \geq (-\bar{p}_- - \gamma)h - C \quad \text{for } x \leq -r \text{ and } h \geq 0.$$  

iii) (Rescaling $w$) For $\varepsilon > 0$, we set

$$w^\varepsilon(t, x) = \varepsilon w\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right),$$

then (along a subsequence $\varepsilon_n \to 0$) we have that $w^\varepsilon$ converges locally uniformly towards a function $W = W(x)$ which satisfies

$$\left\{ \begin{array}{ll} |W(x) - W(y)| & \leq C|x - y| \quad \text{for all } x, y \in \mathbb{R}, \\ \bar{H}(W_x) &= \bar{A} \quad \text{for all } x \in \mathbb{R} \setminus \{0\}, \end{array} \right.$$  

In particular, we have (with $W(0) = 0$)

$$W(x) = \bar{p}_+ x 1_{\{x > 0\}} + \bar{p}_- x 1_{\{x < 0\}}.$$  

iv) (Uniqueness of the flux limiter $\bar{A}$) We define the following set of functions

$$\mathcal{S} = \{ w \text{ s.t. } \exists m \in \text{Lip}(\mathbb{R}) \text{ and } C \geq 0 \text{ s.t. } ||w - m||_{L^\infty(\mathbb{R})} \leq C \}.$$  

Then we have

$$\bar{A} = \inf \{ \lambda \in \mathbb{R} : \exists w \in \mathcal{S} \text{ solution of (27)} \}.$$  

v) (Monotonicity of the flux-limiter $\bar{A}$) Let $\phi_1, \phi_2 : \mathbb{R}^+ \times \mathbb{R} \to [0, 1]$ be two functions satisfying (A6). Let $\bar{A}_1$ and $\bar{A}_2$ be their respective flux limiters given by Theorem 2.3. If, for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, we have

$$\phi_1(t, x) \leq \phi_2(t, x),$$

then

$$\bar{A}_1 \geq \bar{A}_2.$$  

3) Link between the system of ODEs and the PDE: finally, the main result of this work (Theorem 2.1) is obtained by the comparison principle for equation (22). In fact, if $u_0(x) = -\frac{x}{h_1}1_{\{x \leq 0\}} - \frac{x}{h_2}1_{\{x > 0\}}$ with $h_1, h_2 \geq h_0$, we know from step 1 that $\rho^\varepsilon$ is a solution of (17). Moreover, we can prove that

$$|\rho^\varepsilon(0, x) - u_0(x)| \leq f(\varepsilon)$$

with $f(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence, using the comparison principle for equation (22), we obtain that

$$u^\varepsilon(t, x) - f(\varepsilon) \leq \rho^\varepsilon(t, x) \leq u^\varepsilon(t, x) + f(\varepsilon)$$

where $u^\varepsilon$ is the unique viscosity solution of (22) with $u_0(x) = -\frac{x}{h_1}1_{\{x \leq 0\}} - \frac{x}{h_2}1_{\{x > 0\}}$. Passing to the limit as $\varepsilon \to 0$ in (33) and using step ii), we obtain our result.
2.5. **Notations.** To each operator $M_p$ (resp. $M$), we associate the operator $\tilde{M}_p$ (resp. $\tilde{M}$) which is defined in the same way except that the function $E$ is replaced by the function $\tilde{E}$, defined by

$$\tilde{E}(z) = \begin{cases} 
0 & \text{if } z > 0 \\
1/2 & \text{if } -1 < z \leq 0 \\
3/2 & \text{if } z \leq -1.
\end{cases}$$

(34)  

**Remark 6.** Using the fact that $E$ and $V$ are bounded, we get that for every function $U$ and every $x \in \mathbb{R}$, we have

$$-M_0 = -\frac{3}{2} V_{\max} \leq M[U](x) \leq 0.$$

(35)  

We also use the following notations for the upper and lower semi-continuous envelopes of a locally bounded function $u$:

$$u^*(t, x) = \limsup_{s \to t, y \to x} u(s, y) \quad \text{and} \quad u_*(t, x) = \liminf_{s \to t, y \to x} u(s, y).$$

2.6. **Organization of the article.** Section 3 contains the definition of the viscosity solutions for the problems we consider in the entire article and it also contains some results for those problems. Section 4 contains the proof of the point 1) of Subsection 2.4. Point 2 of Subsection 2.4 is divided into two sections: in Section 5, we construct correctors and in Section 6 we prove the convergence result. Finally, Section 7 contains the proof of the main Theorem 2.1 i.e. point 3) of Subsection 2.4.

3. Viscosity solutions for (17) and (5).

3.1. **Definitions.** In order to give a general definition for all the non-local problems we consider, we will give the definition for the following equation, with $p \in \mathbb{R}$, for all $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$\begin{cases}
\begin{align*}
&u_t + \psi(x) \cdot M_p[u(t, \cdot)](x) \cdot \phi(t, x) \cdot |p + u_x| + (1 - \psi(x)) \cdot \Pi(u_x) = 0, \\
&u(0, x) = u_0(x),
\end{align*}
\end{cases}$$

(36)  

with $\psi : \mathbb{R} \to [0, 1]$ a Lipschitz continuous function.

**Definition 3.1 (Viscosity solutions for (36)).** Let $T > 0$. An upper semi-continuous function (resp. lower semi-continuous) $u : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a viscosity subsolution (resp. super-solution) of (36) on $[0, T] \times \mathbb{R}$, if $u(0, x) \leq u_0(x)$ (resp. $u(0, x) \geq u_0(x)$) and for all $(t, x) \in (0, T) \times \mathbb{R}$ and for all $\varphi \in C^2([0, T] \times \mathbb{R})$ such that $u - \varphi$ reaches a maximum (resp. a minimum) at the point $(t, x)$, we have

$$\varphi_t(t, y) + \psi(x) \cdot \phi(t, x) \cdot M_p[u(t, \cdot)](x) \cdot |p + \varphi_x(t, x)| + (1 - \psi(x)) \Pi(\varphi_x(t, x)) \leq 0$$

(resp.

$$\varphi_t(t, x) + \psi(x) \cdot \phi(t, x) \cdot \tilde{M}_p[u(t, \cdot)](x) \cdot |p + \varphi_x(t, x)| + (1 - \psi(x)) \Pi(\varphi_x(t, x)) \geq 0.$$  

We say that a function $u$ is a viscosity solution of (36) if $u^*$ and $u_*$ are respectively a sub-solution and a super-solution of (36).

**Remark 7.** We use this definition in order to have a stability result for the non-local term. We refer to [39] for such kind of definition and to [18, Proposition 4.2] for the corresponding stability result.
Definition 3.2 (Class of test functions for (5)). We denote by \(J_\infty := (0, +\infty) \times \mathbb{R}\), \(J_\infty^+ := (0, +\infty) \times [0, +\infty)\) and \(J_\infty^- := (0, \infty) \times (-\infty, 0]\). We define a class of test functions on \(J_\infty\) by
\[C^1(J_\infty) = \{\varphi \in C(J_\infty), \text{ the restriction of } \varphi \text{ to } J_\infty^+ \text{ and to } J_\infty^- \text{ is } C^1\}.

Definition 3.3 (Viscosity solutions for (5)). Let \(H\) be given by (2) and \(A \in \mathbb{R}\). An upper semi-continuous (resp. lower semi-continuous) function \(u: [0, +\infty) \times \mathbb{R} \to \mathbb{R}\) is a viscosity sub-solution (resp. super-solution) of (5) if \(u(0, x) \leq u_0(x)\) (resp. \(u(0, x) \geq u_0(x)\)) and for all \((t, x) \in J_\infty\) and for all \(\varphi \in C^1(J_\infty)\) such that \(u \leq \varphi\) (resp. \(u \geq \varphi\)) in a neighbourhood of \((t, x) \in J_\infty\) and \(u(t, x) = \varphi(t, x)\), we have
\[
\varphi_t(t, x) + H(\varphi_x(t, x)) \leq 0 \quad \text{if } x \neq 0,
\]
\[
\varphi_t(t, x) + F(A)(\varphi_x(t, 0^-), \varphi_x(t, 0^+)) \leq 0 \quad \text{if } x = 0.
\]
We say that a function \(u\) is a viscosity solution of (5) if \(u^*\) and \(u^-\) are respectively a sub-solution and a super-solution of (5). We refer to this solution as an \(A\)-flux limited solution.

3.2. Results for viscosity solutions of (36).

Proposition 2 (Comparison principle for (36)). Assume (A0) and (A). Let \(u\) be a sub-solution of (36) and \(v\) be a super-solution of (36). Let us also assume that there exists a constant \(K > 0\) such that for all \((t, x) \in [0, T] \times \mathbb{R}\),
\[u(t, x) \leq u_0(x) + Kt \quad \text{and} \quad -v(t, x) \leq -u_0(x) + Kt.\] (37)
Then we have \(u(t, x) \leq v(t, x)\) for all \((t, x) \in [0, T] \times \mathbb{R}\).

Proof. The only difficulty in proving the comparison principle comes from the non-local term, but in our case the proof is similar to the proof of [18, Theorem 4.4] and we skip it.

We now give a comparison principle on bounded sets, to do this, we define for a given point \((t_0, x_0) \in (0, T) \times \mathbb{R}\) and for \(\tau, R > 0\), the set
\[Q_{\tau, R}(t_0, x_0) = (t_0 - \tau, t_0 + \tau) \times (x_0 - R, x_0 + R).
\]

Theorem 3.4 (Comparison principle on bounded sets for (36)). Assume (A). Let \(u\) be a sub-solution of (36) and let \(v\) be a super-solution of (36) on the open set \(Q_{\tau, R} \subset (0, T) \times \mathbb{R}\). We assume that \(u\) (resp. \(v\)) is upper semi-continuous (resp. lower semi-continuous) on \(Q_{\tau, R}\). Also assume that
\[u \leq v \quad \text{outside } Q_{\tau, R},\]
then
\[u \leq v \quad \text{on } Q_{\tau, R}.
\]

Proof. The proof of this theorem is similar to the one of Proposition 2, so we skip it.
Lemma 3.5 (Existence of barriers for (36)). Assume (A0) and (A). There exists a constant $K_1 > 0$ such that
\[ u^+(t, x) = K_1 t + u_0(x) \quad \text{and} \quad u^-(t, x) = u_0(x), \]
are respectively super and sub-solutions of (36).

Proof. We define $K_1 = M_0 \cdot (|p| + k_0) + |H_0|$. Let us prove that $u^+$ is a super-solution of (36). Using assumption (A0) and the form of the non-local operator and of $\overline{H}$, we have
\[
\phi(t, x) \psi(x) M_p[u_0](x) |p + (u_0)_x| + (1 - \psi(x)) \overline{H}((u_0)_x) \geq -M_0 |p + (u_0)_x| + H_0 \\
\geq -M_0 (|p| + k_0) - |H_0| \\
= -K_1,
\]
where we used (35) and (4). The proof for $u^-$ is simpler, it uses (35) and (4),
\[
\phi(t, x) \psi(x) M_p[u_0](x) \cdot |p + (u_0)_x| + (1 - \psi(x)) \cdot \overline{H}((u_0)_x) \leq 0.
\]
\[ \square \]

Applying Perron’s method (see [27, Proof of Theorem 6], [4] or [23] to see how to apply Perron’s method for problems with non-local terms), joint to the comparison principle, we obtain the following result.

Theorem 3.6 (Existence and uniqueness of viscosity solutions for (36)). Assume (A0) and (A). Then, there exists a unique continuous solution $u$ of (36) which satisfies (for some constant $K_1$)
\[ u_0(x) \leq u(t, x) \leq u_0(x) + K_1 t \]

3.3. Results for viscosity solutions of (5). Now we recall an equivalent definition (see [25, Theorem 2.5]) for sub and super solutions at the junction. We will also consider the following problem,
\[ u_t + \overline{H}(u_x) = 0 \quad \text{for } t \in (0, T) \text{ and } x \in \mathbb{R} \setminus \{0\}. \tag{38} \]

Theorem 3.7 (Equivalent definition for sub/super-solutions). Let $\overline{H}$ given by (2) and consider $A \in [H_0, +\infty)$ with $H_0$ defined in (4). Given arbitrary solutions $p^A_\pm \in \mathbb{R}$ of
\[ \Pi \left( p^A_\pm \right) = \Pi^+ \left( p^A_\pm \right) = A = \Pi^- \left( p^A_\pm \right) = \Pi \left( p^A_\pm \right), \tag{39} \]
let us fix any time independent test function $\phi^0(x)$ satisfying
\[
\phi^0_\pm(0^\pm) = p^A_\pm.
\]
Given a function $u : (0, T) \times \mathbb{R} \to \mathbb{R}$, the following properties hold true.

i) If $u$ is an upper semi-continuous sub-solution of (38) and satisfies
\[ u(t, 0) = \limsup_{(s, y) \to (t, 0), y \in \mathbb{R}^*_+} u(s, y) = \limsup_{(s, y) \to (t, 0), y \in \mathbb{R}^*_+} u(s, y), \tag{40} \]
then $u$ is a $H_0$-flux limited sub-solution.

ii) Given $A > H_0$ and $t_0 \in (0, T)$, if $u$ is an upper semi-continuous sub-solution of (38) and satisfies (40) and if for any test function $\varphi$ touching $u$ from above at $(t_0, 0)$ with
\[ \varphi(t, x) = \psi(t) + \phi^0(x), \tag{41} \]
for some $\psi \in C^1(0, +\infty)$, we have
\[ \varphi_t + F_A (\varphi_x(t_0, 0^-), \varphi_x(t_0, 0^+)) \leq 0 \quad \text{at } (t_0, 0), \]
then \( u \) is an \( A \)-flux limited sub-solution at \((t_0,0)\).

iii) Given \( t_0 \in (0,T) \), if \( u \) is a lower semi-continuous super-solution of (38) and if for any test function \( \varphi \) satisfying (41) touching \( u \) from above at \((t_0,0)\) we have
\[
\varphi_t + F_A (\varphi_x(t_0,0^-), \varphi_x(t_0,0^+)) \geq 0 \quad \text{at} \quad (t_0,0),
\]
then \( u \) is an \( A \)-flux limited super-solution at \((t_0,0)\).

Proof. The proof of Theorem 3.7 can be founded in [25, Theorem 2.5].

3.4. Control of the oscillations for (22).

Theorem 3.8 (Control of the oscillations). Let \( T > 0 \). Assume (A0)-(A) and let \( u \) be a solution of (22), with \( \varepsilon = 1 \). Then for all \( x,y \in \mathbb{R}, x \geq y \) and for all \( t \in [0,T] \), we have
\[
-k_0(x-y) - 1 \leq u(t,x) - u(t,y) \leq 0,
\]
with \( k_0 \) defined in (21).

Proof. In this proof we used the barriers given by Lemma 3.5 (with \( p = 0 \) and \( \psi \equiv 1 \)), which means that the solution \( u \) of (22) with \( \varepsilon = 1 \) satisfies for all \( (t,x) \in [0,\infty) \times \mathbb{R} \),
\[
0 \leq u(t,x) - u_0(x) \leq M_0 k_0 t.
\]
In the rest of the proof we will use the following notation:
\[
\Omega = \{(t,x,y) \in [0,T) \times \mathbb{R}^2 \text{ s.t. } x \geq y\}.
\]

Proof of the upper inequality for the control of the space oscillations. We introduce,
\[
N = \sup_{(t,x,y) \in \Omega} \{u(t,x) - u(t,y)\}.
\]
We want to prove that \( N \leq 0 \). We argue by contradiction and assume that \( N > 0 \).

Step 1: the test function. For \( \eta, \alpha > 0 \), small parameters, we define
\[
\varphi(t,x,y) = u(t,x) - u(t,y) - \frac{\eta}{T - t} - \alpha x^2 - \alpha y^2.
\]
Using (43), we have that
\[
\varphi(t,x,y) \leq u_0(x) - u_0(y) + 2M_0 k_0 T - \alpha(x^2 + y^2) \leq -\alpha(x^2 + y^2) + 2M_0 k_0 T,
\]
where we used assumption (A0) for the second inequality. Therefore we have
\[
\lim_{\max|x|,|y| \to +\infty} \varphi(t,x,y) = -\infty.
\]
Since \( \varphi \) is upper-semi continuous, it reaches a maximum at a point that we denote by \((\bar{t}, \bar{x}, \bar{y}) \in \Omega \). Classically we have for \( \eta \) and \( \alpha \) small enough,
\[
\begin{cases}
0 < \frac{N}{2} \leq \varphi(\bar{t}, \bar{x}, \bar{y}), \\
\alpha |\bar{x}|, \alpha |\bar{y}| \to 0 \text{ as } \alpha \to 0.
\end{cases}
\]

Step 2: \( \bar{t} > 0 \) and \( \bar{x} > \bar{y} \). By contradiction, assume first that \( \bar{t} = 0 \). Then we have
\[
\frac{\eta}{T} < u_0(\bar{x}) - u_0(\bar{y}) \leq 0,
\]
where we used that \( u_0 \) is non-increasing, and we get a contradiction. The fact that \( \bar{x} > \bar{y} \), comes directly from the fact that \( \varphi(\bar{t}, \bar{x}, \bar{y}) > 0 \).
Step 3: utilisation of the equation. By doing a duplication of the time variable and passing to the limit in this duplication parameter, we get that
\[
\frac{\eta}{(T - t)^2} \leq \dot{M}[u(\bar{t}, \cdot)](\bar{y}) \cdot |2\alpha \bar{y}| \cdot \phi(\bar{t}, \bar{y}) - M[u(\bar{t}, \cdot)](\bar{x}) \cdot \phi(\bar{t}, \bar{x}) \cdot |2\alpha \bar{x}|
\leq 2M_0 \cdot \alpha(|\bar{x}| + |\bar{y}|).
\]
Passing to the limit as \(\alpha\) goes to 0, we obtain a contradiction.
Proof of the lower inequality for the control of the space oscillations. Let us introduce,
\[
N = \sup_{(t, x, y) \in \Omega} \left\{ u(t, y) - u(t, x) - 1 - k_0(x - y) \right\}.
\]
We want to prove that \(N \leq 0\). We argue by contradiction and assume that \(N > 0\).
Step 1: the test function. For \(\alpha, \eta > 0\), small parameters we consider the function
\[
\varphi(t, x, y) = u(t, y) - u(t, x) - 1 - k_0(x - y) - \alpha(x^2 + y^2) - \frac{\eta}{T - t}.
\]
We have that
\[
\varphi(t, x, y) \leq u_0(y) - u_0(\bar{x}) - \alpha(x^2 + y^2) + 2M_0k_0T - k_0(x - y) - 1
\leq -\alpha(x^2 + y^2) + 2M_0k_0T.
\]
Therefore, we have
\[
\lim_{|x|, |y| \to +\infty} \varphi(t, x, y) = -\infty.
\]
Using the fact that \(\varphi\) is upper-semi continuous we deduce that \(\varphi\) reaches a maximum at a finite point that we denote \((\bar{t}, \bar{x}, \bar{y}) \in \Omega\). Classically we have for \(\eta\) and \(\alpha\) small enough,
\[
\left\{ \begin{array}{l}
0 < \frac{N}{2} \leq \varphi(\bar{t}, \bar{x}, \bar{y}), \\
\alpha |\bar{x}|, \alpha |\bar{y}| \to 0 \text{ as } \alpha \to 0.
\end{array} \right.
\]
Step 2: \(\bar{t} > 0\) and \(\bar{x} > \bar{y}\). By contradiction, assume that \(\bar{t} = 0\). Using the fact that \(\varphi(\bar{t}, \bar{x}, \bar{y}) > 0\) and (A0), we have
\[
\frac{\eta}{T} < u(0, \bar{y}) - u(0, \bar{x}) - k_0(\bar{x} - \bar{y}) - 1 \leq -1,
\]
which is a contradiction. Hence \(\bar{t} > 0\). Using that \(\varphi(\bar{t}, \bar{x}, \bar{y}) > 0\), we also deduce that \(\bar{x} > \bar{y}\).
Step 3: Utilisation of the equation. By duplicating the time variable and passing to the limit we have that there exists two real numbers \(a, b\), such that \((a, -k_0 + 2\alpha \bar{y}) \in \overline{B}^+ u(\bar{t}, \bar{y}), (b, -k_0 + 2\alpha \bar{x}) \in \overline{B}^- u(\bar{t}, \bar{x})\) and
\[
a - b = \frac{\eta}{(T - t)^2}.
\]  \(\text{(44)}\)
Using that \(u\) is a sub-solution of (22) (with \(\varepsilon = 1\)), we get
\[
a + M[u(\bar{t}, \cdot)](\bar{y}) \cdot \phi(\bar{t}, \bar{y}) \cdot | - k_0 + 2\alpha \bar{y}| \leq 0.
\]  \(\text{(45)}\)
We claim that
\[
M[u(\bar{t}, \cdot)](\bar{y}) = \int_{\mathbb{R}} J(z)E(u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}))dz - \frac{3}{2}V_{\text{max}} = 0.
\]
Indeed, let \(z \in (h_0, h_{\text{max}}]\). If \(\bar{y} + z \geq \bar{x}\), using that \(u\) is non-increasing in space, we get
\[
u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) \leq u(\bar{t}, \bar{x}) - u(\bar{t}, \bar{y}) \leq -k_0(\bar{x} - \bar{y}) - 1 < -1.
\]
If $\bar{y} + z < \bar{x}$, using the fact that $\varphi(\tilde{t}, \bar{x}, \bar{y} + z) \leq \varphi(\tilde{t}, \bar{x}, \bar{y})$, for $\alpha$ small enough, we obtain
\[ u(\tilde{t}, \bar{y} + z) - u(\tilde{t}, \bar{y}) \leq -k_0 z + \alpha (2z\bar{y} + z^2) \leq -k_0 z + \alpha (2h_{max}\bar{y} + h_{max}^2) < -1. \]
This implies that we have for all $z \in (h_0, h_{max}]$,
\[ E(u(\tilde{t}, \bar{y} + z) - u(\tilde{t}, \bar{y})) = \frac{3}{2}. \]
Injecting this in the non-local term, we deduce the claim.
Finally, the fact that $u_t \geq 0$ implies that $a, b \geq 0$. Therefore, inequality (45) implies
\[ a = 0. \]
Finally, using (44), we obtain
\[ \frac{\eta}{I^2} \leq 0, \]
which is a contradiction. This ends the proof.

4. Proof of Theorem 2.2.

Proof of Theorem 2.2. Theorem 2.2 is a consequence of the following lemma.

Lemma 4.1 (Link between the velocities). Assume (A). Let $((U_i)_i)$ be the solution of (1) with
\[ U_{i+1}(0) - U_i(0) > h_0. \]
Then we have
\[ \dot{U}_i(t) = -M[u(t, \cdot)](U_i(t)) \cdot \phi(t, U_i(t)), \]
where $E$ and $J$ are defined in (16) and $u(t, x)$ is a continuous function such that
\[ \begin{cases} 
  u(t, x) = \rho_*(t, x) = \rho(t, x) & \text{for } x = U_i(t), \ i \in \mathbb{Z}, \\
  u \text{ is decreasing in } x, 
\end{cases} \]
with $\rho$ defined in (9) (with $\varepsilon = 1$).

Proof. We drop the time dependence to simplify the presentation. Let $j \in \mathbb{Z}$. Using the fact that $u(U_j) = -(i + 1)$ and (48), we have for all $z \in [0, +\infty)$,
\[ \begin{cases} 
  0 \geq u(U_i + z) - u(U_i) > u(U_{i+1}) - u(U_i) = -1 & \text{if } z \in [0, U_{i+1} - U_i) \\
  -1 \geq u(U_i + z) - u(U_i) & \text{if } z \in [U_{i+1} - U_i, +\infty). 
\end{cases} \]
Given that $u$ is continuous, this implies that
\[ M[u(U_i)] = \int_0^{U_{i+1} - U_i} \frac{1}{2} J(z)dz + \int_{U_{i+1} - U_i}^{+\infty} \frac{3}{2} J(z)dz - \frac{3}{2} V_{max} = -V(U_{i+1} - U_i). \]
Combining this result with (1), we obtain (47).

Noticing that because of (48), we have for $x = U_i(t), \ i \in \mathbb{Z}$,
\[ \dot{M}[\rho_*(t, \cdot)](x) = \dot{M}[u(t, \cdot)](x) = M[u(t, \cdot)](x), \]
and using Lemma 4.1, and Definition 3.1, we can see that $\rho_*$ is a discontinuous viscosity super-solution of (19). We obtain a similar result for $\rho^*$, therefore, $\rho$ is a discontinuous viscosity solution of (19).
We prove the converse. For the readers convenience we recall Proposition 4.8 from [18] that we will use later. The proof of this proposition remains almost the same in our case the only difference being the definition of the functions $E$ and $\tilde{E}$.

**Lemma 4.2.** Assume that $\theta : \mathbb{R} \to \mathbb{R}$ is a non-decreasing and upper semi-continuous (resp. lower semi-continuous). Assume also that
\[
\theta(v) - v \text{ is } 1-\text{periodic in } v.
\]
Assume that $\varepsilon = 1$ in (17). Consider also a sub-solution (resp. a super-solution) $u$ of (17). Then $\theta(u)$ is also a sub-solution (resp. a super-solution) of (17).

Using Lemma 4.2 we can conclude that $\rho^* = \lceil u \rceil$ (resp. $\rho^* = \lfloor u \rfloor$) is a viscosity super-solution (resp. sub-solution) of
\[
\partial_t \rho - \tilde{c}(t,x)\partial_x \rho = 0 \quad \text{with } \tilde{c}(t,x) = M[u(t,\cdot)](x) \cdot \phi(t,x) = \bar{M}[u(t,\cdot)](x) \cdot \phi(t,x).
\]
Using the fact that $u$ is decreasing in space, we define
\[
U_i(t) = \inf \{ x, u(t,x) \leq -(i+1) \} = (u(t,\cdot))^{-1}(-i-1)
\]
and we consider the functions $t \mapsto U_i(t)$. They are continuous because $u$ is decreasing in $x$ and is continuous in $(t,x)$.

We now prove that the functions $U_i$ are viscosity solutions of (1). Let $\varphi$ be a test function such that $\varphi(t) \leq U_i(t)$ and $\varphi(t_0) = U_i(t_0)$. Let us now define $\hat{\varphi}(t,x) = -(i+1) + \varphi(t) - x$. It satisfies
\[
\hat{\varphi}(t_0,U_i(t_0)) = \rho^*(t_0,U_i(t_0)),
\]
and
\[
\hat{\varphi}(t,x) \leq \rho^*(t,x) \quad \text{for } U_i(t) - 1 < x < U_{i+1}(t).
\]
This implies that
\[
\varphi_i(t_0) + \hat{c}(t_0,U_i(t_0)) \geq 0
\]
\[
\Leftrightarrow \varphi_i(t_0) \geq -\hat{c}(t_0,U_i(t_0)) = -\tilde{c}_i(t_0) = V(U_{i+1}(t_0) - U_i(t_0)) \cdot \hat{\varphi}(t,U_i(t_0)).
\]
This proves that $U_i$ are viscosity super-solutions of (1). The proof for sub-solutions is similar and we skip it. Moreover, since $\tilde{c}_i$ is continuous, we deduce that $U_i \in C^1$ and it is therefore a classical solution of (1).

5. **Proof of Proposition 1 and Theorem 2.4.** In this section, we will construct the correctors far and near the junction point.

5.1. **Proof of Proposition 1.**

*Proof of Proposition 1.* Let us prove that $v = 0$ is an obvious solution of (24) with $\lambda = \overline{H}(p)$, for $p \in [-k_0,0]$. First, let us notice that if $p = 0$ the result is obvious since by definition of $\overline{H}$, we have $\overline{H}(0) = 0$ and $M_0[0](x)$ is finite (for all $x \in \mathbb{R}$) by
definition (see (35)). Let us now consider \( p > 0 \), we have for all \( x \in \mathbb{R} \),
\[
M_p[0](x) = \int_{-\infty}^{+\infty} J(z) E(pz)dz - \frac{3}{2} V_{\max}
\]
\[
= \int_{0}^{+\infty} J(z) E(pz)dz - \frac{3}{2} V_{\max}
\]
\[
= \int_{0}^{1/p} \frac{1}{2} J(z)dz + \int_{-1/p}^{+\infty} \frac{3}{2} J(z)dz - \frac{3}{2} V_{\max}
\]
\[
= \frac{1}{2} \left( V\left( -\frac{1}{p} \right) - V(0) \right) + \frac{3}{2} \left( \lim_{h \to +\infty} V(h) \right) - V\left( -\frac{1}{p} \right) - \frac{3}{2} V_{\max}
\]
\[
= - V\left( -\frac{1}{p} \right),
\]
where we have used assumption (A3) for the second line, the definition of \( E \) and \( J \) (see (16)) for the third and fourth line. Finally, using this result and the definition of \( \mathcal{H} \), we notice that \( \mathcal{H}(p) = M_p[0](x)|p| = \lambda \). The uniqueness of \( \lambda \) is classical (see for instance [15, Proof of Proposition 4.6]) so we skip it.

5.2. Proof of Theorem 2.4. This subsection contains the proof of Theorem 2.4. To do this, we will construct correctors on truncated domains and then pass to the limit as the size of the domain goes to infinity. This idea comes from [1] and [20]. The difficulty in our non-local case is that it is non-standard to well define boundary conditions. In order to overcome this difficulty, we will replace the non-local operator by a local one near the boundary. More precisely, for \( l \in (r, +\infty) \), \( r \ll l \) and \( r \leq R \ll l \), we want to find \( \lambda_{l,R} \), such that there exists a solution \( u^{l,R} \) of
\[
\begin{cases}
  w^{l,R}_t + G_R(t, x, [w^{l,R}_t, R], [w^{l,R}_x, R]) = \lambda_{l,R} & \text{if } (t, x) \in \mathbb{R} \times (-l, l) \\
  w^{l,R}_t + \mathcal{H}(w^{l,R}_x) = \lambda_{l,R} & \text{if } (t, x) \in \mathbb{R} \times \{l\} \\
  w^{l,R} & \text{is 1-periodic in } t.
\end{cases}
\]
with
\[
G_R(t, x, [U], q) = \psi_R(x)\phi(t, x) \cdot M[U](x) \cdot |q| + (1 - \psi_R(x)) \cdot \mathcal{H}(q),
\]
and \( \psi_R \in C^\infty, \psi_R : \mathbb{R} \to [0, 1] \), with
\[
\psi_R \equiv \begin{cases}
  1 & \text{on } [-R, R] \\
  0 & \text{outside } [-R - 1, R + 1],
\end{cases}
\] and \( \psi_R(x) < 1 \forall x \notin [-R, R] \).

To \( G_R \), we associate \( \tilde{G}_R \) which is defined in the same way but the operator \( M \) is replaced by \( \tilde{M} \).

Remark 8. The operator \( G_R \) is used to have a local operator near the boundary and then to well define the boundary conditions.

5.2.1. Comparison principle for a truncated problem.

Proposition 3 (Comparison principle on truncated domains). Let us consider the following problem for \( r < l_1 < l_2 \) and \( \lambda \in \mathbb{R} \), with and \( l_2 >> R \).
\[
\begin{cases}
  v_t + G_R(t, x, [v(t, \cdot)], v_x) \geq \lambda & \text{for } (t, x) \in \mathbb{R} \times (l_1, l_2) \\
  v_t + \mathcal{H}(v_x) \geq \lambda & \text{for } (t, x) \in \mathbb{R} \times \{l_2\} \\
  v(t, x) \geq U_0(t) & \text{for } (t, x) \in \mathbb{R} \times \{l_1\} \\
v \text{ is 1-periodic in } t,
\end{cases}
\]
where $U_0$ is continuous, and for $\varepsilon_0 > 0$
\[
\begin{align*}
  u_t + G_R(t,x,[u(t,\cdot),u_x]) &\leq \lambda - \varepsilon_0 & \text{for } (t,x) \in \mathbb{R} \times (l_1,l_2) \\
  u_t + \overline{H}(u_x) &\leq \lambda - \varepsilon_0 & \text{for } (t,x) \in \mathbb{R} \times \{l_2\} \\
  u(t,x) &\leq U_0(t) & \text{for } (t,x) \in \mathbb{R} \times \{l_1\} \\
  u \text{ is 1-periodic in } t,
\end{align*}
\]  

(53)

Then we have $u \leq v$ in $\mathbb{R} \times [l_1,l_2]$.

**Proof.** The only difficulty in proving this result is the comparison at the boundary $\{l_2\}$. However, for $x$ close to $l_2$, the function $G_R$ is actually the effective Hamiltonian $H$. Therefore, we can proceed as in the proof of [20, Proposition 4.1] and so we skip the proof.

**Remark 9.** We have a similar result for $l_1 < l_2 < -r$ and if for all $x \in [l_2,l_2+h_{max}]$, $u(t,x) \leq v(t,x)$ and the following conditions are imposed at $x = l_1$:
\[
\begin{align*}
  v_t + \overline{H}^{-}(v_x) &\geq \lambda & \text{for } x = l_1, \\
  u_t + \overline{H}^{-}(u_x) &\leq \lambda - \varepsilon_0 & \text{for } x = l_1.
\end{align*}
\]

5.2.2. Existence of correctors on a truncated domain.

**Proposition 4.** (Existence of correctors on truncated domains) There exists a unique $\lambda_{l,R} \in \mathbb{R}$ such that there exists a solution $u^{l,R}$ of (49). Moreover, there exists a constant $C$ (depending only on $k_0$, $V_{max}$ and $|H_0|$), and a Lipschitz continuous function $m^{l,R}$, such that
\[
\begin{align*}
  H_0 &\leq \lambda_{l,R} \leq 0, \\
  |m^{l,R}(x) - m^{l,R}(y)| &\leq C|x - y| & \text{for } x,y \in [-l,l], \\
  |u^{l,R}(t,x) - m^{l,R}(x)| &\leq C & \text{for } (t,x) \in \mathbb{R} \times [-l,l],
\end{align*}
\]

(54)

with $H_0 = \min \overline{H}$.

**Proof.** In order to construct a corrector on the truncated domain, we will classically consider the approximated problem
\[
\begin{align*}
  \delta v^\delta + v^\delta_t + G_R(t,x,[v^\delta(t,\cdot),v^\delta_x]) = 0 & \text{for } (t,x) \in \mathbb{R} \times (-l,l) \\
  \delta v^\delta + v^\delta_t + \overline{H}^-(v^\delta_x) = 0 & \text{for } (t,x) \in \mathbb{R} \times \{-l\} \\
  \delta v^\delta + v^\delta_t + \overline{H}^+(v^\delta_x) = 0 & \text{for } (t,x) \in \mathbb{R} \times \{l\} \\
  v^\delta \text{ is 1-periodic in } t
\end{align*}
\]

(55)

Step 1: construction of barriers. Using that 0 and $\delta^{-1}C_0$ are respectively sub and super-solution of (55) with $C_0 = |H_0|$, the comparison principle and Perron’s method for 1-periodic solutions, we deduce that there exists a continuous viscosity solution, $v^\delta$ of (55) which satisfies
\[
0 \leq v^\delta \leq \frac{C_0}{\delta}.
\]

(56)

Step 2: control of the space oscillations of $v^\delta$.

**Lemma 5.1.** The function $v^\delta$ satisfies for all $t \in \mathbb{R}$ and for all $x,y \in [-l,l]$, $x \geq y$,
\[
-k_0(x - y) - 1 \leq v^\delta(t,x) - v^\delta(t,y) \leq 0,
\]

with $k_0$ defined in (A0).

**Proof of Lemma 5.1.** In the rest of the proof we will use the following notation,
\[
\Omega = \{(t,x,y) \in \mathbb{R} \times [-l,l]^2 \text{ such that } x \geq y\}.
\]
Step 2.1: proof of the upper inequality. Let \( \varepsilon > 0 \). We want to prove that
\[
N = \sup_{(t,x,y) \in \Omega} \left\{ v^{\delta}(t,x) - v^{\delta}(t,y) \right\} \leq 0.
\]
We argue by contradiction and assume that \( N > 0 \). We then consider
\[
N_{\nu} = \sup_{t,s \in \mathbb{R}, x \geq y} \left\{ v^{\delta}(t,x) - v^{\delta}(s,y) - \frac{(t-s)^2}{2\nu} \right\}.
\]
Since \( N > 0 \), we deduce that \( N_{\nu} > 0 \). Remark also that we consider the supremum of a continuous, 1-periodic in \( t \) function, so we deduce that \( N_{\nu} \) is reached at a point \((\bar{t}, \bar{s}, \bar{x}, \bar{y})\). Given that \( N_{\nu} > 0 \), we deduce that \( \bar{x} \neq \bar{y} \) if \( \nu \) is small enough (classically we have that \( |t-s| \to 0 \) as \( \nu \to 0 \)). Therefore, we can use the viscosity inequalities for (55).

- If \((\bar{x}, \bar{y}) \in (-l,l)^2\), we have
  \[
  \begin{align*}
  \delta v^{\delta}(\bar{t}, \bar{x}) + \frac{\bar{t} - \bar{s}}{\nu} + G_R(\bar{x}, [v^{\delta}(\bar{t}, \cdot)], 0) & \leq 0 \\
  \delta v^{\delta}(\bar{s}, \bar{y}) + \frac{\bar{t} - \bar{s}}{\nu} + G_R(\bar{y}, [v^{\delta}(\bar{s}, \cdot)], 0) & \geq 0,
  \end{align*}
  \]
  combining these two inequalities with the fact that \( G_R(x, [U], 0) = 0 \), we obtain
  \[
  \delta (v^{\delta}(\bar{t}, \bar{x}) - v^{\delta}(\bar{s}, \bar{y})) \leq 0.
  \]
- If \( \bar{x} = l \) and \( \bar{y} \in (l, \bar{l}) \), similarly we obtain
  \[
  \delta (v^{\delta}(\bar{t}, \bar{x}) - v^{\delta}(\bar{s}, \bar{y})) \leq 0,
  \]
  where we have used the fact that \( \bar{H}^+(0) = 0 \).
- If \( \bar{x} \in (-l, l) \) and \( \bar{y} = -l \), we obtain
  \[
  \delta (v^{\delta}(\bar{t}, \bar{x}) - v^{\delta}(\bar{s}, \bar{y})) \leq H_0 \leq 0,
  \]
  where we used the fact that \( \bar{H}^-(0) = H_0 \).
- If \( \bar{x} = l \) and \( \bar{y} = -l \), we obtain
  \[
  \delta (v^{\delta}(\bar{t}, \bar{x}) - v^{\delta}(\bar{s}, \bar{y})) \leq H_0 \leq 0.
  \]
  For every value of \( \bar{x} \) and \( \bar{y} \) we obtain a contradiction, therefore we have \( N \leq 0 \).

Step 2.2: proof of the lower inequality. We want to prove that
\[
N = \sup_{(t,x,y) \in \Omega} \left\{ v^{\delta}(t,y) - v^{\delta}(t,x) - k_0(x-y) - 1 \right\} \leq 0.
\]
We argue by contradiction and assume that \( N > 0 \). We then consider
\[
N_{\nu} = \sup_{t,s \in \mathbb{R}, x \geq y} \left\{ v^{\delta}(t,y) - v^{\delta}(s,x) - k_0(x-y) - 1 - \frac{(t-s)^2}{2\nu} \right\}.
\]
Since \( N > 0 \), we get \( N_{\nu} > 0 \). Remark also that we consider the supremum of a continuous, 1-periodic in \( t \) and \( s \) function, so we deduce that \( N_{\nu} \) is reached at a point \((\bar{t}, \bar{s}, \bar{x}, \bar{y})\). Given that \( N_{\nu} > 0 \), we deduce that \( \bar{x} \neq \bar{y} \) if \( \nu \) is small enough (classically we have that \( |t-s| \to 0 \) as \( \nu \to 0 \)). Therefore, we can use the viscosity inequalities for (55).

**Case 1:** \( \bar{y} \in (-l, l) \). If \( \bar{y} \in (-l, l) \), we have
\[
\begin{align*}
\delta v^{\delta}(\bar{t}, \bar{y}) + \frac{\bar{t} - \bar{s}}{\nu} & + \psi_R(\bar{y}) M[v^{\delta}(\bar{t}, \cdot)](\bar{y}) \cdot \phi(\bar{t}, \bar{y}) \cdot | - k_0 | + (1 - \psi_R(\bar{y})) \bar{H}(-k_0) \leq 0.
\end{align*}
\]
We claim that \( M[v^\delta(\bar{t}, \cdot)](\bar{y}) = 0 \).

Indeed, for all \( z > h_0 \), if \( \bar{x} \geq \bar{y} + z \) using the fact that the maximum is reached for \( (\bar{t}, \bar{s}, \bar{x}, \bar{y}) \), we deduce that

\[
v^\delta(\bar{t}, \bar{y} + z) - v^\delta(\bar{t}, \bar{y}) \leq -k_0 z < -1.
\]

On the contrary, if \( \bar{x} \leq \bar{y} + z \), using the fact that \( v^\delta \) is continuous, non-increasing in space, and the fact that \( v^\delta(\bar{s}, \bar{x}) - v^\delta(\bar{t}, \bar{y}) < -1 \), we deduce that

\[
v^\delta(\bar{t}, \bar{y} + z) - v^\delta(\bar{t}, \bar{y}) - \nu \leq v^\delta(\bar{t}, \bar{x}) - v^\delta(\bar{t}, \bar{y}) < -1.
\]

We can therefore, conclude that for all \( z \in (h_0, +\infty) \), \( E(v^\delta(\bar{t}, \bar{y} + z) - v^\delta(\bar{t}, \bar{y})) = -\frac{3}{2} \)

and so we get \( M[v^\delta(\bar{t}, \cdot)](\bar{y}) = 0 \). Using also that \( \mathcal{H}(-k_0) = 0 \), equation (57) becomes

\[
\delta v^\delta(\bar{t}, \bar{y}) + \frac{\bar{t} - \bar{s}}{\nu} \leq 0.
\]

Moreover, whether \( \bar{x} \in (-l, l) \) or \( \bar{x} = l \), since the non-local operator is negative and \( H^+(-k_0) < 0 \), we have that

\[
-\delta v^\delta(\bar{s}, \bar{x}) - \frac{\bar{t} - \bar{s}}{\nu} \leq 0.
\]

We deduce that

\[
\delta \left( v^\delta(\bar{t}, \bar{y}) - v^\delta(\bar{s}, \bar{x}) \right) \leq 0,
\]

which is a contradiction.

**Case 2:** \( \bar{y} = -l \). In this situation, the viscosity inequality becomes

\[
\delta v^\delta(\bar{t}, \bar{y}) + \frac{\bar{t} - \bar{s}}{\nu} + \mathcal{H}(-k_0) \leq 0.
\]

Using the fact that \( \mathcal{H}(-k_0) = \mathcal{H}(-k_0) = 0 \), and as in the previous case, we obtain a contradiction. This ends the proof of the lemma.

Step 3: control of the time oscillations of \( v^\delta \).

**Lemma 5.2.** The function \( v^\delta \) satisfies for all \( x \in [-l, l] \) and for all \( t, s \in \mathbb{R} \),

\[
\left| v^\delta(t, x) - v^\delta(s, x) \right| \leq C_1
\]

with \( C_1 = \frac{3}{2} V_{max} k_0 + |H_0| + 1 \).

**Proof.** Since \( v^\delta \) is 1-periodic in \( t \), it is sufficient to show that for all \( x \in [-l, l] \) and for all \( t, s \in \mathbb{R} \) such that \( t \geq s \), we have that

\[
v^\delta(t, x) - v^\delta(s, x) \leq C_2(t - s) + 1.
\]

with \( C_2 = C_1 - 1 \). In order to prove that, we will fix \( x_0 \in (-l, l) \) and \( s_0 \in \mathbb{R} \), and we will prove that if \( t \geq s_0 \), then

\[
v^\delta(t, x_0) \leq v^\delta(s_0, x_0) + C_2(t - s_0) + 1.
\]

We define

\[
w^\delta(t, x) = v^\delta(s_0, x_0) + C_2(t - s_0) + k_0 |x - x_0| + 1.
\]
Finally, using the comparison principle on \([s_0, +\infty) \times [-l, l]\), we deduce that

\[ v^\delta (t, x) \leq w^\delta (t, x). \]

In particular, for \(x = x_0\), we obtain (59). We deduce that (58) is true even if \(x = \pm l\) because \(v^\delta\) is continuous. The proof is now complete. \(\square\)

Step 4: Lipschitz estimate.

**Lemma 5.3.** There exists a Lipschitz continuous function \(m^\delta\), such that there exists a constant \(C\), (independent of \(l, R\) and \(\delta\)) such that

\[
\begin{align*}
| m^\delta(x) - m^\delta(y) | & \leq C|x - y| \quad \text{for all } x, y \in [-l, l], \\
| v^\delta(t, x) - m^\delta(x) | & \leq C \quad \text{for all } (t, x) \in \mathbb{R} \times [-l, l].
\end{align*}
\]

(60)

**Proof of Lemma 5.3.** Let us define \(m^\delta\) as an affine function in each interval of the form 
\([ih_0, (i + 1)h_0]\), with \(i \in \mathbb{Z}\), such that

\[ m^\delta(ih_0) = v^\delta(0, ih_0) \quad \text{and} \quad m^\delta((i+1)h_0) = v^\delta(0, (i+1)h_0). \]

Since \(m^\delta, v^\delta(0, \cdot)\) are non-increasing and \(|v^\delta(0, (i+1)h_0) - v^\delta(0, ih_0)| \leq k_0h_0 + 1 = 2\), we deduce that \(\forall x \in [ih_0, (i+1)h_0]\),

\[-2 \leq v^\delta(0, (i+1)h_0) - m^\delta(ih_0) \leq v^\delta(0, x) - m^\delta(x) \leq v^\delta(0, ih_0) - m^\delta((i+1)h_0) \leq 2,\]

and for all \(x, y \in [-l, l]\),

\[ | m^\delta(x) - m^\delta(y) | \leq 2k_0|x - y|. \]

Using the time oscillations of \(v^\delta\), we deduce that

\[ | v^\delta(t, x) - m^\delta(x) | \leq C \quad \text{for all } (t, x) \in \mathbb{R} \times [-l, l] \]

with \(C = \frac{3}{2}V_{\max}k_0 + |H_0| + 3\). \(\square\)

Step 4: passing to the limit as \(\delta\) goes to 0. Using (56) and (60), we deduce that there exists \(\delta_n \to 0\) such that

\[ \delta_n v^{\delta_n}(0, 0) \to -\lambda_{l, R} \quad \text{as } n \to +\infty, \]

\[ m^{\delta_n} - m^\delta(0) \to m^{l, R} \quad \text{as } n \to +\infty, \]

the second convergence being locally uniform. Let us consider,

\[ w^{l, R}(t, x) = \limsup_{\delta_n \to 0} (v^{\delta_n} - v^{\delta_n}(0, 0)) \quad \text{and} \quad \omega^{l, R} = \liminf_{\delta_n \to 0} (v^{\delta_n} - v^{\delta_n}(0, 0)). \]

Therefore, we have that \(\lambda_{l, R}, m^{l, R}, \omega^{l, R}\) and \(\omega^{l, R}\) satisfy

\[
\begin{align*}
H_0 & \leq \lambda_{l, R}, \\
|\omega^{l, R} - \omega^{l, R}| & \leq C, \\
m^{l, R} & \leq C, \\
|\omega^{l, R}| & \leq C.
\end{align*}
\]

(61)
By stability of the solutions we have that $\overline{w}^{l,R} - 2C$ and $\underline{w}^{l,R}$ are respectively a sub-solution and a super-solution of (49) and

$$\overline{w}^{l,R} - 2C \leq \underline{w}^{l,R}.$$ 

By Perron’s method we can construct a solution $w^{l,R}$ of (49) and thanks to (56) and (61), $m_l^{l,R}$, $\lambda_l^{l,R}$ and $w_l^{l,R}$ satisfy (54).

The uniqueness of $\lambda_l^{l,R}$ is classical so we skip it. This ends the proof of Proposition 4.

Proposition 5 (First definition of the flux limiter). The following limits exist (up to a subsequence)

$$\left\{ \begin{array}{l} A_R = \lim_{l \to +\infty} \lambda_{l,R} \\ \overline{A} = \lim_{R \to +\infty} \overline{A}_R. \end{array} \right.$$ 

Moreover, we have

$$H_0 \leq \overline{A}_R, \overline{A} \leq 0.$$ 

Proof. This results comes from the fact that we have the following bound on $\lambda_{l,R}$ which is independent of $l$ and $R$ (see Proposition 4),

$$H_0 \leq \lambda_{l,R} \leq 0.$$ 

Proposition 6 (Control of the slopes on a truncated domain). Assume that $l$ and $R$ are big enough. Let $w^{l,R}$ be the solution of (49) given by Proposition 4. We also assume that up to a sub-sequence $\overline{A} = \lim_{R \to +\infty} \lim_{l \to +\infty} \lambda_{l,R} > H_0$. Then there exists $\gamma_0 > 0$ such that for all $\gamma \in (0, \gamma_0)$, there exists a constant $C$ (independent of $l$ and $R$) such that for all $x \geq r$ and $h \geq 0$

$$w^{l,R}(t, x + h) - w^{l,R}(t, x) \geq (p_+ - \gamma)h - C.$$ 

Similarly, for all $x \leq -r$ and $h \geq 0$,

$$w^{l,R}(t, x - h) - w^{l,R}(t, x) \geq (-p_+ - \gamma)h - C.$$ 

Proof. We only prove (63) since the proof for (64) is similar. For $\mu > 0$ small enough, we denote by $p_+^{\mu}$ the real number such that

$$\overline{H}(p_+^{\mu}) = \overline{P}^+(p_+^{\mu}) = \lambda_{l,R} - \mu.$$ 

Using that

$$H_0 < \lambda_{l,R} \leq 0,$$ 

we deduce that $p_+^{\mu}$ exists, is unique and satisfies $-k_0 \leq p_+^{\mu} \leq 0$ for $\mu$ small enough.

Let us now consider the function $w^+ = p_+^{\mu}x$ that satisfies

$$\overline{H}(w^+_x) = \lambda_{l,R} - \mu \quad \text{for} \quad x \in \mathbb{R}.$$
We also have
\[
M[w^+](x) = \int_{\mathbb{R}} J(z)E(p_+^l(x + z) - p_+^l x) dz - \frac{3}{2} V_{\text{max}}
\]
\[
= \int_0^{\infty} \frac{1}{2} J(z) dz + \int_{\mathbb{R}}^{+\infty} \frac{3}{2} J(z) dz - \frac{3}{2} V_{\text{max}}
\]
\[
= -V\left(\frac{1}{p_+^R}\right).
\]
For all \(x \in (r, l)\), using that \(\phi(t, x) = 1\), we deduce that
\[
M[w^+](x) \cdot \phi(t, x) \cdot |w^+_x| = -V\left(\frac{1}{p_+^R}\right) \cdot |p_+^R| = \mathcal{H}(p_+^R) = \lambda_{l,R} - \mu,
\]
and so the restriction of \(w^+\) to \((r, l)\) satisfies
\[
\begin{cases}
  w_+^R + G_R(t, x, [w^+], w_+^R) = \lambda_{l,R} - \mu \quad \text{for } (t, x) \in \mathbb{R} \times (r, l) \\
  w_+^R + \mathcal{H}^+(w_+^R) = \lambda_{l,R} - \mu \quad \text{for } (t, x) \in \mathbb{R} \times \{l\}.
\end{cases}
\]
Let us denote by \(g(t, x) = w^{l,R}(t, x) - w^{l,R}(0, x_0)\) and \(u(x) = w^+(x) - w^+(x_0) - C\), for some \(x_0 \in (r, l)\) and \(C\) defined as in Proposition 4. Then we have
\[
g(t, x_0) \geq -C = u(x_0).
\]
Using that \(g\) is a solution of (52) and \(u\) is a solution of (53) (with \(\varepsilon_0 = \mu\)) joint to the comparison principle (Proposition 3) we get that
\[
w^{l,R}(t, x) - w^{l,R}(0, x_0) = g(t, x) \geq u(x) = p_+^R(x - x_0) - C.
\]
This implies that for all \(h \geq 0\) and for all \(x \in (r, l)\),
\[
w^{l,R}(t, x + h) - w^{l,R}(t, x) \geq p_+^R\ h - C.
\]
Finally, if we choose\( \gamma_0 < |p_0 - \bar{p}_+|\) (with \(p_0\) defined in (4)), then
\[
\mathcal{H}(\bar{p}_+ - \gamma) = \mathcal{H}^+(\bar{p}_+ - \gamma),
\]
and we can choose \(\mu > 0\) such that
\[
p_+^R = \bar{p}_+ - \gamma.
\]
This implies inequality (63). \(\square\)

**Proof of Theorem 2.4.** The proof is performed in two steps.
Step 1: proof of i) and ii). The goal is to pass to the limit as \(l \to +\infty\) and then as \(R \to +\infty\). Using Proposition 4, there exists \(l_n \to +\infty\), such that
\[
m^{l_n,R} - m^{l_n,R}(0) \to m^R \quad \text{as } n \to +\infty,
\]
the convergence being locally uniform. We also define
\[
\begin{align*}
\overline{w}^R(t, x) &= \lim_{l_n \to +\infty} \sup (w^{l_n,R} - w^{l_n,R}(0, 0)) , \\
\underline{w}^R(t, x) &= \lim_{l_n \to +\infty} \inf (w^{l_n,R} - w^{l_n,R}(0, 0)) .
\end{align*}
\]
Thanks to (54), we know that \(\overline{w}^R\) and \(\underline{w}^R\) are finite and satisfy
\[
m^R - C \leq \underline{w}^R \leq \overline{w}^R \leq m^R + C.
\]
By stability of viscosity solutions, \(\overline{w}^R - 2C\) and \(\underline{w}^R\) are respectively a sub and a super-solution of
\[
w_t^R + G_R(x, [w^R(t, \cdot)], w_x^R) = \overline{A}_R \quad \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}
\] (65)
Therefore, using Perron’s method, we can construct a solution $w^R$ of (65) with $m^R, \overline{A}^R$ and $w$ satisfying

\[
\begin{align*}
|m^R(x) - m^R(y)| &\leq C|x - y| \quad \text{for all } x, y \in \mathbb{R}, \\
|w^R(t,x) - m^R(x)| &\leq C \quad \text{for } (t,x) \in \mathbb{R} \times \mathbb{R}, \\
H_0 &\leq \overline{A}^R \leq 0.
\end{align*}
\]  

(66)

Using Proposition 6, if $\overline{A} > H_0$, we know that there exists a $\gamma_0$ and a constant $C$, such that for all $\gamma \in (0, \gamma_0)$,

\[
\begin{align*}
w^R(t,x+h) - w^R(t,x) &\geq (\overline{p}_+ - \gamma)h - C \quad \text{for all } x \geq r, h \geq 0, \\
w^R(t,x-h) - w^R(t,x) &\geq (-\overline{p}_- - \gamma)h - C \quad \text{for all } x \leq -r, h \geq 0.
\end{align*}
\]  

(67)

We now pass to the limit as $R \to +\infty$. We consider (up to some subsequence)

\[
\begin{align*}
\overline{w}(t,x) &= \limsup_{R \to +\infty} (w^R - w^R(0,0)) , \\
\underline{w}(t,x) &= \liminf_{R \to +\infty} (w^R - w^R(0,0)) , \\
\overline{A} &= \lim_{R \to +\infty} \overline{A}^R , \\
m &= \lim_{R \to +\infty} (m^R - m^R(0)).
\end{align*}
\]

The last convergence being locally uniform. Thanks to (66), we know that $\overline{w}$ and $\underline{w}$ are finite and satisfy

\[
m - C \leq \underline{w} \leq \overline{w} \leq m + C.
\]

By stability of viscosity solutions, $\overline{w} - 2C$ and $\underline{w}$ are respectively a sub and a supersolution of (27) with $\lambda = \overline{A}$. Using Perron’s method, we can then construct a solution $w$ of (27) with $\lambda = \overline{A}$ that satisfies (28) and (29)-(30).

Step 2: proof of iii). We are now interested in the rescaled function $w^\varepsilon(t,x) = \varepsilon w \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right)$. Using (29)-(30), we have that

\[
w^\varepsilon(t,x) = \varepsilon m \left( \frac{x}{\varepsilon} \right) + O(\varepsilon).
\]

Therefore, we can find a sequence $\varepsilon_n \to 0$, such that

\[
w^\varepsilon_n \to W \quad \text{locally uniformly as } n \to +\infty,
\]

with $W(0) = 0$. Like in [25], arguing as in the proof of convergence away from the junction point, we have that $W$ satisfies

\[
\overline{H}(W_x) = \overline{A} \quad \text{for } x \neq 0.
\]

For all $\gamma \in (0, \gamma_0)$, we have that if $\overline{A} > H_0$ and $x > 0$,

\[
W_x \geq \overline{p}_+ - \gamma,
\]

where we have used (29)-(30). Therefore we get

\[
W_x = \overline{p}_+ \quad \text{for } x > 0,
\]

this result remains valid even if $\overline{A} = H_0$ (in this particular case $W_x = p_0$). Similarly, we get

\[
W_x = \overline{p}_- \quad \text{for } x < 0.
\]

which implies (31) and (32). This ends the proof of Theorem 2.4. □
Step 3: proof of iv). Up to a sub-sequence, we assume that
\[ A = \lim_{R \to +\infty} \lim_{l \to +\infty} \lambda_{l,R}. \]
We want to prove that
\[ A = \inf E, \]
where
\[ E = \{ \lambda \in [h_0, 0] : \exists w \in S \text{ solution of (27)} \}, \]
with
\[ S = \{ w \text{ s.t. } \exists m \in \text{Lip}(\mathbb{R}) \text{ and a } C > 0 \text{ s.t. } |w(t,x) - m(x)| \leq C \}. \]

We argue by contradiction and assume that there exists a \( \lambda < A \) and a function \( w^\lambda \in S \) solution of (27). We assume that \( w^\lambda(0,0) = 0 \) (if we are not in this situation, we do a translation since we have \( w^\lambda(0,0) \in S \)). Arguing as in the proof of step 2, we deduce that the function
\[ w^\varepsilon(\lambda, t, x) = \varepsilon w^\lambda(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}) \]
has a limit \( W^\lambda \) (with \( W^\lambda(0) = 0 \)) which satisfies
\[ \mathcal{H}(W^\lambda_x) = \lambda \quad \text{for } x > 0, \]
which means that for all \( x > 0 \),
\[ W^\lambda_x \leq p^+_x < p_+ \quad \text{with } \mathcal{H}(p^+_x) = \mathcal{H}^+(p^+_x) = \lambda. \quad (68) \]

Similarly we have for all \( x < 0 \),
\[ W^\lambda_x \geq p^-_x > p_- \quad \text{with } \mathcal{H}(p^-_x) = \mathcal{H}^-(p^-_x) = \lambda. \quad (69) \]

These inequalities imply that for all \( \gamma > 0 \), there exists a constant \( \tilde{C}_\gamma > 0 \) such that
\[ w^\lambda(t, x) \leq \begin{cases} \ (p^+_x + \gamma)x + \tilde{C}_\gamma & \text{for } x > 0, \\ \ (p^-_x - \gamma)x + \tilde{C}_\gamma & \text{for } x < 0, \end{cases} \quad (70) \]

In fact, if \( w^\lambda \) does not satisfies (70), we cannot have (68) and (69). Using point ii) of Theorem 2.4, we get
\[ w^\lambda < w \quad \text{for } |x| \geq \tilde{R} \]
if \( \gamma \) is small enough and \( \tilde{R} \) big enough. This implies that there exists a constant \( C_{\tilde{R}} > 0 \) such that for all \( x \in \mathbb{R} \), we have
\[ w^\lambda(t, x) < w(t, x) + C_{\tilde{R}}. \]

Let us now introduce, \( u(t, x) = w(t, x) + C_{\tilde{R}} - A t \) and \( u_\lambda(t, x) = w^\lambda(t, x) - \lambda t \) both solutions of (17) with \( \varepsilon = 1 \) and \( u_\lambda(0, x) \leq u(0, x) \). Therefore, the comparison principle implies
\[ w^\lambda(t, x) - \lambda t \leq w(t, x) + C_{\tilde{R}} - A t \]
Dividing by \( t \) and passing to the limit as \( t \) goes to infinity, we get
\[ \overline{A} \leq \lambda, \]
which is a contradiction.
Step 4: proof of v). In order to establish the monotonicity, we have to consider the approximated truncated cell problem (55). Let us consider $v_1^\delta$ and $v_2^\delta$ viscosity solutions of (55), respectively for $\phi_1$ and $\phi_2$, with $0 \leq \phi_1 \leq \phi_2$. First, using the fact that the non-local operator is negative, we have

$$G^2_R(t, x, [U], q) \leq G^1_R(t, x, [U], q),$$

with

$$G^i_R(t, x, [U], q) = \phi_i(t, x) \cdot M[U](x) \cdot \psi_R(x) \cdot |q| + (1 - \psi_R(x))\overline{H}(q), \quad \text{for } i = 1, 2.$$

Therefore, we have

$$0 = \delta v_1^\delta + (v_1^\delta)_t + G^1_R(t, x, [v_1^\delta(t, \cdot)], (v_1^\delta)_x) \geq \delta v_1^\delta + (v_1^\delta)_t + G^2_R(t, x, [v_1^\delta(t, \cdot)], (v_1^\delta)_x),$$

meaning that $v_1^\delta$ is a sub-solution of (55) with $\phi_2$. The comparison principle and (56) imply that

$$0 \leq \delta v_1^\delta \leq \delta v_2^\delta \leq |H_0|.$$

Passing to the limit as $\delta \to 0$, we obtain

$$0 \geq \lambda_{1,R} \geq \lambda_{2,R} \geq H_0.$$

Passing to the limit as $t, R \to +\infty$, we get the result.

6. Proof of Theorem 2.3. This section contains the proof of the main homogenization result (Theorem 2.3). This proof relies on the existences of correctors (Proposition 1 and Theorem 2.4).

We begin with two useful lemmas for the proof of Theorem 2.3. The first result is a direct consequence of Perron’s method and Lemma 3.5.

Lemma 6.1 (Barriers uniform in $\varepsilon$). Assume (A0) and (A). There exists a constant $C > 0$ (depending only on $M_0$ and $k_0$) such that for all $t > 0$ and $x \in \mathbb{R}$,

$$|u^\varepsilon(t, x) - u_0(x)| \leq Ct.$$

The following lemma is a direct result of Theorem 3.8.

Lemma 6.2 (Uniform gradient bound). Assume (A0) and (A). Then the solution $u^\varepsilon$ of (22) satisfies for all $t > 0$, for all $x, y \in \mathbb{R}$, $x \geq y$,

$$-k_0(x - y) - \varepsilon \leq u^\varepsilon(t, x) - u^\varepsilon(t, y) \leq 0. \quad (71)$$

Before passing to the proof of Theorem 2.3, let us show how it allows us to obtain the following result

Corollary 1. Assume (A0)-(A). Let $u^0$ be the unique solution of (5), then we have for all $(t, x) \in [0, T] \times \mathbb{R}$,

$$-k_0 \leq u_0^0 \leq 0,$$

with $k_0$ defined in (A0).

Remark 10 (Extension of the effective Hamiltonian). This result implies in particular that in the case of traffic flow, the effective Hamiltonian only needs to be computed for $p \in [-k_0, 0]$. However, for the construction of the correctors it is necessary to work with a coercive Hamiltonian in $\mathbb{R}$ that is why we extend the function $\overline{H}$ in (2).
Proof of Corollary 1. We want to prove that for all \( t \in [0, +\infty) \) and for all \( x, y \in \mathbb{R} \), \( x \geq y \),

\[
-k_0(x - y) \leq u^0(t, x) - u^0(t, y) \leq 0.
\]  

(72)

Using Lemma 6.2, we have that the solution \( u^\varepsilon \) of (22), satisfies for all \( (t, x, y) \in [0, +\infty) \times \mathbb{R} \times \mathbb{R} \), with \( x \geq y \),

\[
-k_0(x - y) - \varepsilon \leq u^\varepsilon(t, x) - u^\varepsilon(t, y) \leq 0.
\]

Now using Theorem 2.3, passing to the limit as \( \varepsilon \to 0 \), we obtain the result. \( \square \)

We now turn to the proof of Theorem 2.3.

Proof of Theorem 2.3. We introduce

\[
\bar{u}(t, x) = \limsup_{\varepsilon \to 0} u^\varepsilon \quad \text{and} \quad \underline{u}(t, x) = \liminf_{\varepsilon \to 0} u^\varepsilon.
\]

(73)

Thanks to Lemma 6.1, we know that these functions are well defined. We want to prove that \( \bar{u} \) and \( \underline{u} \) are respectively a sub-solution and a super-solution of (5). In this case, the comparison principle [25, Theorem 1.4] will imply that \( \bar{u} \leq \underline{u} \). But, by construction, we have \( \underline{u} \leq \bar{u} \), hence we will get \( \underline{u} = \bar{u} = u^0 \), the unique solution of (5). Let us prove that \( \bar{u} \) is a sub-solution of (5) (the proof for \( \underline{u} \) is similar and we skip it). We argue by contradiction and assume that there exists a test function \( \varphi \in C^1(J_{\infty}) \) (in the sense of Definition 3.2), and a point \((\bar{t}, \bar{x}) \in (0, +\infty) \times \mathbb{R} \) such that

\[
\begin{cases}
\bar{u}(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x}) \\
\bar{u} \leq \varphi - 2\eta & \text{on } Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}) \text{ with } \bar{r} > 0 \\
\bar{u} \leq \varphi - \eta & \text{outside } Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}) \text{ with } \eta > 0 \\
\varphi(\bar{t}, \bar{x}) + \overline{H}(\bar{x}, \varphi_\varepsilon(\bar{t}, \bar{x})) = \theta & \text{with } \theta > 0,
\end{cases}
\]

(74)

where

\[
\overline{H}(\bar{x}, \varphi_\varepsilon(\bar{t}, \bar{x})) := \begin{cases}
\underline{H}(\varphi_\varepsilon(\bar{t}, \bar{x})) & \text{if } \bar{x} \neq 0, \\
\overline{F}_\varepsilon(\varphi(\bar{t}, 0^-), \varphi_\varepsilon(\bar{t}, 0^+)) & \text{if } \bar{x} = 0.
\end{cases}
\]

Given Lemma 6.2 and (73), we can assume (up to changing \( \varphi \) at infinity) that for \( \varepsilon \) small enough, we have

\[
u^\varepsilon \leq \varphi - \eta \quad \text{outside } Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).
\]

Using the previous lemmas we get that the function \( \bar{u} \) satisfies for all \( t > 0 \) and \( x, y \in \mathbb{R} \), \( x \geq y \),

\[
|\bar{u}(t, x) - u_0(x)| \leq Ct,
\]

\[
-k_0(x - y) \leq u(t, x) - \bar{u}(t, y) \leq 0.
\]

(75)

First case: \( \bar{x} \neq 0 \). We only consider \( \bar{x} > 0 \), since the other case \( \bar{x} < 0 \) is treated in the same way. We define \( p = \varphi_\varepsilon(\bar{t}, \bar{x}) \) that according to (75) satisfies

\[
-k_0 \leq p \leq 0.
\]

We choose \( \bar{r} \) small enough so that \( \bar{x} - 2\bar{r} > 0 \). Let us prove that the test function \( \varphi \) satisfies in the viscosity sense, the inequality

\[
\varphi_t + \bar{M}^\varepsilon \left[ \frac{\varphi}{\varepsilon}(t, \cdot) \right](x) \cdot \phi \left( \frac{t}{\varepsilon}, -\frac{x}{\varepsilon} \right) \cdot |\varphi_\varepsilon| \geq \frac{\theta}{2} \quad \text{for } (t, x) \in Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).
\]

(76)
Let us notice that for \( \varepsilon \) small enough we have
\[
\phi \left( \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right) = 1 \quad \text{for all } (t, x) \in Q_{\bar{r}}(\bar{t}, \bar{x}).
\]

For all \((t, x) \in Q_{\bar{r}}(\bar{t}, \bar{x})\), we have for \( \bar{r} \) small enough
\[
\phi_t(t, x) + \tilde{M}_\varepsilon \left[ \frac{\phi}{\varepsilon}(t, \cdot) \right] (x) \cdot |\varphi_x| = \varphi_t(\bar{t}, \bar{x}) + o_r(1) + \tilde{M}_\varepsilon \left[ \frac{\phi}{\varepsilon}(t, \cdot) \right] (x) \cdot |\varphi_x| = \theta + o_r(1) + \tilde{M}_\varepsilon \left[ \frac{\phi}{\varepsilon}(t, \cdot) \right] (x) \cdot |p| - \mathcal{H}(p)
\]
\[
=: \Delta,
\]
where we have used (74). We recall that for \(-k_0 \leq p \leq 0\),
\[
\mathcal{H}(p) = M_p[0](0)|p|.
\]

Moreover, for all \(z \in [h_0, h_{\text{max}}]\), and for \(\varepsilon\) and \(\bar{r}\) small enough we have that
\[
\frac{\varphi(t, x + \varepsilon z) - \varphi(t, x)}{\varepsilon} = z\varphi_x(t, y) + \varepsilon z^2 \varphi_{xx}(t, \xi(x, x + \varepsilon z)) \leq \frac{pz + o_r(1) + \varepsilon}{\varepsilon},
\]
where we have used the fact that \(\varphi \in C^2\) and that \(z \in [h_0, h_{\text{max}}]\). Now using the fact that \(\tilde{E}\) is decreasing we have
\[
\tilde{E}(pz + c\varepsilon + o_r(1)) \leq \tilde{E} \left( \frac{\varphi(t, x + \varepsilon z) - \varphi(t, x)}{\varepsilon} \right).
\]

Using this result and replacing the non-local operators in (77) by their definition (see 25), we obtain
\[
\Delta \geq \theta + o_r(1) + |p| \int_{h_0}^{h_{\text{max}}} J(z) \tilde{E}(pz + c\varepsilon + o_r(1))dz
\]
\[
- |p| \int_{h_0}^{h_{\text{max}}} J(z) \tilde{E}(pz)dz.
\]

We can see that if we have \(p = 0\), we obtain directly our result. However, if \(-k_0 \leq p < 0\),
\[
\int_{\mathbb{R}} J(z) \tilde{E}(pz + c\varepsilon + o_r(1))dz
\]
\[
= - V \left( \frac{-1 - c\varepsilon + o_r(1)}{p} \right) - \frac{1}{2} V \left( \frac{- c\varepsilon + o_r(1)}{p} \right) + \frac{3}{2} V_{\text{max}},
\]
\[
\int_{\mathbb{R}} J(z) \tilde{E}(pz)dz = - V \left( \frac{-1}{p} \right) + \frac{3}{2} V_{\text{max}}.
\]

Injecting (79) in (78) and choosing \(\varepsilon\) and \(\bar{r}\), we obtain
\[
\Delta \geq \theta + o_r(1) + |p| \cdot \left[ -V \left( \frac{-1 - c\varepsilon + o_r(1)}{p} \right) + V \left( \frac{-1}{p} \right) \right]
\]
\[
\geq \theta + o_r(1) - ||V'||_{\infty} \cdot (c\varepsilon + o_r(1))
\]
\[
\geq \frac{\theta}{2},
\]
where we have used assumption (A1) for the second line.
Getting a contradiction. By definition, we have for \( \varepsilon \) small enough, 
\[
    u^\varepsilon \leq \varphi - \eta \quad \text{outside} \quad Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).
\]
Using the comparison principle on bounded subsets for (17) (Theorem 3.4), we get 
\[
    u^\varepsilon \leq \varphi - \eta \quad \text{on} \quad Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).
\]
Passing to the limit as \( \varepsilon \to 0 \), we get \( u \leq \varphi - \eta \) on \( Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}) \) and this contradicts the fact that \( \pi(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x}) \).

Second case: \( \bar{x} = 0 \). Using Theorem 3.7, we may assume that the test function has the following form
\[
    \varphi(t, x) = g(t) + \bar{p}_-x1_{\{x<0\}} + \bar{p}_+x1_{\{x>0\}} \quad \text{on} \quad Q_{2\bar{r}, 2\bar{r}}(\bar{t}, 0),
\]
where \( g \) is a \( C^1 \) function defined in \((0, +\infty)\). The last line in condition (74) becomes
\[
    g'(t) + F_{\bar{A}}(\bar{p}_-, \bar{p}_+) = g'(t) + \bar{A} = \theta \quad \text{at} \quad (\bar{t}, 0),
\]
Let us consider the solution \( w \) of (27) provided by Theorem 2.4, and let us denote by
\[
    \varphi^\varepsilon(t, x) = \begin{cases} 
        g(t) + w^\varepsilon(t, x) & \text{on} \quad Q_{2\bar{r}, 2\bar{r}}(\bar{t}, 0), \\
        \varphi(t, x) & \text{outside} \quad Q_{2\bar{r}, 2\bar{r}}(\bar{t}, 0).
    \end{cases}
\]
We would like to prove that this function satisfies in the viscosity sense, for \( \bar{r} \) and \( \varepsilon \) small enough,
\[
    \varphi^\varepsilon(t, x) + M^\varepsilon \left[ \frac{\varphi^\varepsilon}{\varepsilon}(t, \cdot) \right](x) \cdot \phi \left( \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right) \cdot |\varphi^\varepsilon_x| \geq \frac{\theta}{2} \quad \text{on} \quad Q_{\bar{r}, \bar{r}}(\bar{t}, 0).
\]
Let \( h \) be a test function touching \( \varphi^\varepsilon \) from below at \((t_1, x_1) \in Q_{\bar{r}, \bar{r}}(\bar{t}, 0)\), so we have
\[
    w \left( \frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon} \right) = \frac{1}{\varepsilon} (h(t_1, x_1) - g(t_1)),
\]
and
\[
    w(s, y) \geq \frac{1}{\varepsilon} (h(\varepsilon s, \varepsilon y) - g(\varepsilon s)),
\]
for \((s, y)\) in a neighbourhood of \( \left( \frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon} \right) \). Therefore, we have
\[
    h_t(t_1, x_1) - g'(t_1) + M \left[ w \left( \frac{t_1}{\varepsilon}, \cdot \right) \right] \left( \frac{x_1}{\varepsilon} \right) \cdot \phi \left( \frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon} \right) \cdot |h_x(t_1, x_1)| \geq \bar{A}.
\]
This implies that (using (81) and taking \( \bar{r} \) small enough)
\[
    h_t(t_1, x_1) + M \left[ w \left( \frac{t_1}{\varepsilon}, \cdot \right) \right] \left( \frac{x_1}{\varepsilon} \right) \cdot \phi \left( \frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon} \right) \cdot |h_x(t_1, x_1)| \geq \bar{A} + g'(t_1) \geq \frac{\theta}{2}.
\]
Now for \( \varepsilon \) small enough such that \( \varepsilon h_{\text{max}} \leq \bar{r} \), we deduce from the previous inequality and using the fact that \( M \) is a non-local operator with a bounded support, that we have
\[
    h_t(t_3, x_1) + M^\varepsilon \left[ \frac{\varphi^\varepsilon(t_1, \cdot)}{\varepsilon} \right](x_1) \cdot \phi \left( \frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon} \right) \cdot |h_x(t_1, x_1)| \geq \frac{\theta}{2}.
\]
Getting the contradiction. We have that for \( \varepsilon \) small enough
\[
    u^\varepsilon + \eta \leq \varphi = g(t) + \mathbf{p}_- x 1_{x < 0} + \mathbf{p}_+ x 1_{x > 0} \quad \text{on } Q_{2\varepsilon, 2r}(\tilde{t}, 0) \setminus Q_{r, \varepsilon}(\tilde{t}, 0).
\]
Using the fact that \( u^\varepsilon \to W \), and using (32), we have for \( \varepsilon \) small enough
\[
    u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{on } Q_{2\varepsilon, 2r}(\tilde{t}, 0) \setminus Q_{r, \varepsilon}(\tilde{t}, 0).
\]
Combining this with (82), we get that
\[
    u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{outside } Q_{r, \varepsilon}(\tilde{t}, 0),
\]
By the comparison principle on bounded subsets (Theorem 3.4) the previous inequality holds in \( Q_{r, \varepsilon}(\tilde{t}, 0) \). Passing to the limit as \( \varepsilon \to 0 \) and evaluating the inequality in \( \tilde{t}, 0 \), we obtain
\[
    \pi(\tilde{t}, 0) + \frac{\eta}{2} \leq \varphi(\tilde{t}, 0) = \pi(\tilde{t}, 0),
\]
which is a contradiction. \( \square \)

7. Proof of Theorem 2.1. This section is devoted to the proof of Theorem 2.1, which is a direct application of our convergence result, Theorem 2.3.

Proof of Theorem 2.1. We recall that in Theorem 2.1, we have \( u_0(x) = -\frac{x}{h_1} I_{x \leq 0} - \frac{x}{h_2} I_{x > 0} \), with \( h_1, h_2 \geq h_0 \). First, we would like to prove that for all \( \varepsilon > 0 \), we have
\[
    |\rho^\varepsilon(0, x) - u_0(x)| \leq f(\varepsilon) \quad \text{for all } x \in \mathbb{R},
\]
with \( f(\varepsilon) \to 0 \) as \( \varepsilon \) goes to 0. To do this, we define a piece-wise affine function \( v \) satisfying
\[
    \rho^1(0, x) = v(x) \quad \text{for } x = U_i(0), \text{ for all } i \in \mathbb{Z}.
\]
Given that for all \( U_{i+1}(0) - U_i(0) \geq h_0 \), we notice that \( v \) is \( k_0 \)-Lipschitz continuous and by definition of \( \rho^1(0, x) \), we have
\[
    |\rho^1(0, x) - v(x)| \leq 1 \quad \text{for all } x \in \mathbb{R}.
\]

Let us consider the integer \( i_0 \in \mathbb{N} \) defined by
\[
    i_0 = \sup \{ i \in \mathbb{Z}, \ s.t. \ U_i(0) \leq -R \}.
\]
Using the assumption that for all \( i \in \mathbb{Z} \) such that \( U_i(0) \leq -R \) we have \( U_{i+1}(0) - U_i(0) = h_1 \), we deduce that for all \( x \leq U_{i_0}(0) \)
\[
    v(x) = -\frac{x}{h_1} + \frac{U_{i_0}(0)}{h_1} + \rho^1(0, U_{i_0}(0)) = -\frac{x}{h_1} + \frac{U_{i_0}(0)}{h_1} - i_0 - 1.
\]
Let us now consider the integer \( i_1 \in \mathbb{N} \) defined by
\[
    i_1 = \inf \{ i \in \mathbb{Z}, \ s.t. \ U_i(0) \geq R \}.
\]
Now using the assumption that for all \( i \in \mathbb{Z} \) such that \( U_i(0) \geq R \) we have \( U_{i+1}(0) - U_i(0) = h_2 \), we deduce that for all \( x \geq U_{i_1}(0) \)
\[
    v(x) = -\frac{x}{h_2} + \frac{U_{i_1}(0)}{h_2} + \rho^1(0, U_{i_1}(0)) = -\frac{x}{h_2} + \frac{U_{i_1}(0)}{h_2} - i_1 - 1.
\]
Moreover, we recall that for all $\varepsilon > 0$, we have $\rho^\varepsilon(0, x) = \varepsilon \rho^1(0, x/\varepsilon)$, this implies that for all $x \notin [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$,
\begin{equation}
|\rho^\varepsilon(0, x) - u_0(x)| \leq |\rho^\varepsilon(0, x) - \varepsilon v \left( \frac{x}{\varepsilon} \right)| + |\varepsilon v \left( \frac{x}{\varepsilon} \right) - u_0(x)| \\
\leq \varepsilon + \varepsilon \max \left( \left| \frac{U_{i_0}(0)}{h_2} - i_0 - 1 \right|, \frac{U_{i_0}(0)}{h_1} - i_0 - 1 \right). \tag{84}
\end{equation}
Similarly, we have for all $x \in [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$,
\begin{equation}
|\rho^\varepsilon(0, x) - u_0(x)| \leq |\rho^\varepsilon(0, x) - \varepsilon v \left( \frac{x}{\varepsilon} \right)| + |\varepsilon v \left( \frac{x}{\varepsilon} \right) - \varepsilon u_0 \left( \frac{x}{\varepsilon} \right)| \\
\leq \varepsilon + \varepsilon \max_{y \in [U_{i_0}(0), U_{i_1}(0)]} (|v(y) - u_0(y)|), \tag{85}
\end{equation}
where we have used the fact that $\varepsilon u_0(x/\varepsilon) = u_0(x)$. Combining (84) and (85) and choosing
\begin{equation}
f(\varepsilon) = \varepsilon + \varepsilon \max \left( \left| \frac{U_{i_0}(0)}{h_2} - i_0 - 1 \right|, \right.
\left. \max_{y \in [U_{i_0}(0), U_{i_1}(0)]} (|v(y) - u_0(y)|), \frac{U_{i_1}(0)}{h_1} - i_1 - 1 \right) \nonumber
\end{equation}
we deduce (83). Notice also that thanks to (83), we have
\begin{equation}
|(\rho^\varepsilon)^*(0, x) - u_0(x)| \leq f(\varepsilon) + \varepsilon. \tag{86}
\end{equation}
Therefore, we have
\begin{equation}
u_0(x) - f(\varepsilon) \leq \rho^\varepsilon(0, x) \leq (\rho^\varepsilon)^*(0, x) \leq u_0(x) + f(\varepsilon) + \varepsilon.
\end{equation}
Using the fact that $\rho^\varepsilon$ is a viscosity solution of (17) and the comparison principle (Proposition 2) we deduce that (with $u^\varepsilon$ the continuous solution of (17) associated to the initial condition $u_0(x) = -x/h$)
\begin{equation}
u^\varepsilon(t, x) - f(\varepsilon) \leq \rho^\varepsilon(t, x) \leq (\rho^\varepsilon)^*(t, x) \leq u^\varepsilon(t, x) + f(\varepsilon) + \varepsilon,
\end{equation}
where we have used the fact that (17) is invariant by addition of constants to the solutions. Passing to the limit as $\varepsilon \to 0$ and using Theorem 2.3 we get that $\rho^\varepsilon \to u^0$, which ends the proof of Theorem 2.1.

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