

CHARACTERIZATION OF EXTENDED HAMMING AND GOLAY CODES AS PERFECT CODES IN POSET BLOCK SPACES

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ABSTRACT. Alves, Panek and Firer (Error-block codes and poset metrics, *Adv. Math. Commun.*, **2** (2008), 95–111) classified all poset block structures which turn the $[8, 4, 4]$ extended binary Hamming code into a 1-perfect poset block code. However, the proof needs corrections that are supplied in this paper. We provide a counterexample to show that the extended binary Golay code is not 1-perfect for the proposed poset block structures. All poset block structures turning the extended binary and ternary Golay codes into 1-perfect codes are classified.

1. INTRODUCTION

Most of the studies in coding theory have been made in finite-dimensional vector spaces \mathbb{F}_q^n over a finite field \mathbb{F}_q and equipped with a metric, the most common one being the Hamming metric. The error-correction capability of a code is largely determined by its minimum distance. Finding, for integers $n \geq k \geq 1$, a k -dimensional subspace of \mathbb{F}_q^n (called a “linear code of length n over F_q ”) with the largest possible minimum distance is one of the main problems of coding theory. For linear codes in the Hamming space, this problem has an equivalent formulation in terms of the parity-check matrix of the code, as is well known. The resulting problem for matrices was generalized by Niederreiter [20, 21]. With this generalization as basis and using the concept of partially ordered sets, Brualdi, Graves and Lawrence [4] in 1995 introduced the concept of codes with a poset metric. Poset metrics gave rise to many new perfect codes [1, 6, 15, 17], a fact attributed to the increased packing radius with respect to a poset metric. The $[8, 4, 4]$ extended binary Hamming code, the $[24, 12, 8]$ extended binary Golay code and the $[12, 6, 6]$ extended ternary Golay code, though not perfect with respect to the Hamming metric, turn out to be perfect with respect to certain poset metrics. For the construction and uniqueness of the codes mentioned above, see [14].

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Poset metrics can be used to perform unequal error protection for some channels, which takes the semantic aspect of communication into account [8]. The Rosenbloom-Tsfasman metric (RT metric), a particular case of a poset metric in which the poset is a disjoint union of chains of equal size, is useful in the case of interference in several consecutive channels starting from the last, occupied by a priority user [23]. When the poset is a chain, we obtain the Niederreiter-Rosenbloom-Tsfasman metric (NRT-metric), which has many applications ranging from a generalization of the Fourier transform over finite fields to the list decoding of algebraic codes. In [6], Felix and Firer have established that syndrome decoding for hierarchical poset codes offers an exponential gain in comparison to the usual syndrome decoding of linear codes and, further, reduces to a linear map for the case of the NRT metric. Moreover, the syndrome decoding complexity of a general poset code is comparable to that of a hierarchical poset code obtained by using the primary P -decomposition of a code [9].

In 2006, Feng, Xu and Hickernell [7] proposed another generalization of the Hamming metric, called the π -metric or block metric, by partitioning the set of coordinate positions (non-trivially) into a family of blocks and using the Hamming metric on the resulting direct product. This metric has been studied by various authors [19, 22]. The packing radius of a code with respect to the π -metric is smaller than its Hamming packing radius. The study of perfect linear error block codes is intimately related to partial spreads and the concept of mixed perfect codes arising from vector space partitions as discussed in [3, 12, 13]. In fact, a mixed 1-perfect code is nothing but a 1-perfect code with respect to a block metric. It follows from [11, p. 5] that a necessary and sufficient condition for the existence of a 1-perfect $[n, n - k]$ linear error block code over \mathbb{F}_q with partition type $n = \pi(1) + \pi(2) + \dots + \pi(s)$ is that there exists a vector space partition $U_1 \cup U_2 \cup \dots \cup U_s$ of \mathbb{F}_q^n . Also, as explained in [10], a partial spread in $\text{PG}(3, q)$ with deficiency δ gives rise to a mixed 1-perfect code and hence a 1-perfect linear error block code with type $n = \pi(1) + \pi(2) + \dots + \pi(s)$, where $\pi(i) = 1$ for $1 \leq i \leq m$, $\pi(i) = 2$ for $m+1 \leq i \leq s$, $m = \delta(q+1)$ and $s = m + q^2 + 1 - \delta$. The distance between two vectors introduced in [10, p. 279] coincides with the distance in the block metric. Further, the author has obtained a characterization for a partial spread to be maximal in terms of the dual of the code arising from the partial spread.

Unifying the above two metrics, another metric called the poset block metric was introduced in 2008 by Alves, Panek and Firer [2]. According to [2] the problem with the packing radius ameliorates further when the π -metric is weighted by a partial order. The authors of [2] discuss poset block structures which turn classical codes such as the $[8, 4, 4]$ extended binary Hamming code and the $[24, 12, 8]$ extended binary Golay code into perfect codes. Their results are stated in Section 2. It should be pointed out that the proof of [2, Theorem 3.3] and the proof of the remark following [2, Theorem 3.3] require modification. The condition given in [2] for the $[24, 12, 8]$ extended binary Golay code to be 1-perfect with respect to a poset block metric needs to be revised. The present paper discusses the requisite modifications. Further general results on extended Golay codes have been derived.

The remainder is organized as follows. In Section 2, we present basic definitions and provide the modified proof of [2, Theorem 3.3]. In Section 3, a counterexample is provided to show that the condition given in [2] for a poset block structure to turn the $[24, 12, 8]$ extended binary Golay code into a 1-perfect poset block code does not hold. We provide a necessary condition for a poset block structure to turn

the [24, 12, 8] extended binary Golay code into a 1-perfect code. All poset block structures which turn the [12, 6, 6] extended ternary Golay code into a 1-perfect code are classified.

Throughout the paper, the [8, 4, 4] extended binary Hamming code, the [24, 12, 8] extended binary Golay code and the [12, 6, 6] extended ternary Golay code will be denoted by \mathcal{H}_3 , \mathcal{G}_{24} and \mathcal{G}_{12} respectively. Also, $\text{supp}(\cdot)$ and $w(\cdot)$ denote the support and Hamming weight, respectively.

2. BASIC DEFINITIONS AND MODIFICATIONS

In this section, we discuss basic definitions and modifications of a few results on poset block spaces.

A **partial order** on a set is a binary relation “ \leq ” which is reflexive, antisymmetric and transitive. A set with a partial order is called a **poset**. Let $P = ([s], \leq)$ denote a **poset** with underlying set $[s] = \{1, 2, \dots, s\}$. An **ideal** I of P is a subset of $[s]$ such that $i \in I, j \leq i$ implies $j \in I$. For $X \subseteq [s]$, we denote by $\langle X \rangle$ the smallest ideal containing X , called the **ideal generated by** X . If $X = \{i\}$ then we write $\langle i \rangle$ instead of $\langle \{i\} \rangle$. The **t-th level set** $\Gamma^{(t)}(P)$ of P is defined as

$$\Gamma^{(t)}(P) = \{i \in P : |\langle i \rangle| = t\}.$$

Let $\pi : [s] \rightarrow \mathbb{N}$ be a map such that $n = \sum_{i=1}^s \pi(i)$. The map π is said to be a **labeling** of the poset P and the pair (P, π) is called a **poset block structure** over $[s]$. We aim to construct a vector space of dimension n with metric induced by the poset block structure (P, π) . Denote $\pi(i)$ by k_i and take V_i as the \mathbb{F}_q -vector space $V_i = \mathbb{F}_q^{k_i}$ for every $1 \leq i \leq s$. Define V as the external direct sum (direct product)

$$V = V_1 \oplus V_2 \oplus \dots \oplus V_s,$$

which is naturally isomorphic to \mathbb{F}_q^n with $n = k_1 + k_2 + \dots + k_s$. In the sequel we will use V and \mathbb{F}_q^n interchangeably. Each $x \in \mathbb{F}_q^n$ can be uniquely decomposed as $x = (x_1, x_2, \dots, x_s)$ with $x_i \in \mathbb{F}_q^{k_i}, 1 \leq i \leq s$. The **block support** or **π -support** and **(P, π) -weight** of $x \in \mathbb{F}_q^n$ are defined, respectively, as

$$\text{supp}_\pi(x) = \{i : 1 \leq i \leq s, x_i \neq 0\}$$

and

$$w_{(P,\pi)}(x) = |\langle \text{supp}_\pi(x) \rangle|$$

where $|\cdot|$ denotes the cardinality of the given set. For $x, y \in \mathbb{F}_q^n$,

$$d_{(P,\pi)}(x, y) = w_{(P,\pi)}(x - y)$$

defines a metric on \mathbb{F}_q^n called the **poset block metric** or **(P, π) -metric**. The pair $(\mathbb{F}_q^n, d_{(P,\pi)})$ is said to be a **poset block space**. A k -dimensional \mathbb{F}_q -linear subspace C of \mathbb{F}_q^n is said to be a linear $[n, k, d]$ **(P, π) -code** if \mathbb{F}_q^n is equipped with the poset block metric $d_{(P,\pi)}$ and

$$d = d_{(P,\pi)}(C) = \min\{w_{(P,\pi)}(c) : 0 \neq c \in C\}$$

is the **(P, π) -minimum distance** of C . The **weight** of the i -th block in a poset block structure (P, π) is defined as the cardinality of the ideal of P generated by i , i.e., $|\langle i \rangle|$.

For $x \in \mathbb{F}_q^n$ the **(P, π) -ball** with center x and radius r is defined by

$$B_{(P,\pi)}(x, r) = \{y \in \mathbb{F}_q^n : d_{(P,\pi)}(x, y) \leq r\},$$

i.e., the set of all vectors in \mathbb{F}_q^n whose (P, π) -distance from x is less than or equal to r . For $x \in \mathbb{F}_q^n$,

$$(2.1) \quad |B_{(P,\pi)}(x, r)| = 1 + \sum_{i=1}^r \sum_{j=1}^i \sum_{I \in \theta_j(i)} \prod_{m \in \max(I)} (q^{k_m} - 1) \prod_{\ell < m; m \in \max(I)} q^{k_\ell}$$

where $\theta_j(i) = \{I \subseteq P: I \text{ ideal}, |I| = i, |\max(I)| = j\}$ and $\max(I)$ is the set of maximal elements of the ideal I .

Let d be a metric on $V = \mathbb{F}_q^n$ and let C be a subset of V . The **packing radius** $R_d(C)$ of C is the greatest non-negative integer r such that any two balls of radius r centered at (distinct) elements of C are disjoint. The code C is said to be $R_d(C)$ -**perfect** if the union of the balls of radius $R_d(C)$ centered at the elements of C is equal to V .

If the metric is translation invariant, perfect codes are characterized using the Hamming bound. A metric d on \mathbb{F}_q^n is said to be translation invariant if $d(x, y) = d(x + z, y + z)$ for all $x, y, z \in \mathbb{F}_q^n$. For a translation invariant metric d on \mathbb{F}_q^n and an $[n, k]$ linear code $C \subseteq \mathbb{F}_q^n$ with packing radius $R_d(C)$, the Hamming bound can be stated in terms of the volume of a single ball centered at 0 as

$$q^{n-k} \geq |B_d(0, r)|, \quad r = R_d(C).$$

A code whose parameters satisfy the Hamming bound with equality is said to be a perfect code. In the case of the block metric, we have that $R_{d_\pi}(C) = \lfloor \frac{d_\pi - 1}{2} \rfloor$ and hence, the above bound can be stated explicitly as in [7, Theorem 2.1]. However, [7, Theorem 2.1] also defines a set $B'_\pi(x, r)$ based on the ball $B_\pi(x, r)$ as follows:

$$B'_\pi(x, r) = \{y \in B_\pi(x, r): |\text{supp}_\pi(y - x) \cap \{2, 3, \dots, s\}| \leq r - 1\}.$$

The authors have discussed perfect block codes with even minimum distance using the concept of sets $B'_\pi(x, r)$ instead of balls $B_\pi(x, r)$. Such codes are not perfect since they decode incorrectly as shown in the following example.

Consider the block code $C \subseteq \mathbb{F}_2^5$ with partition type $5 = 2 + 1 + 1 + 1$ given by the following parity check matrix:

$$\left[\begin{array}{cc|cc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{array} \right]$$

where $|$ denotes the separation of blocks. The code C is a $[5, 3]$ code with minimum block distance 2. Since $n - k = 2 = k_1$, this code should be a 1-perfect block code by [7, Theorem 2.1]. Suppose the codeword $(11|1|1|0)$ is sent and received on the other end as $(11|1|1|1)$ with a single error. Although the sets $B'_\pi(c, r)$ for $c \in C$ do form a partition of \mathbb{F}_2^5 , the received vector will be decoded incorrectly into $(00|1|1|1)$. Thus, it is not a 1-perfect code. Moreover, perfect block codes with even minimum distance do not give rise to a vector space partition of the space (see [11, p. 5]) as can be checked using this example. Further, the packing radius of such codes is not equal to the covering radius. Since the distance in the block metric is equal to the shortest path distance induced by the distance 1-graph, perfect block codes should have odd minimum distance.

In order to check the perfectness of a code whose packing radius with respect to a translation invariant metric is known, we just need to check whether the code satisfies the Hamming bound with equality. However, for the case of a poset metric, the packing radius of a code can not be computed directly from its minimum distance. Thus, the Hamming bound for a poset code can not be stated in a simplified

manner in terms of the minimum distance of the code. Therefore, in order to study perfect codes with respect to poset metrics, the best approach is to find all the poset structures that turn a given code perfect [15, 16]. The same applies to the case of poset block metrics. In order to characterize the poset block structures which turn a given code 1-perfect, we need to ensure that there exists a $(n - k)$ -subset of $[n]$ that does not contain any codeword of C or equivalently, forms an information set for the dual code of C . The converse implications are nontrivial and require number theoretic analysis.

In what follows, let $V_X = \{v \in \mathbb{F}_q^n : \text{supp}_\pi(v) \subseteq X\}$, where $X \subseteq [s]$. Alves, Firer and Panek [2] proved the following result.

Result 2.1. [2, Theorem 3.3] Let X' be a subset of $[8]$ with $|X'| = 4$ such that $|\text{supp}(c) \cap X'| \leq 3$ for every $c \in \mathcal{H}_3$ with $w(c) = 4$, π a label on $[s]$ such that

$$\pi(1) + \pi(2) + \dots + \pi(s) = 8$$

and $V = V_1 \oplus V_2 \oplus \dots \oplus V_s$ with V_j isomorphic to $\mathbb{F}_2^{\pi(j)}$ for all $j \in [s]$, where $V_i = V_{X'}$ for some $i \in [s]$. Then an order P turns the extended binary Hamming code \mathcal{H}_3 into a 1-perfect code if and only if $\Gamma^{(1)}(P) = \{i\}$ where $\pi(i) = 4$ and the block V_i does not contain any codeword of minimum weight.

Alternatively, a poset block structure turns the extended binary Hamming code \mathcal{H}_3 into a 1-perfect code if and only if it has a unique block of weight 1 with 4 elements that does not contain any non-zero codeword of \mathcal{H}_3 .

In the proof of Result 2.1 [2, p. 101], it has been shown that if (P, π) is a poset block structure that turns \mathcal{H}_3 into a 1-perfect code, then the poset has just one block of weight 1, i.e., $|\Gamma^{(1)}(P)| = 1$. While proving the same by contradiction, the authors have discussed the case when $|\Gamma^{(1)}(P)| = 3$. In this case, the authors have argued that there exists a non-zero codeword c in \mathcal{H}_3 with $\text{supp}(c) = \{i_1, i_2, i_3, i_4\} \subseteq \Gamma^{(1)}(P)$ and hence, $w_{(P,\pi)}(c) = 1$. However, it should be noted that $|\Gamma^{(1)}(P)| = 3$ and $\text{supp}(c) \subseteq [8]$ whereas $\Gamma^{(1)}(P) \subseteq [s]$. Therefore, $\text{supp}(c)$ is not comparable to $\Gamma^{(1)}(P)$. Moreover, $w_{(P,\pi)}(c) = 1$ only if $\text{supp}(c) = \{j\}$ where $j \in \Gamma^{(1)}(P)$. However, the dimension of V_j can be at most 3 whereas $\text{supp}(c)$ has 4 elements. Thus, the argument given to discard the case when $|\Gamma^{(1)}(P)| = 3$ is not correct.

In light of the above, we provide the modified proof of the claim that $|\Gamma^{(1)}(P)| \neq 3$ as follows:

Modified Proof. Suppose that $|\Gamma^{(1)}(P)| = 3$ and let k_1, k_2, k_3 be the dimensions of the corresponding block spaces. Since the code is 1-perfect, therefore $|B_{(P,\pi)}(0, 1)| = 2^4$ which implies that $2^{k_1} + 2^{k_2} + 2^{k_3} = 18$. Also, we have $\sum_{i=1}^r k_i \leq 7$. It follows that $(k_1, k_2, k_3) = (3, 3, 1)$ (up to a permutation) and the only remaining block has weight 2. Let H be a parity check matrix of \mathcal{H}_3 . It follows that the union of the first two blocks of H would have 6 columns whereas H has only 4 rows. Therefore, the first two blocks are linearly dependent and hence, there exists $c \in \mathcal{H}_3$ with $\text{supp}_\pi(c) = \{1, 2\}$, say $c = (c_1|c_2|0|0)$. Take $x = (c_1|0|0|0) \in \mathbb{F}_2^8$. Then $x \in B_{(P,\pi)}(0, 1) \cap B_{(P,\pi)}(c, 1)$, which is a contradiction to the fact that \mathcal{H}_3 is 1-perfect. Therefore $|\Gamma^{(1)}(P)| \neq 3$. \square

We now discuss the error in the proof of Result 2.2 as given in [2] as follows:

Result 2.2. [2] If P , in Result 2.1, is a chain, the extended Hamming code \mathcal{H}_3 is one of the codes described in [2, Proposition 3.1](as it should be).

Alves, Panek and Firer [2] argue in the proof of Result 2.2 that $S = T|_{V_1}$ is a linear isomorphism from V_1 onto W . Since $V_1 = \ker(T)$, S is the zero map which contradicts that S is a linear isomorphism. We need to consider the map $T|_{\mathcal{H}_3}$. Since no non-zero codeword of \mathcal{H}_3 is contained in V_1 , therefore $T|_{\mathcal{H}_3}$ is a linear isomorphism. The required linear transformation from W to V_1 is S^{-1} .

3. EXTENDED GOLAY CODES

In this section, we first present the modification of the following result:

Result 3.1. [2, p. 103] The extended binary Golay code \mathcal{G}_{24} is 1-perfect for any poset block structure satisfying the following condition: It has a unique block of weight 1 with 12 elements that does not contain any codeword of minimum weight of the Golay code.

We provide a counter example to prove that \mathcal{G}_{24} is not 1-perfect for the poset block structures given in Result 3.1 [2].

Example 3.2. Let Y be the support of any weight 12 codeword (say c) of \mathcal{G}_{24} . If $\text{supp}(c') \subseteq \text{supp}(c)$ for some codeword c' of \mathcal{G}_{24} with the Hamming weight 8, then their difference $c - c'$ would be a codeword of weight less than 8, which is a contradiction. Hence, Y does not contain the support of any minimum weight codeword of \mathcal{G}_{24} . Consider a label $\pi: [s] \rightarrow \mathbb{N}$ such that $\pi(1) + \pi(2) + \dots + \pi(s) = 24$ and $\pi(i) = 12$ for some $i \in [s]$ with $V_i = V_Y$. This poset block structure satisfies the condition given in Result 3.1. Take any non-zero element v of V_i , other than c . Then

$$v \in B_{(P,\pi)}(0,1) \cap B_{(P,\pi)}(c,1)$$

which contradicts the fact that \mathcal{G}_{24} is a 1-perfect poset block code.

The following theorem presents the modification of Result 3.1 [2] and prove its converse to provide a characterization of poset block structures which turn the extended binary Golay code into a 1-perfect code.

Theorem 3.3. *A poset block structure turns the extended binary Golay code \mathcal{G}_{24} into a 1-perfect code if and only if it has a unique block of weight 1 with 12 elements that does not contain any non-zero codeword of \mathcal{G}_{24} .*

Proof. Firstly, we show the existence of such structure. We have that the number of 12 element subsets of [24] is $\binom{24}{12}$, which implies that the number of 12 element subsets of [24] containing a particular 8 subset is $\binom{16}{4}$. Since the number of weight 8 codewords of \mathcal{G}_{24} is 759, therefore the number of 12 element subsets of [24] containing support of some weight 8 codeword of \mathcal{G}_{24} is at most $759 \times \binom{16}{4}$. Using the fact that the number of weight 12 codewords of \mathcal{G}_{24} is 2576, we get

$$|\{X \subseteq [24]: |X| = 12, \text{supp}(c) \not\subseteq X \text{ for all } 0 \neq c \in \mathcal{G}_{24}\}| \geq \binom{24}{12} - 2576 - 759 \binom{16}{4} > 0.$$

Let Y be a subset of [24] with $|Y| = 12$ that does not contain the support of any non-zero codeword of \mathcal{G}_{24} . We consider a label $\pi: [s] \rightarrow \mathbb{N}$ such that $\pi(1) + \pi(2) + \dots + \pi(s) = 24$ and $\pi(i) = 12$ for some $i \in [s]$ with $V_i = V_Y$. Note that $1 < s \leq 23$. If $\Gamma^{(1)}(P) = \{i\}$, then $B_{(P,\pi)}(0,1) = \{x \in \mathbb{F}_2^{24}: \text{supp}(x) \subseteq Y\}$. Hence $|B_{(P,\pi)}(0,1)| = 2^{|Y|} = 2^{12}$. For any non-zero codeword c of \mathcal{G}_{24} , we have

$w_{(P,\pi)}(c) \geq 2$ since $\Gamma^{(1)}(P) = \{i\}$ with $V_i = V_Y$ and $\text{supp}(c)$ must have an element not contained in Y . Let, if possible,

$$x \in B_{(P,\pi)}(0, 1) \cap B_{(P,\pi)}(c, 1) \text{ for some } 0 \neq c \in \mathcal{G}_{24}.$$

We have $x \in B_{(P,\pi)}(0, 1)$ which implies that $\text{supp}(x) \subseteq Y$. Also, $x \in B_{(P,\pi)}(c, 1)$ implies that $\text{supp}(c - x) \subseteq Y$. Combining the two, we get $\text{supp}(c) \subseteq Y$, which is a contradiction. Therefore $B_{(P,\pi)}(0, 1) \cap B_{(P,\pi)}(c, 1) = \phi$ for all $0 \neq c \in \mathcal{G}_{24}$. Since $|\mathcal{G}_{24}| \cdot |B_{(P,\pi)}(0, 1)| = 2^{24}$, we have that \mathcal{G}_{24} is a 1-perfect poset block code.

Conversely, let (P, π) be a poset block structure that turns the extended binary Golay code \mathcal{G}_{24} into a 1-perfect code. If there is a minimal coordinate $i \in \Gamma^{(1)}(P)$ such that $\dim(V_i) = k_i > 12$, then

$$|B_{(P,\pi)}(0, 1)| \geq 1 + (2^{k_i} - 1) = 2^{k_i} > 2^{12}$$

and hence \mathcal{G}_{24} cannot be 1-perfect, since $|\mathcal{G}_{24}| = 2^{12}$ and $2^{12} \cdot 2^{k_i} > 2^{24}$. Thus, we have

$$(3.1) \quad k_i \leq 12 \text{ for all } i \in \Gamma^{(1)}(P).$$

Suppose that $|\Gamma^{(1)}(P)| = r > 1$, say $\Gamma^{(1)}(P) = \{1, 2, \dots, r\}$. Let k_1, k_2, \dots, k_r be the dimension of the corresponding block spaces. Using (2.1), we get

$$|B_{(P,\pi)}(0, 1)| = 1 - r + \sum_{i=1}^r 2^{k_i}.$$

Since the code is 1-perfect, therefore

$$1 - r + \sum_{i=1}^r 2^{k_i} = 2^{12}.$$

Thus, we have

$$(3.2) \quad \sum_{i=1}^r 2^{k_i} = 4095 + r \text{ where } 2 \leq r \leq 24.$$

Since L.H.S. is multiple of 2, R.H.S. should also be a multiple of 2. Therefore the cases when r is even would be discarded. Without loss of generality, assume that $k_1 \geq k_2 \geq \dots \geq k_r$. From (3.1), we have $1 \leq k_i \leq 12$ for all $i \in \Gamma^{(1)}(P)$. If $k_i = 12$ for some $i \in \Gamma^{(1)}(P)$ then we get $|B_{(P,\pi)}(0, 1)| > 2^{12}$ since $r > 1$, which contradicts that \mathcal{G}_{24} is a 1-perfect code. Thus, $1 \leq k_i \leq 11$ for each $1 \leq i \leq r$. Also, we have $\sum_{i=1}^r k_i \leq 23$ since $r < s$. The following cases need to be considered.

Case 1. For $r = 3$, (3.2) reduces to

$$2^{k_1} + 2^{k_2} + 2^{k_3} = 4098.$$

The above equation holds only if $(k_1, k_2, k_3) = (11, 11, 1)$ in which case there is a unique block with weight 2. Let H be a parity check matrix of \mathcal{G}_{24} . It follows that the union of the first two blocks of H would have 22 columns whereas H has only 12 rows. Therefore, the first two blocks are linearly dependent and hence, there exists $c \in \mathcal{G}_{24}$ with $\text{supp}_\pi(c) = \{1, 2\}$, say $c = (c_1|c_2|0|0)$. Take $x = (c_1|0|0|0) \in \mathbb{F}_2^{24}$. Then $x \in B_{(P,\pi)}(0, 1) \cap B_{(P,\pi)}(c, 1)$, which is a contradiction to the fact that \mathcal{G}_{24} is 1-perfect. Therefore, this case is not possible.

Case 2. For $r \in \{2n+1: 2 \leq n \leq 11\}$, (3.2) is a partition of the integer $4095+r$ into r parts. It follows that at least one part must be greater than or equal to $\frac{4095+r}{r}$.

Since $r \leq 23$ and each part is a power of 2, we have $k_1 \geq 8$. Substituting the value of k_1 in (3.2), $8 \leq k_1 \leq 11$ and using the above argument, we obtain $7 \leq k_2 \leq k_1$. Similarly, we have $6 \leq k_3 \leq k_2$. Since $\sum_{i=1}^r k_i \leq 23$ and $r \in \{2n+1: 2 \leq n \leq 11\}$, the only possibility is when $(k_1, k_2, k_3, k_4, k_5) = (8, 7, 6, 1, 1)$. Equation (3.2) does not hold for this case. Therefore, we get $|\Gamma^{(1)}(P)| = 1$.

Let $\Gamma^{(1)}(P) = \{i\}$. Using (3.1), we get $k_i \leq 12$. For $k_i < 12$, since $|B_{(P,\pi)}(0, 1)| = 2^{k_i}$, it follows that the poset block structure does not turn \mathcal{G}_{24} into a 1-perfect code. Therefore, we have $k_i = 12$. If V_i contains a non-zero codeword, say c , of \mathcal{G}_{24} then $w_{(P,\pi)}(c) = 1$, which contradicts that \mathcal{G}_{24} is 1-perfect. Hence, V_i does not contain any non-zero codeword of \mathcal{G}_{24} . \square

We now investigate the poset block structures that turn \mathcal{G}_{12} into a 1-perfect poset block code. Firstly, in the following example we show that the packing radius of \mathcal{G}_{12} with respect to a block metric (when the poset is an antichain) having one block of dimension 6 is zero.

Example 3.4. Let $\pi: [s] \rightarrow \mathbb{N}$ be a label such that $\pi(1) + \pi(2) + \dots + \pi(s) = 12$ and define $m_j = \pi(1) + \pi(2) + \dots + \pi(j)$ for $j \in [s]$ and $m_0 = 0$. Let $V = V_1 \oplus V_2 \oplus \dots \oplus V_s$ be a vector space such that $\dim(V_i) = \pi(i)$ for each $1 \leq i \leq s$. Also, $v \in V_j$ if and only if $\text{supp}(v) \subseteq \{m_{j-1} + 1, \dots, m_{j-1} + \pi(j) - 1, m_j\}$. Let

$$\mathcal{B} = \{\text{supp}(c): c \in \mathcal{G}_{12}, w(c) = 6\}$$

be the set of the supports of all minimal weight codewords of \mathcal{G}_{12} and $\mathcal{P} = [12]$. The pair $(\mathcal{P}, \mathcal{B})$ is a 5-(12, 6, 1) design, that is, given a subset $X \subseteq \mathcal{P}$ with 5 elements, there is a unique block $\text{supp}(c) \in \mathcal{B}$ such that $X \subseteq \text{supp}(c)$. Suppose there is some $i \in [s]$ such that $\pi(i) = 6$. Since the supports of weight 6 codewords in \mathcal{G}_{12} form a 5-(12, 6, 1) design, there is a minimal weight codeword $c \in \mathcal{G}_{12}$ satisfying

$$|\text{supp}(c) \cap \{m_{i-1} + 1, m_{i-1} + 2, \dots, m_i\}| \geq 5$$

which implies that $w_\pi(c) \leq 2$ and hence

$$R_{d_\pi}(\mathcal{G}_{12}) = \left\lfloor \frac{d_\pi - 1}{2} \right\rfloor = 0.$$

It is clear that \mathcal{G}_{12} cannot be 1-perfect for a block metric having one block of dimension 6. Therefore, we introduce poset structure on $[s]$ so as to define poset block metric which turns \mathcal{G}_{12} into a 1-perfect poset block code. In the following theorem, a necessary and sufficient condition for \mathcal{G}_{12} to be 1-perfect code has been provided.

Theorem 3.5. *A poset block structure turns the extended ternary Golay code \mathcal{G}_{12} into a 1-perfect code if and only if it has a unique block of weight 1 with 6 elements which is not the support of a codeword of \mathcal{G}_{12} .*

Proof. Firstly, we prove that \mathcal{G}_{12} is a 1-perfect poset block code when the poset block structure has a unique block of weight 1 with 6 elements which is not the support of a codeword of \mathcal{G}_{12} . We have that the number of 6 element subsets of $[12]$ is $\binom{12}{6}$ and the number of weight 6 codewords of \mathcal{G}_{12} is 264. Therefore

$$|\{X \subseteq [12]: |X| = 6, \text{supp}(c) \not\subseteq X \text{ for all } 0 \neq c \in \mathcal{G}_{12}\}| = \binom{12}{6} - 264 > 0.$$

Let $Y \subseteq [12]$ with $|Y| = 6$ be such that it is not the support of any weight 6 codeword of \mathcal{G}_{12} . Consider a label $\pi: [s] \rightarrow \mathbb{N}$ such that

$$\pi(1) + \pi(2) + \dots + \pi(s) = 12$$

and $\pi(i) = 6$ for some $i \in [s]$ with $V_i = V_Y$. Note that $1 < s \leq 7$. If $\Gamma^{(1)}(P) = \{i\}$, then $B_{(P,\pi)}(0, 1) = 3^{|Y|} = 3^6$. On the lines of proof of Theorem 3.3, we can prove that

$$B_{(P,\pi)}(0, 1) \cap B_{(P,\pi)}(c, 1) = \phi \text{ for all } 0 \neq c \in \mathcal{G}_{12}.$$

Since $|\mathcal{G}_{12}| \cdot |B_{(P,\pi)}(0, 1)| = 3^{12}$, we have that \mathcal{G}_{12} is a 1-perfect poset block code.

Conversely, let (P, π) be a poset block structure that turns \mathcal{G}_{12} into a 1-perfect code. If there is a minimal coordinate $i \in \Gamma^{(1)}(P)$ such that $\dim(V_i) = k_i > 6$, then

$$|B_{(P,\pi)}(0, 1)| \geq 1 + (3^{k_i} - 1) = 3^{k_i} > 3^6$$

and hence \mathcal{G}_{12} cannot be 1-perfect, since $|\mathcal{G}_{12}| = 3^6$ and $3^6 \cdot 3^{k_i} > 3^{12}$. Thus, we have

$$(3.3) \quad k_i \leq 6 \text{ for each } i \in \Gamma^{(1)}(P).$$

Suppose that $|\Gamma^{(1)}(P)| = r > 1$, say $\Gamma^{(1)}(P) = \{1, 2, \dots, r\}$. Let k_1, k_2, \dots, k_r be the dimension of the corresponding block spaces. Then

$$|B_{(P,\pi)}(0, 1)| = 1 - r + \sum_{i=1}^r 3^{k_i}.$$

Since the code is 1-perfect, therefore

$$1 - r + \sum_{i=1}^r 3^{k_i} = 3^6.$$

This implies that $r \equiv 1 \pmod 3$. Thus, we get

$$(3.4) \quad \sum_{i=1}^r 3^{k_i} = 728 + r \text{ where } r = 4 \text{ or } 7.$$

From (3.3), we have $1 \leq k_i \leq 6$ for all $i \in \Gamma^{(1)}(P)$. If $k_i = 6$ for some $i \in \Gamma^{(1)}(P)$ then $|B_{(P,\pi)}(0, 1)| > 3^6$, which contradicts that \mathcal{G}_{12} is a 1-perfect code. Thus, we have $1 \leq k_i \leq 5$. Also, $\sum_{i=1}^r k_i \leq 11$ since $r < s$. Equation (3.4) is a partition of the integer $728 + r$ into r parts and therefore, at least one part must be greater than or equal to $\frac{728}{r}$. Since $r \leq 7$ and each part is a power of 3, we have $k_1 \geq 5$, which gives us $k_1 = 5$. Now (3.4) reduces to

$$(3.5) \quad \sum_{i=2}^r 3^{k_i} = 485 + r \text{ where } r = 4 \text{ or } 7.$$

Using the same argument for (3.5), we get $k_2 = 5$. Now $k_1 = k_2 = 5$ and $k_i \geq 1$ for $3 \leq i \leq r$, which contradicts that $\sum_{i=1}^r k_i \leq 11$. Therefore $|\Gamma^{(1)}(P)| = 1$.

Let $\Gamma^{(1)}(P) = \{i\}$. Using (3.3), we get $k_i \leq 6$. For $k_i < 6$, since $|B_{(P,\pi)}(0, 1)| = 3^{k_i}$, it follows that the poset block structure does not turn \mathcal{G}_{12} into a 1-perfect code. Therefore $k_i = 6$. If V_i contains a non-zero codeword, say c , of \mathcal{G}_{12} , then $w_{(P,\pi)}(c) = 1$ which contradicts that \mathcal{G}_{12} is 1-perfect. Hence, V_i does not contain any non-zero codeword of \mathcal{G}_{12} . \square

CONCLUSION

The key feature of perfect linear $[n, k, d]$ codes is the perfect packing property of the decoding spheres that yield a complete bounded distance decoding algorithm. This algorithm corrects all error patterns of weight not exceeding the packing radius of the code, which for poset block codes may be strictly larger than $\lfloor (d-1)/2 \rfloor$.

It seems promising to explore perfect poset block codes further, due to possible applications that employ unequal error protection to capture semantic aspects of communication. Also it may be worth studying the canonical systematic form for the generator matrix of a hierarchical poset block code, which would enable us to characterize perfect poset block codes as well as extend the known syndrome decoding algorithm for hierarchical poset codes to the case of hierarchical poset block codes.

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