THE CAUCHY PROBLEM FOR A GENERALIZED NOVIKOV EQUATION

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Abstract. We establish the local well-posedness for a generalized Novikov equation in nonhomogeneous Besov spaces. Besides, we obtain a blow-up criteria and provide a sufficient condition for strong solutions to blow up in finite time.

1. Introduction. In this paper we consider the Cauchy problem for the following generalized Novikov equation:

\[
\begin{aligned}
(1 - \partial_x^2)u_t &= (1 + \partial_x)(2u_x^2u_{xx} - uu_xu_{xx} - u^3 - u^2u_{xx} - 2uu_x^2), \\
u(0,x) &= u_0(x), \quad t \geq 0, \quad x \in \mathbb{R},
\end{aligned}
\]

which can be rewritten as

\[
\begin{aligned}
m_t + (2u_x^2 - uu_x - u^2)m_x &= (4u_x - u)m^2 + (2u_x - u)(u_x - u)m, \\
m &= u - uu_x, \quad m(0,x) = (1 - \partial_x^2)u_0(x) = m_0(x), \quad t \geq 0, \quad x \in \mathbb{R}.
\end{aligned}
\]

The equation (1.1) was proposed by Novikov in [41]. It was shown that (1.1) possesses an infinite hierarchy of quasi-local higher symmetries in [41]. It is of the form

\[
(1 - \partial_x^2)u_t = F(u, u_x, u_{xx}, u_{xxx}),
\]

where $F$ is a homogeneous polynomial. The most celebrated integrable member of (1.3) is the Camassa-Holm (CH) equation:

\[
(1 - \partial_x^2)u_t = 3u u_x - 2u_xu_{xx} - uu_{xxx}.
\]

The CH equation can be regarded as a shallow water wave equation [5, 17]. It is completely integrable [4, 7, 15, 18], and it also has a bi-Hamiltonian structure [27], and

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admits exact peaked solitons of the form $ce^{-|x-ct|}$ with $c > 0$, which are orbitally stable \[20\]. It is worth mentioning that the peakons are suggested by the form of the Stokes water wave of greatest height, see discussions in \[9, 10, 13, 14, 40, 42\]. The local well-posedness for the Cauchy problem of the CH equation in Sobolev spaces and Besov spaces was discussed in \[11, 21\]. It was shown that there exist global strong solutions to the CH equation and finite time blow-up strong solutions to the CH equation \[8, 11, 12\]. The existence and uniqueness of global weak solutions were also proven in \[19, 48\]. The global conservative and dissipative solutions of the CH equation were discussed in \[2, 3\].

The second important integrable member of \(1.3\) is the Degasperis-Procesi (DP) equation \[23\]:

\[
(1 - \partial^2_x)u_t = 4uu_x - 3u_xu_{xx} - u_{xxx}.
\]

The DP equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the CH equation \[24\]. It is integrable and has a bi-Hamiltonian structure \[22\]. It also has travelling wave solutions \[32, 44\]. The Cauchy problems of the DP equation has been studied extensively. The local well-posedness, blow-up phenomena, global strong solutions and global weak solutions were studied in \[6, 25, 26, 28, 36, 37, 51, 52, 53, 54\]. Different from the CH equation, the DP equation has not only peakon solutions \[16, 22\] and periodic peakon solutions \[54\] but also shock peakons \[38\] and periodic shock waves \[26\].

The third integrable example of \(1.3\) is the Novikov equation:

\[
(1 - \partial^2_x)u_t = 3uu_x + u^2u_{xxx} - 4u^2u_x.
\]

Compared with the above two equation, it has cubic nonlinearity. It also possesses a bi-Hamiltonian structure and admits exact peakon solutions $u(t, x) = \sqrt{c}e^{-|x-ct|}$ with $c > 0$ in \[30\]. The local well-posedness was studied in \[46, 47, 49, 50\]. The global existence of strong solutions was established in \[46\] under some sign conditions and the blow-up phenomena of the strong solutions were shown in \[50\]. The global weak solutions were studied in \[45\].

The aim of this paper is to establish the local well-posedness for the Cauchy problem of \(1.2\) in Besov spaces, and to get blow-up results for strong solutions to \(1.2\). Our paper is organized as follows. In Section 2, we introduce some preliminaries which will be used in sequel. In Section 3, we prove the local well-posedness of \(1.2\) in $B^s_{p,r}$ with $s > \max(\frac{1}{2}, \frac{1}{p})$ or $(s = \frac{1}{p}, 1 \leq p \leq 2, r = 1)$ in the sense of Hadamard (i.e. \(1.2\) has a unique local solution in $B^s_{p,r}$ with continuity with respect to the initial data). The main approach is based on the Littlewood-Paley theory and transport equations theory. In Section 4, we obtain a blow-up criteria and give a sufficient condition for strong solutions to blow up in finite time.

2. Preliminaries. In this section, we first recall the Littlewood-Paley decomposition and Besov spaces.

**Proposition 2.1.** \[1\] Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{5}{4}\}$. There exist radial functions $\chi$ and $\varphi$, valued in the interval $[0, 1]$, belonging respectively to $\mathcal{D}(B(0, \frac{3}{4}))$ and $\mathcal{D}(\mathcal{C})$, and such that

\[
\forall \xi \in \mathbb{R}^d, \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1,
\]

\[
\forall \xi \in \mathbb{R}^d \setminus \{0\}, \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1,
\]
\(|j - j'| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-j} \cdot) \cap \text{Supp } \varphi(2^{-j'} \cdot) = \emptyset,\)
\(j \geq 1 \Rightarrow \text{Supp } \chi(\cdot) \cap \text{Supp } \varphi(2^{-j} \cdot) = \emptyset.\)

The set \(\tilde{C} = B(0, \frac{2}{3}) + \tilde{C}\) is an annulus, and we have
\(|j - j'| \geq 5 \Rightarrow 2^j \tilde{C} \cap 2^{j'} \tilde{C} = \emptyset.\)

Further, we have
\(\forall \xi \in \mathbb{R}^d, \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j} \xi) \leq 1,\)
\(\forall \xi \in \mathbb{R}^d \setminus \{0\}, \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j} \xi) \leq 1.\)

Denote \(\mathcal{F}\) by the Fourier transform and \(\mathcal{F}^{-1}\) by its inverse. Let \(u\) be a tempered distribution in \(S'(\mathbb{R}^d)\). For all \(j \in \mathbb{Z}\), define
\[
\Delta_j u = 0 \text{ if } j \leq -2, \quad \Delta_{-1} u = \mathcal{F}^{-1}(\chi F u),
\]
\[
\Delta_j u = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) F u) \text{ if } j \geq 0, \quad S_j u = \sum_{j' < j} \Delta_{j'} u.
\]

Then we have the following Littlewood-Paley decomposition:
\[
u = \sum_{j \in \mathbb{Z}} \Delta_j u \text{ in } S'(\mathbb{R}^d).
\]

Let \(s \in \mathbb{R}, 1 \leq p, r \leq \infty\). The nonhomogeneous Besov space \(B^s_{p, r}(\mathbb{R}^d)\) is defined by
\[
B^s_{p, r}(\mathbb{R}^d) = \{ u \in S'(\mathbb{R}^d) : \|u\|_{B^s_{p, r}(\mathbb{R}^d)} = \left\| (2^j)^s \|\Delta_j u\|_{L^p} \right\|_{l^r(\mathbb{Z}) < \infty} \}.
\]

Some properties about Besov spaces are as follows.

**Proposition 2.2.** [1][2] Let \(s \in \mathbb{R}, 1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty\).

1. \(B^s_{p, r}\) is a Banach space, and is continuously embedded in \(S'\).
2. If \(r < \infty\), then \(\lim_{j \to \infty} \|S_j u - u\|_{B^s_{p, r}} = 0\). If \(p, r < \infty\), then \(C_0^\infty\) is dense in \(B^s_{p, r}\).
3. If \(p_1 \leq p_2\) and \(r_1 \leq r_2\), then \(B^s_{p_1, r_1} \hookrightarrow B^{s - \frac{1}{p_1}}_{p_2, r_2}\). If \(s_1 < s_2\), then the embedding \(B^{s_1}_{p_2, r_2} \hookrightarrow B^{s_2}_{p_1, r_1}\) is locally compact.
4. \(B^s_{p, r} \hookrightarrow L^\infty \Leftrightarrow s > \frac{d}{p} \) or \(s = \frac{d}{p}, r = 1\). \(B^s_{2, 2} = H^s\).
5. Fatou property: if \((u_n)_{n \in \mathbb{N}}\) is a bounded sequence in \(B^s_{p, r}\), then an element \(u \in B^s_{p, r}\) and a subsequence \((u_{n_k})_{k \in \mathbb{N}}\) exist such that
\[
\lim_{k \to \infty} u_{n_k} = u \text{ in } S' \text{ and } \|u\|_{B^s_{p, r}} \leq \text{Cl} \inf_{k \to \infty} \|u_{n_k}\|_{B^s_{p, r}}.
\]
6. Let \(m \in \mathbb{R}\) and \(f\) be a \(S^m\)-multiplier (i.e. \(f\) is a smooth function and satisfies \(\forall \alpha \in \mathbb{N}^d, \exists C = C(\alpha), \text{ such that } |\partial^\alpha f(\xi)| \leq C(1 + |\xi|)^{m-|\alpha|}, \forall \xi \in \mathbb{R}^d\)). Then the operator \(f(D) = \mathcal{F}^{-1}(fF)\) is continuous from \(B^s_{p, r}\) to \(B^{s-m}_{p, r}\).

We introduce two useful interpolation inequalities.

**Proposition 2.3.** [1][2] (1) If \(s_1 < s_2, \theta \in (0, 1), \) and \((p, r)\) is in \([1, \infty]^2\), then we have
\[
\|u\|_{B^{s_1 + (1-\theta)s_2}_{p_2, r_2}} \leq \|u\|_{B^{s_1}_{p_1, r_1}}^\theta \|u\|_{B^{s_2}_{p_2, r_2}}^{1-\theta}.
\]
(2) If \( s \in \mathbb{R}, \ 1 \leq p \leq \infty, \ \varepsilon > 0, \) a constant \( C = C(\varepsilon) \) exists such that
\[
\|u\|_{B_{p,r}^{r,1}} \leq C\|u\|_{B_{p,\infty}^{r}} \ln \left( 1 + \frac{\|u\|_{B_{p,\infty}^{r}}}{\|u\|_{B_{p,\infty}^{r}}} \right).
\]

Proposition 2.4. \([1]\) Let \( s \in \mathbb{R}, \ 1 \leq p, r \leq \infty. \)
\[
\begin{align*}
B_{p,r}^{s} \times B_{p',r'}^{s'} & \rightarrow \mathbb{R}, \\
(u, \phi) & \mapsto \sum_{|\Delta_j u, \Delta_j \phi| \leq 1} \langle \Delta_j u, \Delta_j \phi \rangle,
\end{align*}
\]
defines a continuous bilinear functional on \( B_{p,r}^{s} \times B_{p',r'}^{s'} \). Denote by \( Q_{p',r'} \), the set of functions \( \phi \) in \( S' \) such that \( \|\phi\|_{B_{p',r'}^{s'}} \leq 1 \). If \( u \) is in \( S' \), then we have
\[
\|u\|_{B_{p,r}^{s}} \leq C \sup_{\phi \in Q_{p',r'}^{s'}} \langle u, \phi \rangle.
\]

We have the following continuity properties for the product of two functions:

Lemma 2.5. \([1, 31]\) (1) For any \( s > 0 \) and any \( (p, r) \) in \([1, \infty]^2\), the space \( L^\infty \cap B_{p,r}^{s} \) is an algebra, and a constant \( C = C(s, d) \) exists such that
\[
\|uv\|_{B_{p,r}^{s}} \leq C(\|u\|_{L^\infty} \|v\|_{B_{p,r}^{s}} + \|u\|_{B_{p,r}^{s}} \|v\|_{L^\infty}).
\]

(2) If \( 1 \leq p, r \leq \infty, \ s_1 \leq s_2, \ s_2 > \frac{d}{p} (s_2 \geq \frac{d}{p} \text{ if } r = 1) \) and \( s_1 + s_2 > \max(0, \frac{2d}{p} - d) \), there exists \( C = C(s_1, s_2, p, r, d) \) such that
\[
\|uv\|_{B_{p,r}^{s_1}} \leq C\|u\|_{B_{p,r}^{s_1}} \|v\|_{B_{p,r}^{s_2}}.
\]

(3) If \( 1 \leq p \leq 2 \), there exists \( C = C(p, d) \) such that
\[
\|uv\|_{B_{p,\infty}^{\frac{d}{p} - d}} \leq C\|u\|_{B_{p,\infty}^{\frac{d}{p} - d}} \|v\|_{B_{p,1}^{\frac{d}{p}}}.\]

Here is the Osgood lemma, a generalization of the Gronwall lemma.

Lemma 2.6. \([1]\) Let \( \rho \) be a measurable function from \([t_0, T]\) to \([0, a] \), \( \gamma \) a locally integrable function from \([t_0, T]\) to \( \mathbb{R}^+ \), and \( \mu \) a continuous and nondecreasing function from \([0, a]\) to \( \mathbb{R}^+ \). Assume that for some \( c \geq 0 \), the function \( \rho \) satisfies
\[
\rho(t) \leq c + \int_{t_0}^{t} \gamma(t') \mu(\rho(t')) dt' \quad \text{for a.e. } t \in [t_0, T].
\]

If \( c > 0 \), then for a.e. \( t \in [t_0, T] \),
\[
-\mathcal{M}(\rho(t)) + \mathcal{M}(c) \leq \int_{t_0}^{t} \gamma(t') dt' \quad \text{with} \quad \mathcal{M}(x) = \int_{x}^{a} \frac{dr}{\mu(r)}.
\]

If \( c = 0 \), and \( \mu \) satisfies \( \int_{0}^{a} \frac{dr}{\mu(r)} = \infty \), then \( \rho = 0 \), a.e.

Remark 2.7. \([39]\) For example, when \( \mu(r) = r(1 - \ln r), \ r \in [0, 1] \), we have \( \mathcal{M}(x) = \ln(1 - \ln x) \), and \( \rho(t) \leq ce^{x - f'_0 \gamma(t')} dt' \), if \( c > 0 \). We will use this result later.

We shall present here some results for the following transport equation.
\[
\begin{align*}
& \frac{d}{dt} f + v \cdot \nabla f = g, \ x \in \mathbb{R}^d, \ t > 0, \\
& f(0, x) = f_0(x).
\end{align*}
\] (2.1)
Lemma 2.8. [1] Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. There exists a constant $C$ such that for all solutions $f \in L^\infty([0,T]; \mathcal{B}_{p,r}^s)$ of (2.1) in one dimension with initial data $f_0$ in $\mathcal{B}_{p,r}^s$, and $g \in L^1([0,T]; \mathcal{B}_{p,r}^s)$, we have, for a.e. $t \in [0,T]$, 
\[ \|f(t)\|_{\mathcal{B}_{p,r}^s} \leq e^{CV(t)} \left( \|f_0\|_{\mathcal{B}_{p,r}^s} + \int_0^t e^{-CV(t')} \|g(t')\|_{\mathcal{B}_{p,r}^s} dt' \right) \]
with 
\[ V'(t) = \left\{ \begin{array}{ll}
\|\nabla v\|_{\mathcal{B}_{p,r}^s}^{-1}, & \text{if } s > \max\left(-\frac{1}{2}, \frac{1}{p} - 1\right), \\
\|\nabla v\|_{\mathcal{B}_{p,r}^s}^{-1}, & \text{if } s > \frac{1}{p} \text{ or } (s = \frac{1}{p}, p < \infty, r = 1),
\end{array} \right. \]
and when $s = \frac{1}{p} - 1$, $1 \leq p \leq 2$, $r = \infty$, and $V'(t) = \|\nabla u\|_{\mathcal{B}_{p,r}^s}^{-1}$.

Lemma 2.9. [55] Let $s > 0$, $1 \leq p, r \leq \infty$. Define $R_j = \{v \cdot \nabla, \Delta_j\} f$. There exists a constant $C$ such that 
\[ \left\| \left(2^{js} \|R_j\|_{L^p}\right) \right\|_{L^r(\mathbb{R})} \leq C(\|\nabla v\|_{L^\infty} \|f\|_{\mathcal{B}_{p,r}^s} + \|\nabla v\|_{\mathcal{B}_{p,r}^s} \|f\|_{L^\infty}). \]
Hence, if $f$ solves the equation (2.1), we have 
\[ \|f(t)\|_{\mathcal{B}_{p,r}^s} \leq \|f_0\|_{\mathcal{B}_{p,r}^s} + C \int_0^t (\|\nabla v\|_{L^\infty} \|f\|_{\mathcal{B}_{p,r}^s} + \|\nabla v\|_{\mathcal{B}_{p,r}^s} \|f\|_{L^\infty} + g\|_{\mathcal{B}_{p,r}^s} dt'). \]

Lemma 2.10. [1] Let $1 \leq p \leq p_1 \leq \infty$, $1 \leq r \leq \infty$, and $s > -\min(\frac{1}{p_1}, \frac{1}{p})$. Let $f_0 \in \mathcal{B}_{p,r}^s$, $g \in L^1([0,T]; \mathcal{B}_{p,r}^s)$, and $v$ be a time-dependent vector field such that $v \in L^\infty([0,T]; B_{-M,\infty}^s)$ for some $\rho > 1$ and $M > 0$, and 
\[ \|\nabla v\|_{L^\infty} \leq (1 + \frac{d}{p_1}), \quad \text{if } s < 1 + \frac{d}{p_1}, \]
\[ \|\nabla v\|_{L^\infty} \leq (1 + \frac{d}{p_1}), \quad \text{if } s > 1 + \frac{d}{p_1} \text{ or } (s = 1 + \frac{d}{p_1} \text{ and } r = 1). \]
Then the equation (2.1) has a unique solution $f$ in 
- the space $C([0,T]; \mathcal{B}_{p,r}^s)$, if $r < \infty$, 
- the space $\left( \bigcap_{s' < s} C([0,T]; \mathcal{B}_{p,r}^{s'}) \right) \cap C_w([0,T]; \mathcal{B}_{p,r}^s)$, if $r = \infty$.

Lemma 2.11. [33] Let $1 \leq p \leq \infty$, $1 \leq r < \infty$, and $s > \frac{d}{p}$ (or $s = \frac{d}{p}$, $p < \infty$, $r = 1$). Denote $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Let $(v^n)_{n \in \mathbb{N}} \subset C([0,T]; \mathcal{B}_{p,r}^s)$. Assume that $(f^n)_{n \in \mathbb{N}}$ in $C([0,T]; \mathcal{B}_{p,r}^s)$ is the solution to 
\[ \left\{ \begin{array}{l}
f_t^n + v^n \cdot \nabla f^n = g, \quad x \in \mathbb{R}^d, \quad t > 0, \\
f^n(0, x) = f_0(x)
\end{array} \right. \]
with initial data $f_0 \in \mathcal{B}_{p,r}^s$, $g \in L^1([0,T]; \mathcal{B}_{p,r}^s)$, and that for some $\alpha \in L^1([0,T])$, 
\[ \sup_{n \in \bar{\mathbb{N}}} \|v^n(t)\|_{\mathcal{B}_{p,r}^{\alpha+1}} \leq \alpha(t). \]
If $v^n \to v^\infty$ in $L^1([0,T]; \mathcal{B}_{p,r}^s)$, then $f^n \to f^\infty$ in $C([0,T]; \mathcal{B}_{p,r}^s)$.

3. Local well-posedness. In this section, we establish local well-posedness of (1.2) in Besov spaces. First we introduce the following function spaces.

Definition 3.1. Let $T > 0$, $s \in \mathbb{R}$, and $1 \leq p, r \leq \infty$. Set 
\[ E_{p,r}(T) \triangleq \left\{ \begin{array}{ll}
C([0,T]; \mathcal{B}_{p,r}^s) \cap C^1([0,T]; \mathcal{B}_{p,r}^{s+1}), & \text{if } r < \infty, \\
C_w([0,T]; \mathcal{B}_{p,r}^s) \cap C^{0,1}([0,T]; \mathcal{B}_{p,r}^{s-1}), & \text{if } r = \infty.
\end{array} \right. \]
Our main result is stated as follows.
Theorem 3.2. Let $1 \leq p, r \leq \infty$, $s \in \mathbb{R}$ and let $(s, p, r)$ satisfy the condition $s > \max(\frac{1}{2}, \frac{1}{p})$ or $(s = \frac{1}{p}, 1 \leq p \leq 2, r = 1)$. Assume $m_0 \in B^{s}_{p,r}$. Then there exists a time $T > 0$ such that (1.2) has a unique solution $u$ in $E_{p,r}^n(T)$. Moreover the solution depends continuously on the initial data.

In order to prove Theorem 3.2 we proceed as the following steps.

**Step 1.** Starting from $m^0 \triangleq 0$, we define by induction a sequence $(m^n)_{n \in \mathbb{N}}$ of smooth functions by solving the following linear transport equations:

\[
\begin{cases}
  m^{n+1}_t + [2(u^n)_x^n - u^n u^n_n - (u^n)^2]m^{n+1}_x \\
  m^n = u^n - u^n_{xxx}, \quad m^{n+1}|_{t=0} = S_{n+1}m_0.
\end{cases}
\]  

(3.1)

Define $G^n = 2(u^n)_x^n - u^n u^n_n - (u^n)^2$, $F^n = (4u^n - u^n)(m^n)^2 + (2u^n - u^n)(u^n_n - u^n)m^n$. We assume that $m_n \in L^\infty([0, T]; B^{s}_{p,r})$ for all $T > 0$. Note that under the assumptions on $(s, p, r)$, $B^{s}_{p,r}$ and $B^{s+1}_{p,r}$ are algebras. We have

\[
\begin{align*}
  \|G^n_x\|_{B^{s+1}_{p,r}} &\leq \|G^n\|_{B^{s+1}_{p,r}} \leq C(\|u^n\|_{B^{s+1}_{p,r}}^2 + \|u^n\|_{B^{s+1}_{p,r}} \|u^n\|_{B^{s+1}_{p,r}} + \|u^n\|_{B^{s+1}_{p,r}}^2) \\
  &\leq C(\|m^n\|_{B^{s}_{p,r}}^2 + \|m^n\|_{B^{s}_{p,r}} \|m^n\|_{B^{s}_{p,r}} + \|m^n\|_{B^{s}_{p,r}}^2) \leq C\|m^n\|_{B^{s}_{p,r}}^2, \\
  \|F^n\|_{B^{s}_{p,r}} &\leq \|4u^n - u^n\|_{B^{s}_{p,r}} \|m^n\|_{B^{s}_{p,r}}^2 + \|2u^n - u^n\|_{B^{s}_{p,r}} \|u^n\|_{B^{s}_{p,r}} \|m^n\|_{B^{s}_{p,r}} \\
  &\leq C(\|m^n\|_{B^{s}_{p,r}} \|m^n\|_{B^{s}_{p,r}}^2 + \|m^n\|_{B^{s}_{p,r}} \|m^n\|_{B^{s}_{p,r}}^2) \leq C\|m^n\|_{B^{s}_{p,r}}^3.
\end{align*}
\]  

(3.2)

(3.3)

So $G^n, F^n \in L^\infty([0, T]; B^{s}_{p,r})$. Hence applying Lemma 2.10 ensures that (3.1) has a global solution $m^{n+1}$ which belongs to $E_{p,r}^n(T)$ for all $T > 0$.

**Step 2.** By Lemma 2.8 together with (3.2) and (3.3), we have

\[
\begin{align*}
  \|m^{n+1}(t)\|_{B^{s}_{p,r}} &\leq e^{C \int_0^t \|G^n\|_{B^{s}_{p,r}} dt'} \left(\|m_0\|_{B^{s}_{p,r}} + \int_0^t e^{-C \int_0^{t'} \|G^n\|_{B^{s}_{p,r}} dt''} \|F^n\|_{B^{s}_{p,r}} dt''\right) \\
  &\leq C e^{C \int_0^t \|m^n\|_{B^{s}_{p,r}}^2 dt'} \left(\|m_0\|_{B^{s}_{p,r}} + \int_0^t e^{-C \int_0^{t'} \|m^n\|_{B^{s}_{p,r}}^2 dt''} \|m^n\|_{B^{s}_{p,r}}^3 dt''\right). \quad (3.4)
\end{align*}
\]

We fix a $T > 0$ such that $4C^3T\|m_0\|_{B^{s}_{p,r}}^2 < 1$ and suppose that

\[
\forall t \in [0, T], \quad \|m^n(t)\|_{B^{s}_{p,r}} \leq \frac{C\|m_0\|_{B^{s}_{p,r}}}{(1 - 4C^3T\|m_0\|_{B^{s}_{p,r}}^2)^{\frac{1}{2}}}, \quad (3.5)
\]

Plugging (3.5) into (3.4) yields

\[
\begin{align*}
  \|m^{n+1}(t)\|_{B^{s}_{p,r}} &\leq C\|m_0\|_{B^{s}_{p,r}} \left(1 - 4C^3T\|m_0\|_{B^{s}_{p,r}}^2\right)^{-\frac{1}{2}} \times \left(1 + C^3\|m_0\|_{B^{s}_{p,r}}^2 \int_0^t (1 - 4C^3T\|m_0\|_{B^{s}_{p,r}}^2)^{-\frac{1}{2}} dt'ight) \\
  &\leq \frac{C\|m_0\|_{B^{s}_{p,r}}}{(1 - 4C^3T\|m_0\|_{B^{s}_{p,r}}^2)^{\frac{1}{2}}},
\end{align*}
\]
Therefore, \((m^n)_{n \in \mathbb{N}}\) is bounded in \(L^\infty([0,T]; B^s_{p,r}).\)

**Step 3.** When \(s = \text{max}\left(\frac{1}{2}, \frac{1}{p}\right)\) or \(s = \frac{1}{p}, 1 \leq p \leq 2, r = 1\), some estimates we shall use are slightly different, so we have to discuss separately.

**Case 1.** \(s > \text{max}\left(\frac{1}{2}, \frac{1}{p}\right)\).

We are going to show that \((m^n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \(L^\infty([0,T]; B^s_{p,r}).\)

For that purpose, we have, for all \((n, k) \in \mathbb{N}^2,\)

\[
(m^{n+k+1} - m^{n+1})_t + G^{n+k}(m^{n+k+1} - m^{n+1})_x = (G^n - G^{n+k})m^{n+1}_x + F^{n+k} - F^n,
\]

where

\[
G^n - G^{n+k} = -2(u^n + u^{n+k})\delta^n_x + \delta^n_x u^n + u^{n+k}\delta^n_x + (u^n + u^{n+k})\delta^n_x,
\]

\[
F^{n+k} - F^n = (4\delta^n_x - \delta^n_x)(m^{n+k})^2 + (2\delta^n_x - \delta^n_x)(u^{n+k} - u^n)m^{n+k} + (4u^n - u^n)(m^{n+k} - m^n)(m^{n+k} + m^n) + (2u^n - u^n)(\delta^n_x - \delta^n_x)m^{n+k} + (2u^n - u^n)(u^n - u^n + m^n)(m^{n+k} - m^n)
\]

with \(\delta^n_x = u^{n+k} - u^n.\)

Applying Lemma 2.8 yields, for any \(t \in [0,T],\)

\[
||m^{n+k+1} - m^{n+1}(t)||_{B^{s}_{p,r}} \leq \epsilon C \int_0^t ||G^{n+k}||_{B^{s}_{p,r}} \, dt' \left( ||S^{n+k+1}_m - S^{n+1}_m||_{B^{s}_{p,r}} + \int_0^t ||G^n - G^{n+k}||_{B^{s}_{p,r}} \, dt'' \, \left( ||m^{n+1}_x||_{B^{s}_{p,r}} + ||F^{n+k} - F^n||_{B^{s}_{p,r}} \right) \right)
\]

By the fact \(B^s_{p,r} \text{ is an algebra and Lemma 2.5 (2)},\)

\[
||G^n - G^{n+k}||_{B^{s}_{p,r}} \leq 2\|u^n + u^{n+k}\|_{B^{s}_{p,r}} \|\delta^n_x\|_{B^{s}_{p,r}} + \|\delta^n_x u^n\|_{B^{s}_{p,r}} + \|u^{n+k}\|_{B^{s}_{p,r}} \|\delta^n_x\|_{B^{s}_{p,r}}
\]

and

\[
||F^{n+k} - F^n||_{B^{s}_{p-r}} \leq 4\|\delta^n_x - \delta^n_x\|_{B^{s}_{p,r}} \|m^{n+k}\|^2_{B^{s}_{p,r}} + \|\delta^n_x - \delta^n_x\|_{B^{s}_{p,r}} \|u^{n+k} - u^n\|_{B^{s}_{p,r}} \|m^{n+k}\|_{B^{s}_{p,r}}
\]

Since \((m^n)_{n \in \mathbb{N}}\) is bounded in \(L^\infty([0,T]; B^s_{p,r}),\) for all \(t \in [0,T],\) we finally have

\[
||(m^{n+k+1} - m^{n+1})(t)||_{B^{s}_{p-r}}
\]
By Lemma 2.5 (3), we deduce that
\[ \text{case } 2. \]
\[ \text{Gronwall inequality entails that } \]
\[ \text{g} \in \text{Fatou's lemma, we obtain} \]
\[ \text{The Gronwall inequality entails that } \]
\[ \text{is a Cauchy sequence in } B_{p,r}^{s-1}. \]

**Case 2.** \( s = \frac{1}{p} \), \( 1 \leq p \leq 2, \) \( r = 1. \)

From Lemma 2.8, comparing with Case 1, we do not have the estimate for the norm \( B_{p,1}^{s-1} \) but \( B_{p,\infty}^{s-1} \). More precisely, we only have
\[ (m^{n+k+1} - m^{n+1})(t) \leq C \int_0^t C_x \frac{1}{\|G^{n+k}\|_{B_{p,\infty}^{s-1}}} dt' \]
\[ + \int_0^t e^{C \int_0^t \|G^{n+k}\|_{B_{p,\infty}^{s-1}}} dt'' \]
\[ (\|m^{n+k} - m^n\|_{B_{p,\infty}^{s-1}} + \|m^{n+k}\|_{B_{p,\infty}^{s-1}}) \]
\[ C \int_0^t \|G^{n+k}\|_{B_{p,\infty}^{s-1}} + \int_0^t \|G^{n+k} - G^n\|_{B_{p,\infty}^{s-1}} dt' \]
\[ \|F^{n+k} - F^n\|_{B_{p,\infty}^{s-1}} \]
\[ \leq C (\|m^{n+k}\|_{B_{p,\infty}^{s-1}} \frac{1}{\|G^{n+k}\|_{B_{p,\infty}^{s-1}}} + \|m^n\|_{B_{p,\infty}^{s-1}} \frac{1}{\|G^{n+k}\|_{B_{p,\infty}^{s-1}}}) \]
\[ \leq C \left( \left\| m^{n+k+1} \right\|_{B_{p,1}^{s-1}} - m^{n+1} \right\|_{B_{p,1}^{s-1}} + \int_0^t \left\| m^{n+k} \right\|_{B_{p,1}^{s-1}} dt' \right) \]
\[ \leq C \left( \left\| m^{n+k+1} \right\|_{B_{p,1}^{s-1}} - m^{n+1} \right\|_{B_{p,1}^{s-1}} + \int_0^t \left\| m^{n+k} \right\|_{B_{p,1}^{s-1}} dt' \right) \]
Noting that the function \( x \ln(e + \frac{C}{x}) \) is nondecreasing in \( (0, \infty) \), from (3.10) and (3.11), it follows that
\[
\|m^{n+k+1} - m^{n+1}\|_{L^T(B_{\beta,\infty}^{\frac{1}{p}-1})} \\
\leq C \left( \|S_{n+k+1}m_0 - S_{n+1}m_0\|_{B_{p,1}^{\frac{1}{p}}} \\
+ \int_0^t \|m^{n+k} - m^n\|_{L^T(B_{\beta,\infty}^{\frac{1}{p}-1})} \ln \left(e + \frac{C}{\|m^{n+k} - m^n\|_{L^T(B_{\beta,\infty}^{\frac{1}{p}-1})}}\right) dt' \right).
\]

Let \( g(t) \equiv \limsup_{n \to \infty} \|m^{n+k} - m^n\|_{L^T(B_{\beta,\infty}^{\frac{1}{p}-1})} \). The above inequality can be written as
\[
g(t) \leq C \int_0^t g(t') \ln \left(e + \frac{C}{g(t')}\right) dt'.
\]
Hence Lemma 2.6 implies that \( g(t) = 0 \), and \((m^n)_{n \in \mathbb{N}}\) is a Cauchy sequence in \( C([0, T]; B_{\beta,\infty}^{\frac{1}{p}-1}) \) and converges to some limit function \( m \) in \( C([0, T]; B_{\beta,\infty}^{\frac{1}{p}-1}) \).

**Step 4.** We have to check that \( m \) belongs to \( E_{\beta,p,r}^s(T) \) and satisfies (1.2). Since \((m^n)_{n \in \mathbb{N}}\) is bounded in \( L^\infty([0, T]; B_{\beta,r}^s) \), the Fatou property for the Besov spaces guarantees that \( m \) also belongs to \( L^\infty([0, T]; B_{\beta,r}^s) \). Now, taking advantage of interpolation inequalities implies that \((m^n)_{n \in \mathbb{N}}\) converges to \( m \) in \( C([0, T]; B_{\beta,r}^{s'}) \) for any \( s' < s \). It is then easy to pass to the limit in (3.1) and to conclude that \( m \) is indeed a solution of (1.2) in the sense of distributions.

Finally, because \( m \) belongs to \( L^\infty([0, T]; B_{\beta,r}^s) \), the right-hand side of (1.2) also belongs to \( L^\infty([0, T]; B_{\beta,r}^s) \). Hence, according to Lemma 2.10, \( m \) belongs to \( C([0, T]; B_{\beta,r}^{s'}) \) (resp., \( C_w([0, T]; B_{\beta,r}^s) \)) if \( r < \infty \) (resp., \( r = \infty \)). Again using the equation (1.2), we see that \( m_k \) is in \( C([0, T]; B_{\beta,r}^{s-1}) \) if \( r \) is finite, and in \( L^\infty([0, T]; B_{\beta,r}^{s-1}) \) otherwise, so \( m \) belongs to \( E_{\beta,p,r}^s(T) \).

**Step 5.** Next we prove the uniqueness of solutions to (1.2). The proof is based on the way we have in Step 3. Suppose that \((m_1, m_2)\) are two solutions of (1.2). We obtain
\[
\partial_t(m_1 - m_2) + G_1 \partial_x(m_1 - m_2) = (G_2 - G_1) \partial_x m_2 + F_1 - F_2,
\]
where for \( i = 1, 2 \),
\[
G_i = 2(\partial_x u_i)^2 - u_i \partial_x u_i - u_i^2, \\
F_i = (4 \partial_x u_i - u_i)m_i^2 + (2 \partial_x u_i - u_i)(\partial_x u_i - u_i)m_i, \\
u_i = (1 - \partial_x^2)^{-1}m_i.
\]

**Case 1.** \( s > \max\left(\frac{1}{2}, \frac{1}{p}\right) \).

By virtue to Lemma 2.8, we have
\[
\|(m_1 - m_2)(t)\|_{B_{\beta,r}^{s-1}} \leq C \int_0^t \|\partial_x G_1\|_{B_{\beta,r}^{s'}} dt' \left(\|(m_1 - m_2)(0)\|_{B_{\beta,r}^{s-1}} + \|F_1 - F_2\|_{B_{\beta,r}^{s-1}}\right). \tag{3.12}
\]
By a similar calculation as in Step 3, we get
\[
\|(G_2 - G_1)\partial_t m_2\|_{B^{q,-1}_{p,r}} \leq C\|m_2\|_{B^{q}_{p,r}} (\|m_1\|_{B^{q}_{p,r}} + \|m_2\|_{B^{q}_{p,r}})\|m_1 - m_2\|_{B^{q,-1}_{p,r}}. \tag{3.13}
\]
\[
\|F_1 - F_2\|_{B^{q,-1}_{p,r}} \leq C(\|m_1\|_{B^{q}_{p,r}}^2 + \|m_2\|_{B^{q}_{p,r}}^2)\|m_1 - m_2\|_{B^{q,-1}_{p,r}}. \tag{3.14}
\]
Plugging (3.13), (3.14) into (3.12) yields that
\[
\|(m_1 - m_2)(t)\|_{B^{q,-1}_{p,r}} \leq e^{\int_0^t C f_0' \|\partial_t G_1\|_{B^{q}_{p,r}} \, dt'} (\|(m_1 - m_2)(0)\|_{B^{q,-1}_{p,r}})
+ C\int_0^t e^{-\int_s^t C f_0' \|\partial_t G_1\|_{B^{q}_{p,r}} \, dt''} (\|(m_1\|_{B^{q}_{p,r}} + \|m_2\|_{B^{q}_{p,r}})\|m_1 - m_2\|_{B^{q,-1}_{p,r}} \, dt').
\]
Using Gronwall’s inequality, we finally get
\[
\|(m_1(t) - m_2(t))\|_{B^{q,-1}_{p,r}} \leq \|(m_1(0) - m_2(0))\|_{B^{q,-1}_{p,r}} e^{\int_0^t C f_0'(\|m_1\|_{B^{q}_{p,r}} + \|m_2\|_{B^{q}_{p,r}}) \, dt'}. \tag{3.15}
\]
\textbf{Case 2.} \(s = \frac{1}{p}, \ 1 \leq p \leq 2, \ r = 1.\)

According to Lemma 2.7, we have
\[
\|(m_1 - m_2)(t)\|_{B^{q,-1}_{p,r}} \leq e^{\int_0^t \|\partial_x G_1\|_{B^{q}_{p,r}} \, dt'} (\|(m_1 - m_2)(0)\|_{B^{q,-1}_{p,r}})
+ \int_0^t e^{-\int_s^t \|\partial_x G_1\|_{B^{q}_{p,r}} \, dt''} (\|(G_2 - G_1)\partial_x m_2\|_{B^{q,-1}_{p,r}} + \|F_1 - F_2\|_{B^{q,-1}_{p,r}}) \, dt'). \tag{3.16}
\]
Similarly, we deduce that
\[
\|(G_2 - G_1)\partial_x m_2\|_{B^{q,-1}_{p,r}} \leq C\|m_2\|_{B^{q}_{p,r}} (\|m_1\|_{B^{q}_{p,r}} + \|m_2\|_{B^{q}_{p,r}})\|m_1 - m_2\|_{B^{q,-1}_{p,r}}. \tag{3.17}
\]
\[
\|F_1 - F_2\|_{B^{q,-1}_{p,r}} \leq C(\|m_1\|_{B^{q}_{p,r}}^2 + \|m_2\|_{B^{q}_{p,r}}^2)\|m_1 - m_2\|_{B^{q,-1}_{p,r}}. \tag{3.18}
\]
Plugging (3.17), (3.18) into (3.16), and using the boundedness of \(m_i\) with respect to \(t\), we have
\[
\|(m_1 - m_2)(t)\|_{B^{q,-1}_{p,r}} \leq C (\|(m_1 - m_2)(0)\|_{B^{q}_{p,r}} + \int_0^t \|m_1 - m_2\|_{B^{q,-1}_{p,r}} \, dt').
\]
Applying Proposition 2.3 (2), it follows that
\[
\|(m_1 - m_2)(t)\|_{B^{q,-1}_{p,r}} \leq C (\|(m_1 - m_2)(0)\|_{B^{q}_{p,r}} + \int_0^t \|m_1 - m_2\|_{B^{q,-1}_{p,r}} \, dt').
\]
Now let \(h(t) = \|(m_1 - m_2)(t)\|_{B^{q,-1}_{p,r}}\). From above, \(h\) satisfies
\[
h(t) \leq C(h(0) + \int_0^t h(t') \ln (e + \frac{C}{h(t')}) \, dt') \leq C(h(0) + \int_0^t h(t') (1 - \ln \frac{C}{h(t')}) \, dt').
\]
By virtue of Remark 2.7, we finally get
\[
\|(m_1(t) - m_2(t))\|_{B^{q,-1}_{p,r}} \leq C\|m_1(0) - m_2(0)\|_{\frac{1}{q} - \frac{1}{p} \leq \frac{1}{q}}. \tag{3.19}
\]
Therefore the uniqueness is a straightforward conclusion of the above inequalities (3.15) and (3.19). Moreover, an interpolation argument ensures that continuity with respect to the initial data holds for the norm $C([0, T]; B^{s}_{p,r})$ whenever $s' < s$.

**Step 6.** Finally, we end up with a proposition about continuity until the exponent $s$, in the proof of which Lemma 2.11 is necessary.

**Proposition 3.3.** Let $(s, p, r)$ be as the statement of Theorem 3.2. Denote $\tilde{N} = N \cup \{\infty\}$. Suppose that $(m^{n})_{n \in \tilde{N}}$ is the corresponding solution to (1.2) given by Theorem 3.2 with the initial data $m^{0}_{n} \in B^{s}_{p,r}$. If $m^{0}_{n} \rightarrow m^{\infty}_{n}$ in $B^{s}_{p,r}$ then $m^{n} \rightarrow m^{\infty}$ in $C([0, T]; B^{s}_{p,r})$ (resp., $C_{w}([0, T]; B^{s}_{p,r})$) if $r < \infty$ (resp., $r = \infty$), with $T > 0$ satisfying $4C^{3}T \sup_{n \in \tilde{N}} ||m^{0}_{n}||_{B^{s}_{p,r}}^{2} < 1$.

**Proof.** According to the proof of the existence, we find for all $n \in \tilde{N}$, $t \in [0, T], \quad ||m^{n}(t)||_{B^{s}_{p,r}} \leq C||m^{0}_{n}||_{B^{s}_{p,r}}^{2} \left(1 - 4C^{3}t||m^{0}_{n}||_{B^{s}_{p,r}}^{2}\right)$. So $(m^{n})_{n \in \tilde{N}}$ is bounded in $L^{\infty}([0, T]; B^{s}_{p,r})$. We splitting $m^{n} = y^{n} + z^{n}$ with $(y^{n}, z^{n})$ satisfying

$$\begin{cases} y_{t}^{n} + G^{n}y_{x}^{n} = F^{n}, \\ y_{t}^{n}|_{t=0} = m_{0}^{n}, \end{cases} \quad \begin{cases} z_{t}^{n} + G^{n}z_{x}^{n} = F^{n} - F^{\infty}, \\ z_{t}^{n}|_{t=0} = m_{0}^{n} - m_{0}^{\infty}, \end{cases}$$

where for $n \in \tilde{N}$,

$$G^{n} = 2(u^{n}_{x})^{2} - u^{n} u_{x}^{n} - (u^{n})^{2}, \quad F^{n} = (4u^{n} - u^{n})(u^{n})^{2} + (2u^{n} - u^{n})(u^{n} - u^{n})m^{n}. \quad (3.20)$$

Obviously we have

$$||G^{n}||_{B^{s}_{p,r}+1} \leq C||m^{n}||_{B^{s}_{p,r}}^{2} \quad \text{and} \quad ||G^{n} - G^{\infty}||_{B^{s}_{p,r}} \leq C||m^{n} - m^{\infty}||_{B^{s}_{p,r}}^{2}. \quad (3.20)$$

We have already known $m^{n} \rightarrow m^{\infty}$ in $L^{\infty}([0, T]; B^{s}_{p,r})$. By (3.20), $G^{n}$ satisfy the condition of Lemma 2.11. This leads to that $y^{n} \rightarrow y^{\infty}$ in $C([0, T]; B^{s}_{p,r})$ if $r < \infty$.

According to Lemma 2.8 we have for all $n \in \tilde{N}$ and $t \in [0, T],

$$||z^{n}(t)||_{B^{s}_{p,r}} \leq \int_{0}^{t} e^{-CJ^{n}_{t'}||G^{n}_{x}||_{B^{s}_{p,r}}dt'} \left(||m_{0}^{n} - m_{0}^{\infty}||_{B^{s}_{p,r}} + \int_{0}^{t} e^{-CJ^{n}_{t'}||G^{n}_{x}||_{B^{s}_{p,r}}dt'}||F^{n} - F^{\infty}||_{B^{s}_{p,r}}dt'\right). \quad (3.21)$$

As before, we deduce that

$$||G^{n}_{x}||_{B^{s}_{p,r}} \leq C||m^{n}||_{B^{s}_{p,r}}^{2} + ||F^{n} - F^{\infty}||_{B^{s}_{p,r}} \leq C(||m^{n}||_{B^{s}_{p,r}}^{2} + ||m^{n} - m^{\infty}||_{B^{s}_{p,r}}). \quad (3.22)$$

Plugging (3.22) into (3.21), and using the boundedness of $m^{n}$, we obtain

$$||z^{n}(t)||_{B^{s}_{p,r}} \leq C\left(||m_{0}^{n} - m_{0}^{\infty}||_{B^{s}_{p,r}} + \int_{0}^{t} ||m^{n} - m^{\infty}||_{B^{s}_{p,r}}dt'\right).$$

Note that $y^{\infty} = m^{\infty}$, $z^{\infty} = 0$. Hence we have

$$||z^{n}(t)||_{B^{s}_{p,r}} \leq C\left(||m_{0}^{n} - m_{0}^{\infty}||_{B^{s}_{p,r}} + \int_{0}^{t} (||y^{n} - y^{\infty}||_{B^{s}_{p,r}} + ||z^{n}||_{B^{s}_{p,r}})dt'\right).$$

Using Gronwall's inequality yields that

$$||z^{n}(t)||_{B^{s}_{p,r}} \leq C\left(||m_{0}^{n} - m_{0}^{\infty}||_{B^{s}_{p,r}} + \int_{0}^{t} e^{-Ct'}||y^{n} - y^{\infty}||_{B^{s}_{p,r}}dt'\right).$$
Therefore we have proven that when \( r < \infty \), \( z^n \to 0 \) in \( C([0, T]; B^s_{p,r}) \), and thus \( m^n \to m^\infty \) in \( C([0, T]; B^s_{p,r}) \).

As for the case \( r = \infty \), we have weak continuity. In fact, for fixed \( \phi \in B^{s-}\infty_{p,1} \), we write

\[
\langle m^n(t) - m^\infty(t), \phi \rangle = \langle m^n(t) - m^\infty(t), S_j \phi \rangle + \langle m^n(t) - m^\infty(t), \phi - S_j \phi \rangle.
\]

By duality, we have

\[
|\langle m^n(t) - m^\infty(t), \phi \rangle| \
\leq \|m^n(t) - m^\infty(t)\|_{B^{p-\infty}_{p,1}} \|S_j \phi\|_{B^{1-\infty}_{p,1}} + \|m^n(t) - m^\infty(t)\|_{B^s_{p,\infty}} \|\phi - S_j \phi\|_{B^{s-}\infty_{p,1}}.
\]

Using the fact that \( m^n \to m^\infty \) in \( L^\infty([0, T]; B^{s-\infty}_{p,\infty}) \), and \( S_j \phi \to \phi \) in \( B^{s-\infty}_{p,1} \) and \( \langle m^n \rangle_{n \in \mathbb{N}} \) is bounded in \( L^\infty([0, T]; B^s_{p,r}) \), it is now easy to conclude that \( \langle m^n(t) - m^\infty(t), \phi \rangle \to 0 \) uniformly on \([0, T]\).

\[ \square \]

4. Blow up. First we prove a conservation law for (1.2).

**Lemma 4.1.** Let \( u_0 \in H^s, s > \frac{5}{2} \) and let \( T^* \) be the maximal existence time of the corresponding solution \( u \) to (1.2). Then for any \( t \in [0, T^*) \), we have

\[
\|u(t)\|_{H^1}^2 = \int_R (u^2 + u_x^2) dx = \|u_0\|_{H^1}^2.
\]

**Proof.** Arguing by density, it suffices to consider the case where \( u \in C_0^\infty(\mathbb{R}) \). The equation (1.2) can be rewrite as

\[
m_t + [(2u_x^2 - uu_x - u^2)m]_x = u_x(u_x - u)m.
\]

(4.1)

Multiplying (4.1) by \( u \) and integrating by parts, we deduce that

\[
\frac{1}{2} \frac{d}{dt} \int_R (u^2 + u_x^2) dx = \int_R u u_t dx = \int_R u_x (2u_x^2 - uu_x - u^2) dx + uu_x(u_x - u) m dx
\]

\[
= \int_R u_x(u_x - u)m(2u_x + 2u) dx = \int_R 2u_x(u_x^2 - u^2) m dx
\]

\[
= \int_R 2u_x^3 m - 2u_x u^2 m dx = \int_R 2u_x^3 u + 2u_x u_x u^2 dx
\]

\[
= \int_R 2u_x^3 u - 2u_x^3 u dx = 0.
\]

\[ \square \]

Next we state a blow-up criterion for (1.2).

**Lemma 4.2.** Let \( m_0 \in B^s_{p,r} \) with \((s,p,r)\) being as the statement of Theorem 3.2, and \( T^* \) be the maximal existence time of the corresponding solution \( m \) to (1.2). If \( T^* < \infty \), then

\[
\int_0^{T^*} \|m(t')\|_{L^\infty}^2 dt' = \infty.
\]

**Proof.** Applying Lemma 2.9, we have

\[
\|m(t)\|_{B^s_{p,r}} \leq C(\|m_0\|_{B^s_{p,r}} + \int_0^t \|G_x\|_L^\infty \|m\|_{B^s_{p,r}} + \|G_x\|_{B^s_{p,r}} \|m\|_{L^\infty} + \|F\|_{B^s_{p,r}} dt'),
\]

where \( G = 2u_x^2 - uu_x - u^2, \ F = (4u_x - u)m^2 + (2u_x - u)(u_x - u)m \).

(4.2)
Note that the operator $(1 - \partial_x^2)^{-1}$ coincides with the convolution by the function $x \mapsto \frac{1}{2} e^{-|x|}$, which implies that $\|u\|_{L^\infty}$, $\|u_x\|_{L^\infty}$ and $\|u_{xx}\|_{L^\infty}$ can be bounded by $\|m\|_{L^\infty}$. Then
\[
\|G_x\|_{L^\infty} = \|4u_x u_{xx} - u_x^2 - uu_{xx} - 2uu_x\|_{L^\infty} \leq C\|m\|_{L^\infty}^2. \tag{4.3}
\]
As $s > 0$, by Lemma 2.5 we have
\[
\|G_x\|_{B^s_{p,r}} \leq C\|G\|_{B^{s+1}_{p,r}} \\
\leq C(\|u_x\|_{B^{s+1}_{p,r}} \|u_x\|_{L^\infty} + \|u\|_{B^{s+1}_{p,r}} \|u_x\|_{L^\infty} + \|u\|_{B^{s+1}_{p,r}} \|u\|_{L^\infty}) \\
\leq C\|m\|_{B^s_{p,r}} (\|u\|_{L^\infty} + \|u_x\|_{L^\infty}) \leq C\|m\|_{B^s_{p,r}} \|m\|_{L^\infty}, \tag{4.4}
\]
and
\[
\|F\|_{B^s_{p,r}} \leq 2\|u_x - u\|_{B^s_{p,r}} \|m\|_{L^\infty}^2 + 2\|u_x - u\|_{L^\infty} \|m\|_{B^s_{p,r}} \|m\|_{L^\infty} \\
+ 2\|u_x - u\|_{B^s_{p,r}} \|u_x - u\|_{L^\infty} \|m\|_{B^s_{p,r}} \|m\|_{L^\infty} \\
+ 2\|u_x - u\|_{L^\infty} \|u_x - u\|_{L^\infty} \|m\|_{B^s_{p,r}} \|m\|_{L^\infty} \\
\leq C\|m\|_{B^s_{p,r}} \|m\|_{L^\infty}^2. \tag{4.5}
\]
Plugging (4.3), (4.4) and (4.5) into (4.2), we get
\[
\|m(t)\|_{B^s_{p,r}} \leq C(\|m_0\|_{B^s_{p,r}} + \int_0^t \|m\|_{L^\infty}^2 \|m\|_{B^s_{p,r}} \, dt'). \tag{4.6}
\]
Hence Gronwall’s inequality yields
\[
\|m(t)\|_{B^s_{p,r}} \leq C\|m_0\|_{B^s_{p,r}} e^{C\int_0^t \|m\|_{L^\infty}^2 \, dt'}.
\]
If $T^*$ is finite, and $\int_0^{T^*} \|m\|_{L^\infty}^2 \, dt' < \infty$, then $m \in L^\infty([0, T^*); B^s_{p,r})$. By the proof of Step 1 in Theorem 3.2, we can extend the solution $m$ beyond $T^*$, which contradicts the assumption that $T^*$ is the maximal existence time.

Consider the ordinary differential equation:
\[
\begin{align*}
q_t(t, x) = (2u_x^2 - uu_x - u^2)(t, q(t, x)), & \quad t \in [0, T), \\
q(0, x) = x, & \quad x \in \mathbb{R}.
\end{align*} \tag{4.7}
\]
If $m \in B^s_{p,r}$ with $(s, p, r)$ as in Theorem 3.2 then $2u_x^2 - uu_x - u^2 \in C([0, T); C^{0,1})$.

By the classical results in the theory of ordinary differential equation, we infer that (4.7) has a unique solution $q \in C^1((0, T) \times \mathbb{R}; \mathbb{R})$ such that the map $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with
\[
q_x(t, x) = \exp \left( \int_0^t (4u_x u_{xx} - u_x^2 - uu_{xx} - 2uu_x)(t', q(t', x)) \, dt' \right) > 0.
\]

Now the following theorem shows that under particular condition for the initial data, the corresponding solution of (1.2) will blow up in finite time.

**Theorem 4.3.** Let $m_0 \in H^s$, $s > \frac{1}{2}$. Assume $m_0(x) \geq 0$ for all $x \in \mathbb{R}$ and $m_0(x_0) > 0$, $(u_0 - 4\partial_x u_0)(x_0) < 0$ for some $x_0 \in \mathbb{R}$, and that
\[
\frac{\|u_0\|_{H^1}}{10\sqrt{2m_0(x_0)}} \leq \exp(\frac{(u_0 - 4\partial_x u_0)(x_0)}{5\sqrt{2} |u_0|_{H^1}}) - (u_0 - 4\partial_x u_0)(x_0) < 1.
\]
Then the corresponding solution $m$ of (1.2) blows up in finite time.
4.1, we find

Proof. Arguing by density, now we assume \( s > \frac{3}{2} \). From (1.2), we know

\[
\frac{d}{dt} m(t, q(t, x)) = [(4u_x - u)m^2 + (2u_x - u)(u_x - u)m](t, q(t, x)),
\]

hence

\[
m(t, q(t, x)) = m_0(x) \exp \left( \int_0^t [(4u_x - u)m + (2u_x - u)(u_x - u)](t', q(t', x)) dt' \right),
\]

which implies that \( m \) doesn’t change sign.

The equation (1.2) can be written as

\[
u_t + (2u_x^2 - uu_x - u^2) v_x = (1 - \partial_x^2)^{-1}(1 + \partial_x)(-4u_x^2 u_{xx} + uu_x u_{xx} + u^2 u_x + 2u^3_x).
\]

Differentiating with respect to \( x \), we have

\[
u_{xt} + (2u_x^2 - uu_x - u^2) v_{xx} = (1 - \partial_x^2)^{-1}(1 + \partial_x)(-4u_x^2 u_{xwx} - 3uu_x u_{xxx} - u^3_x + 4uu_x^2 + u^2 u_{xx}x).
\]

Let \( v = u - 4u_x \). Then \( v \) satisfies

\[
v_t + (2u_x^2 - uu_x - u^2) v_x = (1 - \partial_x^2)^{-1}(1 + \partial_x)(13uu_x u_{xx} + 6u_x^3 - 16uu_x^2 - 4u_x^2 u_{xx} + u^2 u_x)
\]

\[= -(\frac{13}{2}u_x^2 + \frac{1}{3}u^3) + (1 - \partial_x^2)^{-1}(1 + \partial_x)(\frac{1}{3}u^3 - \frac{19}{2}u_x^2 - \frac{1}{2}u_x^3 + 4u_x^2 + 4u^2 - 4u^2)^m).
\]

Remember that \( m_0(x) \geq 0 \) for all \( x \in \mathbb{R} \), so \( m(t, x) \geq 0 \) for all \( (t, x) \in [0, \infty) \times \mathbb{R} \), which leads to \( u(t, x) \geq 0 \) and \( |u_x(t, x)| \leq |u(t, x)| \) for all \( (t, x) \in [0, \infty) \times \mathbb{R} \), since

\[
u(t, x) = (1 - \partial_x^2)^{-1}m(t, x) = \frac{1}{2} \int_{-\infty}^\infty e^{-|x-y|} m(t, y) dy,
\]

\[
u_x(t, x) = \partial_x(1 - \partial_x^2)^{-1}m(t, x) = -\frac{1}{2} \int_{-\infty}^\infty \text{sign}(x-y) e^{-|x-y|} m(t, y) dy.
\]

Hence

\[
v_t + (2u_x^2 - uu_x - u^2) v_x \leq (1 - \partial_x^2)^{-1}(1 + \partial_x)(\frac{1}{3}u^3 - \frac{1}{2}u_x^3 + 4u^2m)
\]

\[\leq \int_{-\infty}^\infty e^{-|x-y|} (\frac{1}{3}u^3 + \frac{1}{2}|u_x|^3 + 4u^2 m)(t, y) dy
\]

\[\leq 2\|u\|^3_{L^\infty} + 4\|u\|^2_{L^\infty} \int_{-\infty}^\infty e^{-|x-y|} m(t, y) dy \leq 10\|u\|^3_{L^\infty}.
\]

From this, together with the Sobolev inequality \( \|u\|_{L^\infty} \leq \frac{1}{\sqrt{2}} \|u\|_{H^1} \) and Lemma 4.1, we find

\[
v_t + (2u_x^2 - uu_x - u^2) v_x \leq \frac{5}{\sqrt{2}} \|u\|^3_{H^1} = \frac{5}{\sqrt{2}} \|u_0\|^3_{H^1} := c.
\]

Thanks to (4.8), (4.9), at \( x_0 \), we have

\[
\frac{d}{dt} v(t, q(t, x_0)) \leq c;
\]

\[
\frac{d}{dt} m(t, q(t, x_0)) = -vm^2(t, q(t, x_0)) + (2u_x - u)(u_x - u)m(t, q(t, x_0)).
\]

Integrating from 0 to \( t \) in (4.10), we have

\[
v(t, q(t, x_0)) \leq ct + v(0, x_0) = ct + (u_0 - 4\partial_x u_0)(x_0) = ct + b,
\]

where \( b = (u_0 - 4\partial_x u_0)(x_0) \).
Combined with (4.11), (4.12), we get

\[-\frac{1}{m^2} \frac{d}{dt} m(t, q(t, x_0)) = v - \frac{1}{m} (2u_x - u)(u_x - u)(t, q(t, x_0))
= v - \frac{1}{m} ((u_x - u)^2 + u_x^2 - uu_x)(t, q(t, x_0))
\leq v + \frac{uu_x}{m}(t, q(t, x_0)) \leq ct + b + \frac{\|u_0\|^2_{H^1}}{2m}(t, q(t, x_0))
= ct + b + \frac{a}{m(t, q(t, x_0))}, (4.13)\]

where $a = \frac{1}{2}\|u_0\|_{H^1}$.

Denoting $M(t) = m(t, q(t, x_0))$, we rewrite (4.13) as

\[\frac{d}{dt}(\frac{1}{M(t)}) \leq ct + b + \frac{a}{M(t)}.\]

Multiplying $e^{-at}$ and integrating from 0 to $t$ yields

\[\frac{1}{M(t)} \leq e^{at}(\frac{1}{M(0)} - (\frac{b}{a} + \frac{c}{a^2})(e^{-at} - 1) - \frac{c}{a}e^{-at}t). \quad (4.14)\]

Let

\[f(t) = \frac{1}{M(0)} - (\frac{b}{a} + \frac{c}{a^2})(e^{-at} - 1) - \frac{c}{a}e^{-at}t.\]

Then

\[f'(t) = (b + \frac{c}{a})e^{-at} - \frac{c}{a}e^{-at} + ce^{-at}t = e^{-at}(ct + b).\]

Because $c > 0$, so the function $f$ is strictly decreasing on the interval $(-\infty, -\frac{b}{c})$ and strictly increasing on $(-\frac{b}{c}, \infty)$, and $f$ get its minimum at $t = -\frac{b}{c}$. According to the assumption, we have

\[f(0) = \frac{1}{M(0)} = \frac{1}{m_0(x_0)} > 0 \quad \text{and} \quad -\frac{b}{c} = -\frac{1}{c}(u_0 - 4\partial_xu_0)(x_0) > 0,
\]

\[f(-\frac{b}{c}) = \frac{1}{M(0)} - (\frac{b}{a} + \frac{c}{a^2})(e^{\frac{b}{c}} - 1) + \frac{b}{a}e^{\frac{b}{c}} = \frac{1}{M(0)} - \frac{c}{a^2}(e^{\frac{b}{c}} - 1) + \frac{b}{a}\]

\[= \frac{c}{a^2}(\frac{a^2}{cM(0)} - (e^{\frac{b}{c}} - 1 - \frac{ab}{c}))\]

\[= \frac{c}{a^2} \left(\|u_0\|_{H^1} - \left(\exp\left(\frac{u_0 - 4\partial_xu_0(x_0)}{5\sqrt{2}\|u_0\|_{H^1}}\right) - \frac{u_0 - 4\partial_xu_0(x_0)}{5\sqrt{2}\|u_0\|_{H^1}} - 1\right)\right)
\leq 0.\]

It follows that there exists a unique point $0 < T \leq -\frac{b}{c}$ such that $f(T) = 0$. So from (4.14) we know $M(t) \to \infty$ as $t \to T$. By Lemma (4.2), the solution $m$ must blow up in finite time.

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REFERENCES


THE CAUCHY PROBLEM FOR A GENERALIZED NOVIKOV EQUATION


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