OPTION PRICING FORMULAS
FOR GENERALIZED FUZZY STOCK MODEL

CUILIAN YOU* AND LE BO

College of Mathematics and Information Science
Hebei University, Baoding 071002, China

(Communicated by Hailiang Yang)

Abstract. Fuzzy stock model has been studied by many scholars in recent years, in which option pricing problem is the most important part. In this paper, we studied option pricing for a new generalized fuzzy stock model. Based on credibility theory, pricing formulas of European option and American option were obtained.


Asset pricing plays an important role in financial markets, especially the option pricing of stock model. In the early 1970s, Black and Scholes [1] and, independently, Metron [12] used geometric Brownian motion to determine the prices of stock options. Then stochastic financial mathematics developed. However, in the
real world, many unexpected things will happen, for example, natural disasters such as typhoons and tsunamis and human disasters such as wars and terrorist attacks. When these events occur, existing historical data cannot describe stock prices, in which case, human involvement is required. In addition, the bankruptcy and merger of the company will also affect stock prices. Therefore, it is meaningful to introduce a fuzzy process in the case of absence of historical data, so it is necessary to study the fuzzy stock model. Fuzzy calculus was introduced to finance by Liu [10] in 2008. Liu [10] presented an alternative assumption that stock price follows geometric Liu process. Based on this assumption, Liu’s stock model was formulated to describe fuzzy financial market, which just a fuzzy counterpart of Black-Scholes stock model. Considering the option pricing problem is a fundamental problem in financial market, many scholars have done some research works around different stock model. Qin and Li [14] presented the European option pricing formula for Liu’s stock model. Gao and Chen [7] investigated the European option pricing formula for generalized Liu’s stock model. Hu [8] gave the power option pricing model by assuming the underlying stock price follows Geometric fractional Liu process. Gao and Gao [6] considered a new model of credibilistic option pricing, which is called Gao’s stock model, it is similar to Black-Karasiski model, and the European option pricing formula was studied. Peng [13] presented a more general stock model and corresponding American option pricing formula. In this paper, under the framework of credibility theory, we will investigate the European option pricing formulas and American option pricing formulas for generalized Gao’s stock model.

The rest of the paper is organized as follows. Some preliminary concepts and properties of credibility theory, Liu process and fuzzy stock models are recalled in Section 2. The European option pricing formulas for generalized Gao’s stock model are discussed in Section 3, and the American option pricing formulas for generalized Gao’s stock model are derived in Section 4. At last, a brief summary is provided in Section 5.

2. Preliminaries. In this section, we will introduce some basic knowledge in credibility theory which will be used in the following sections.

A fuzzy variable is a (measurable) function from a credibility space $(\Theta, \mathcal{P}, \text{Cr})$ to the set of real numbers.

**Definition 2.1.** (Liu and Liu [11]) Let $\xi$ be a fuzzy variable. Then the expected value of $\xi$ is defined by

$$E[\xi] = \int_{0}^{+\infty} \text{Cr}\{\xi \geq r\}dr - \int_{-\infty}^{0} \text{Cr}\{\xi \leq r\}dr,$$

provided that at least one of the two integrals is finite.

**Definition 2.2.** (Liu [10]) Let $T$ be an index set and let $(\Theta, \mathcal{P}, \text{Cr})$ be a credibility space. A fuzzy process is a function from $T \times (\Theta, \mathcal{P}, \text{Cr})$ to the set of real numbers.

That is, a fuzzy process $X(t, \theta)$ is a function of two variables such that the function $X(t^*, \theta)$ is a fuzzy variable for each $t^*$. For each fixed $\theta^*$, the function $X(t, \theta^*)$ is called a sample path of the fuzzy process. A fuzzy process $X(t, \theta)$ is said to be sample-continuous if the sample path is continuous for almost all $\theta$. Instead of longer notation $X(t, \theta)$, we use the symbol $X_t$ in the following sections.

**Definition 2.3.** (Liu [10]) A fuzzy process $C_t$ is said to be Liu process if

(i) $C_0 = 0.$
(ii) $C_t$ has stationary and independent increments,
(iii) $C_{s+t} - C_s$ is a normally distributed fuzzy variable with expected value $et$ and variance $\sigma^2 t^2$ whose membership function is

$$
\mu(x) = 2(1 + \exp\left(\frac{\pi |x - et|}{\sqrt{6} \sigma t}\right))^{-1}, x \in \mathcal{R}.
$$

Note that Liu process is said to be a standard Liu process if $e = 0$ and $\sigma = 1$.

The fuzzy process $X_t = \exp(C_t)$ is called a geometric Liu process.

**Definition 2.4.** (Liu Integral, Liu [10]) Let $X_t$ be a fuzzy process and let $C_t$ be a standard Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$
\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.
$$

Then the fuzzy integral of fuzzy process $X_t$ with respect to $C_t$ is

$$
\int_a^b X_t\,dC_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i}(C_{t_{i+1}} - C_{t_i}),
$$

provided that the limit exists almost surely and is a fuzzy variable.

Traditional fuzzy differential equations are, in fact, differential equations with fuzzy parameters or fuzzy initial conditions, such as Kaleva [9], Ding, Ma, and Kandel [5] and etc. Different from fuzzy differential equations above studied by other researchers, a new kind of fuzzy differential equation was introduced by Liu [10].

**Definition 2.5.** (Fuzzy Differential Equation, Liu [10]) Suppose $C_t$ is a standard Liu process, and $f$, $g$ are some given functions. Then

$$
dX_t = f(t, X_t)\,dt + g(t, X_t)\,dC_t
$$

is called a fuzzy differential equation.

A solution is a fuzzy process $X_t$ that satisfies above equality identically in $t$.

**Definition 2.6.** (General Linear Fuzzy Differential Equation, You, Wang and Huo [19]) Let $C_t$ be a standard Liu process, and let $X_t$ be an unknown fuzzy process. Suppose that $u_{1t}$, $u_{2t}$, $v_{1t}$, $v_{2t}$ are some given continuous fuzzy process. Then the equation

$$
dX_t = (u_{1t}X_t + u_{2t})\,dt + (v_{1t}X_t + v_{2t})\,dC_t
$$

is called a general linear fuzzy differential equation.

**Theorem 2.7.** (You, Wang and Huo [19]) The solution of the general linear fuzzy differential equation is

$$
X_t = U_t(X_0 + \int_0^t \frac{u_{2s}}{U_s} \,ds + \int_0^t \frac{v_{2s}}{U_s} \,dC_s),
$$

where $X_0$ is the initial value and

$$
U_t = \exp\left(\int_0^t u_{1s} \,ds + \int_0^t v_{1s} \,dC_s\right).
$$

Gao and Gao [6] presented a model as a counterpart of Black-Karasinski model, which is called Gao’s stock model.
Definition 2.8. (Gao and Gao [6]) Let $X_t$ be the bond price, and $Y_t$ the stock price. Then Gao’s stock model is
\[
\begin{align*}
\frac{dX_t}{X_t} &= r dt, \\
\frac{dY_t}{Y_t} &= a(b - Y_t) dt + \sigma dC_t,
\end{align*}
\] (1)
where $r$ is the riskless interest rate, $a$, $b$, $\sigma$ are constants, and $C_t$ is a standard Liu process.

This model reflects the phenomenon of mean reversion. According to the mean reversion theory, stock price always rise and fall around the mean.

On the basis of Gao’s stock model, we have the definition of generalized Gao’s stock model.

Definition 2.9. (Qin and Li [15]) Let $X_t$ be the bond price, and $Y_t$ the stock price. The generalized Gao’s stock model is
\[
\begin{align*}
\frac{dX_t}{X_t} &= r_t dt, \\
\frac{dY_t}{Y_t} &= a_t(b_t - Y_t) dt + \sigma_t dC_t,
\end{align*}
\] (2)
where $r_t$, $a_t$, $b_t$, $\sigma_t$ are deterministic functions of time $t$, and $C_t$ is a standard Liu process.

According to Theorem 2.7, the solution of fuzzy differential equation $dY_t = a_t(b_t - Y_t) dt + \sigma_t dC_t$ is
\[
Y_t = \exp(-\int_0^t a_s ds)(Y_0 + \int_0^t \exp(\int_0^s a_x dx)a_x b_x ds + \int_0^t \exp(\int_0^s a_x dx)\sigma_x dC_x).
\]

3. European option pricing formulas. An option is a financial instrument which gives the holder a right without being under obligation to trade the underlying asset at or by expiry date for a certain prescribed price known as exercise or strike price. In this section, European option pricing formula for the stock model (2) will be investigated.

A European call option gives the holder the right, but not the obligation, to buy a stock at a specified time for a specified price. Next, we consider European call option pricing problem. Considering model (2), assume that a European call option has strike price $K$ and expiration time $T$. Then the payoff from buying a European call option is $(Y_T - K)^+$. The present value of this payoff is $\exp(-\int_0^T r_t dt)(Y_T - K)^+$. Thus the European call option price should be $E[\exp(-\int_0^T r_t dt)(Y_T - K)^+]$.

Definition 3.1. European call option price $f$ of model (2) is defined as
\[
f(Y_0, K, a_t, b_t, \sigma_t, r_t) = \exp(-\int_0^T r_t dt)E[(\exp(-\int_0^T a_t dt)(Y_0 + \int_0^T \exp(\int_0^t a_s ds)a_t b_t dt + \int_0^T \exp(\int_0^t a_s ds)\sigma_t dC_t) - K)^+],
\]
where $K$ is the strike price at time $T$.

Based on credibility theory, we derived the following theorem.
Theorem 3.2. (European Call Option Pricing Formula) The European call option pricing formula for the model (2) is given by

\[ f(Y_0, K, a_t, b_t, \sigma_t, r_t) = \exp\left( -\int_0^T (r_t + a_t)dt \right) \int_0^{+\infty} Cr\{ \exp\left( \int_0^t a_s ds \right) a_t b_t dt \]

+ \int_0^T \exp\left( \int_0^t a_s ds \right) \sigma_t dC_t \geq u \} du,

where \( L = K \exp\left( \int_0^T a_t dt \right) - Y_0 \).

Proof. According to the definition of expected value of fuzzy variable, we have

\[
\begin{align*}
    f(Y_0, K, a_t, b_t, \sigma_t, r_t) &= \exp\left( -\int_0^T r_t dt \right) E\left[ (\exp\left( -\int_0^T a_t dt \right) (Y_0 + \int_0^T \exp\left( \int_0^t a_s ds \right) a_t b_t dt \right) \\
    & + \int_0^T \exp\left( \int_0^t a_s ds \right) \sigma_t dC_t - K)^+ \right] \\
    &= \exp\left( -\int_0^T r_t dt \right) \int_0^{+\infty} Cr\{ (\exp\left( -\int_0^T a_t dt \right) (Y_0 + \int_0^T \exp\left( \int_0^t a_s ds \right) a_t b_t dt \right) \\
    & + \int_0^T \exp\left( \int_0^t a_s ds \right) \sigma_t dC_t - K)^+ \right] \geq r} dr \\
    &= \exp\left( -\int_0^T r_t dt \right) \int_0^{+\infty} Cr\{ \exp\left( -\int_0^T a_t dt \right) (Y_0 + \int_0^T \exp\left( \int_0^t a_s ds \right) a_t b_t dt \right) \\
    & + \int_0^T \exp\left( \int_0^t a_s ds \right) \sigma_t dC_t - K \geq r + K} dr \\
    &= \exp\left( -\int_0^T r_t dt \right) \int_0^{+\infty} Cr\{ \exp\left( -\int_0^T a_t dt \right) (Y_0 + \int_0^T \exp\left( \int_0^t a_s ds \right) a_t b_t dt \right) \\
    & + \int_0^T \exp\left( \int_0^t a_s ds \right) \sigma_t dC_t \geq r + K} - Y_0 \exp(\int_0^T a_t dt) \} dr \\
    &= \exp\left( -\int_0^T r_t dt \right) \int_0^{+\infty} Cr\{ \exp\left( \int_0^t a_s ds \right) a_t b_t dt \\
    & + \int_0^T \exp\left( \int_0^t a_s ds \right) \sigma_t dC_t \geq (r + K)\exp(\int_0^T a_t dt) - Y_0} dr \\
    &= \exp\left( -\int_0^T (r_t + a_t) dt \right) \int_0^{+\infty} Cr\{ \exp\left( \int_0^t a_s ds \right) a_t b_t dt \\
    & + \int_0^T \exp\left( \int_0^t a_s ds \right) \sigma_t dC_t \geq u \} du,
\end{align*}
\]

where \( L = K \exp\left( \int_0^T a_t dt \right) - Y_0 \).

The theorem is verified. \( \square \)
A European put option gives the holder the right, but not the obligation, to sell a stock at a specified time for a specified price. Next, European put option pricing problem for stock model (2) will be considered. We assume that a European put option has strike price $K$ and expiration time $T$. Then the payoff from buying a European put option is $(K - Y_T)^+$. Considering the time value of money, the present value of this payoff is $\exp(-\int_0^T r_t dt)(K - Y_T)^+$. Thus the European put option price should be $E[\exp(-\int_0^T r_t dt)(K - Y_T)^+]$.

**Definition 3.3.** European put option price $f$ of model (2) is defined as

$$f(Y_0, K, a_t, b_t, \sigma_t, r_t) = \exp(-\int_0^T r_t dt)E[(K - (\exp(-\int_0^T a_t dt)(Y_0 + \int_0^T \exp(\int_0^t a_s ds)a_t b_t dt)$$

$$+ \int_0^T \exp(\int_0^t a_s ds)\sigma_t dC_t))^+] = \exp(-\int_0^T r_t dt)\int_{-\infty}^{+\infty} Cr\{[(K - (\exp(-\int_0^T a_t dt)(Y_0 + \int_0^T \exp(\int_0^t a_s ds)a_t b_t dt)$$

$$+ \int_0^T \exp(\int_0^t a_s ds)\sigma_t dC_t))^+ \geq r\}dr$$

where $L = K\exp(\int_0^T a_s ds) - Y_0$.

**Theorem 3.4.** (European put Option Pricing Formula) The European put option pricing formula for the model (2) is given by

$$f(Y_0, K, a_t, b_t, \sigma_t, r_t) = \exp(-\int_0^T r_t dt)(K - Y_T)^+.$$

Proof. According to the definition of expected value of fuzzy variable, we have

$$f(Y_0, K, a_t, b_t, \sigma_t, r_t) = \exp(-\int_0^T r_t dt)E[(K - (\exp(-\int_0^T a_t dt)(Y_0 + \int_0^T \exp(\int_0^t a_s ds)a_t b_t dt$$

$$+ \int_0^T \exp(\int_0^t a_s ds)\sigma_t dC_t))^+] = \exp(-\int_0^T r_t dt)\int_{-\infty}^{+\infty} Cr\{[(K - (\exp(-\int_0^T a_t dt)(Y_0 + \int_0^T \exp(\int_0^t a_s ds)a_t b_t dt)$$

$$+ \int_0^T \exp(\int_0^t a_s ds)\sigma_t dC_t))^+ \geq r\}dr$$

$$= \exp(-\int_0^T r_t dt)\int_{-\infty}^{+\infty} Cr\{exp(-\int_0^T a_t dt)(Y_0 + \int_0^T \exp(\int_0^t a_s ds)a_t b_t dt$$

$$+ \int_0^T \exp(\int_0^t a_s ds)\sigma_t dC_t) \geq r\}dr$$
American call option price

Definition 4.1. Next, we consider American call option pricing problem for the stock model (2). The holder of American call option will decide to buy the asset at any time before the option expires, in the same way, the holder of American put option will decide to sell the asset at any time when the price of the underlying asset at the expiry time $T$ is less than the strike price $K$. The holder of the American call option will decide to buy the asset at any time before the option expires, in the same way, the holder of American put option will decide to sell the asset at any time when the price of the underlying asset at the expiry time $T$ is higher than the strike price $K$. The holder of American call option will decide to buy the asset at any time before the option expires, in the same way, the holder of American put option will decide to sell the asset at any time when the price of the underlying asset at the expiry time $T$ is lower than the strike price $K$.

In order to calculate the American call option price, we derived the following theorem.

Definition 4.1. American call option price $f$ of model (2) is defined as

$$f(Y_0, K, a_t, b_t, \sigma_t, r_t) = \max_{0 \leq t \leq T} E[\exp(-\int_0^t r_s ds)((\exp(-\int_0^t a_s ds)(Y_0 + \int_0^t \exp(\int_0^s a_x dx)a_s b_s dx)
+ \int_0^t \exp(\int_0^s a_x dx)\sigma_s dC_t) - K)]^+,$$

where $K$ is the strike price at time $t$.

In order to calculate the American call option price, we derived the following theorem.

Theorem 4.2. (American Call Option Pricing Formula) The American call option pricing formula for the model (2) is given by

$$f(Y_0, K, a_t, b_t, \sigma_t, r_t) = \max_{0 \leq t \leq T} \exp(-\int_0^t r_s ds) \int_0^T^\infty \Cr{\exp(-\int_0^t a_s ds)(Y_0 + \int_0^t \exp(\int_0^s a_x dx)a_s b_s dx)}$$
According to the definition of expected value of fuzzy variable, we have

\[
f(Y_0, K, a_t, b_t, \sigma_t, r_t)
\]

\[
= \max_{0 \leq t \leq T} E[\exp(-\int_0^t r_s ds)(Y_t - K)^+]
\]

\[
= \max_{0 \leq t \leq T} E[\exp(-\int_0^t r_s ds)((\exp(-\int_0^t a_s ds)(Y_0 + \int_0^t \exp(\int_0^s a_x dx)a_x b_x ds
\]

\[
+ \int_0^t \exp(\int_0^s a_x dx)\sigma_x dC_s) - K))^+]
\]

\[
= \max_{0 \leq t \leq T} \int_0^T Cr(\exp(-\int_0^t r_s ds)((\exp(-\int_0^t a_s ds)(Y_0 + \int_0^t \exp(\int_0^s a_x dx)a_x b_x ds
\]

\[
+ \int_0^t \exp(\int_0^s a_x dx)\sigma_x dC_s) - K)^+] \geq r]du
\]

\[
= \max_{0 \leq t \leq T} \exp(-\int_0^t r_s ds) \int_0^\infty \exp(-\int_0^t a_s ds)(Y_0 + \int_0^t \exp(\int_0^s a_x dx)a_x b_x ds
\]

\[
+ \int_0^t \exp(\int_0^s a_x dx)\sigma_x dC_s) \geq r + K) \geq r]dr
\]

\[
= \max_{0 \leq t \leq T} \exp(-\int_0^t r_s ds) \int_K^\infty \exp(-\int_0^t a_s ds)(Y_0 + \int_0^t \exp(\int_0^s a_x dx)a_x b_x ds
\]

\[
+ \int_0^t \exp(\int_0^s a_x dx)\sigma_x dC_s) \geq u]du.
\]

The theorem is obtained. \( \square \)

Next, we consider American put option pricing problem for the stock model (2).

**Definition 4.3.** American put option price \( f \) of model (2) is defined as

\[
f(Y_0, K, a_t, b_t, \sigma_t, r_t)
\]

\[
= \max_{0 \leq t \leq T} E[\exp(-\int_0^t r_s ds)(K - (\exp(-\int_0^t a_s ds)(Y_0 + \int_0^t \exp(\int_0^s a_x dx)a_x b_x ds
\]

\[
+ \int_0^t \exp(\int_0^s a_x dx)\sigma_x dC_s))^+],
\]

where \( K \) is the strike price at time \( T \).

In order to calculate the American put option price, we derived the following theorem.

**Theorem 4.4.** (American Put Option Pricing Formula) The American put option pricing formula for the model (2) is given by
According to the definition of expected value of fuzzy variable, we have

\[
f(Y_0, K, a_t, b_t, \sigma_t, r_t) = \max_{0 \leq t \leq T} \exp(- \int_0^t r_s ds) \int_{-\infty}^{K} \mathcal{C} \{ \exp(- \int_0^t a_s ds)(Y_0 + \int_0^t \exp(\int_0^s a_x dx) a_s b_s ds + \int_0^t \exp(\int_0^s a_x dx) \sigma_s dC_s) \} \] \]

Proof. According to the definition of expected value of fuzzy variable, we have

\[
f(Y_0, K, a_t, b_t, \sigma_t, r_t) = \max_{0 \leq t \leq T} E[\exp(- \int_0^t r_s ds)(K - \exp(- \int_0^t a_s ds)(Y_0 + \int_0^t \exp(\int_0^s a_x dx) a_s b_s ds + \int_0^t \exp(\int_0^s a_x dx) \sigma_s dC_s)]^+] \]

\[
= \max_{0 \leq t \leq T} \int_0^{+\infty} \mathcal{C} \{ \exp(- \int_0^t a_s ds)(K - \exp(- \int_0^t a_s ds)(Y_0 + \int_0^t \exp(\int_0^s a_x dx) a_s b_s ds + \int_0^t \exp(\int_0^s a_x dx) \sigma_s dC_s)) \} \geq r \} dr \]

\[
= \max_{0 \leq t \leq T} \exp(- \int_0^t r_s ds) \int_0^{+\infty} \mathcal{C} \{ K - \exp(- \int_0^t a_s ds)(Y_0 + \int_0^t \exp(\int_0^s a_x dx) a_s b_s ds + \int_0^t \exp(\int_0^s a_x dx) \sigma_s dC_s) \} \geq r \} dr \]

\[
= \max_{0 \leq t \leq T} \exp(- \int_0^t r_s ds) \int_0^{+\infty} \mathcal{C} \{ \exp(- \int_0^t a_s ds)(Y_0 + \int_0^t \exp(\int_0^s a_x dx) a_s b_s ds + \int_0^t \exp(\int_0^s a_x dx) \sigma_s dC_s) \} \leq r + K \} dr \]

\[
= \max_{0 \leq t \leq T} \exp(- \int_0^t r_s ds) \int_{-\infty}^{K} \mathcal{C} \{ \exp(- \int_0^t a_s ds)(Y_0 + \int_0^t \exp(\int_0^s a_x dx) a_s b_s ds + \int_0^t \exp(\int_0^s a_x dx) \sigma_s dC_s) \} \leq u \} du. \]

The theorem is verified. \( \square \)

5. Conclusions. The main results in this paper can be summarized as follows:
(a) The European call option and put option were defined according to generalized Gao’s stock model, and the corresponding pricing formulas were proved; (b) The American call option and put option were defined according to generalized Gao’s stock model, and the corresponding pricing formulas were derived.

REFERENCES


Received June 2018; revised June 2018.
E-mail address: yycclian@163.com
E-mail address:bole0104@163.com