OPTIMAL INVESTMENT AND DIVIDEND FOR AN INSURER UNDER A MARKOV REGIME SWITCHING MARKET WITH HIGH GAIN TAX

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Abstract. This study examines the optimal investment and dividend problem for an insurer with CRRA preference. The insurer’s goal is to maximize the expected discounted accumulated utility from dividend before ruin and the insurer subjects to high gain tax payment. Both the surplus process and the financial market are modulated by an external Markov chain. Using the weak dynamic programming principle (WDPP), we prove that the value function of our control problem is the unique viscosity solution to coupled Hamilton–Jacobi–Bellman (HJB) equations with first derivative constraints. Solving an auxiliary problem without regime switching, we prove that, it is optimal for the insurer in a multiple-regime market to adopt the policies in the same way as in a single-regime market. The regularity of the viscosity solution on its domain is proved and thus the HJB equations admits classical solution. A numerical scheme for the value function is provided by the Markov chain approximation method, two numerical examples are given to illustrate the impact of the high gain tax and regime switching on the optimal policies.

1. Introduction. This study examines the optimal investment and dividend problem of an insurer subjected to the payment of high gain tax. The optimal dividend problem was pioneered by De Finetti [19] and has since then been studied extensively. For example, Gerber and Shiu [27] studied this problem under a drifted Brownian motion model, while Cai et al. [13] examined it under an Ornstein–Uhlenbeck type model with credit and debit interest. Bai and Guo [5] considered the classical model with transaction costs and Yao et al. [58] considered the dual risk model with fixed and proportional costs. Yin and Wen [59] studied this problem for spectrally positive Lévy processes, Zhao et al. [62] studied optimal dividend in a dual risk model with random time horizon or capital injection. Chen et al. [14]
studied optimal dividend strategies with time-inconsistent preferences and Barth et al. [7] studied dividend optimization under stochastic short rate process, providing a finite element approximation and time-marching scheme. One common feature of the aforementioned studies is that the object of the insurer is to maximize the expected discount accumulative dividend payment in an infinite time horizon or in a random time horizon. Alternatively, other studies examine the optimal dividend problem for maximizing the discount expected utility from dividend. For example, see Hubalek and Schachermayer [32] for this problem under a drifted Brownian motion and power utility, Grandits et al. [29] for drifted Brownian motion and exponential utility, Cadenillas et al. [12] for a mean reverting process.

In comparison to separate studies on optimal dividend or optimal investment, few studies examine both optimal investment and dividend for an insurer. With regard to the scope of the actuarial risk model, Højgaard and Taksar [31] studied the optimal portfolio policy of an insurer with dividend policy under the drifted Brownian motion model. Azcue and Muler [3] studied optimal investment and dividend problem under the classical risk model and showed that the value function is the smallest viscosity solution to the associated second order integro-differential HJB equation. The study also provided the regularity of the viscosity solution. Jin et al. [37], Jin and Yin [36] studied the numerical methods for the optimal dividend and investment problem of an insurer with jump diffusion surplus process and singular control.

Although optimal investment and dividend problems do not attract much attention in actuary research, its counterpart in mathematical finance, the optimal investment/consumption problem has always been extensively studied. An example of early work is Merton [41]. For more recent research, we refer the reader to the monograph of Rogers [47] and the references therein. It usually assumes that the coefficients of the risky asset returns are constant and the financial market is frictionless (i.e. no transaction cost, no tax, no restrictions on short or long, etc.). However, in reality, the returns from the risky assets might not be constant. Therefore, it is interesting to consider asset models with non-constant coefficients. Among the various models with non-constant coefficients, Markov regime switching models provide a natural and convenient way to describe the impact of structural changes in macroeconomic conditions and business cycles on price dynamics (c.f. Clements and Krozig [17]). For recent research on optimal investment/consumptions under Markov regime switching model, see Bäuerle and Rieder [8], Elliott and Kopp [22], Sass and Haussmann [49], Yiu et al. [60], Zou and Cadenillas [66], Zou and Cadenillas [67]. At the same time, a significant number of studies concentrate on optimal investment/consumption problem with market frictions, including friction from high gain tax. The high gain tax is paid according to the following rule: whenever the maximum profit up to today, the so-called high-watermark, exceeds the previously attained historic maximum, the management institution charges a fixed proportion of the profit (relative to the previous maximum). Stiglitz [52] discussed the possibility and necessity of charging high gain tax and suggested that the impact of the tax is not adequately summarized by a single number, such as ‘effective tax rate’, representing the average ratio of tax payments to capital gains. Moreover, the impact of the tax cannot be assessed by examining only reported capital gains and losses. It is worthwhile to investigate the impact of tax payment on the behaviour of decision-makers in the long runs. Janecek and Sirbu [33] studied the optimal investment and consumption of a fund manager subjected to high gain tax payment over an infinite
time horizon and demonstrated the existence and regularity of the solution to the HJB equation using Perron’s method and the “homotheticity property” of CRRA utility. One advantage of the method in [33] is that it is not necessary to prove that the value function satisfies the dynamic programming principle (DPP).

In this paper, we study the optimal investment and dividend problem of an insurer under a regime switching model with high gain tax. Briefly, the main feature of the model in this study is the regime switching on structure and constraints on state space. In fact, the optimal dividend problem with regime switching and/or constraints has attracted significant attention in the past decade. For example, Jiang and Pistorius [35] investigated an optimal dividend distribution under a Markov regime switching model and provided an explicit characterization of the value function as the fixed point of a contraction. The fixed point method was also applied by Zhu [64] for singular optimal dividend control in the regime-switching Cramér–Lundberg model with credit and debit interest, and by Fu et al. [25] for portfolio optimization in a regime-switching market with derivatives. Zhu and Chen [65] studied dividend optimization for regime-switching general diffusions and found two interesting, exclusive optimization results, depending on the configuration of the model parameters. The aforementioned papers assume that the regime switching process is observable. It is natural to consider that the regime is unobservable, that is, the hidden-Markov model (HMM). Szölgyenyi [53] studied the optimal dividend problem under the HMM framework. Using a filtering method, an analytic characterization of the optimal value function was given and a numerical study covering various scenarios is presented for a clear picture of how dividends should be paid out. This HMM framework was also applied in Leobacher et al. [40] for the valuation problem of an (insurance) company under partial information. The most optimal dividend problems are under the diffusion or jump-diffusion frameworks, to overcome the shortcoming of infinite money flows. Pospelov and Radionov [44] calculated an optimal dividend policy when cash surplus follows the telegraph process and solve this problem using variational inequalities, demonstrating that the optimal dividend policy is defined by two thresholds.

With regard to dividend problem with constraints, see Choulli et al. [15] for optimal dividend distribution with constraints on risk control in a diffusion model, Paulsen [43] for solvency constraints, Avram et al. [2] for penalty constraints, Reppen et al. [46] for bankruptcy constraints and Tan et al. [55] for debt constraints. In this study, the surplus process of the insurer is specified by an arithmetic Brownian motion with Markov regime switching. The object of the insurer is to maximize the expected accumulated discounted utility from dividend up to ruin. Notably, while similar to the consumption/investment problem of mathematical finance, the investment/dividend optimization problem under actuary models for an insurer possesses a special feature, preventing it from being treated as a particular case of consumption/investment models (c.f. Taksar [54]). The special feature is that the insurer has extra premium income and must pay claims; therefore, the insurer has to bear potential loss beyond the financial market. In later arguments, we will find that this special feature prevents us from adopting the method used in [33]. The main dividend distribution methods consist of restricted dividend (bounded dividend rate) strategy and barrier dividend strategy (c.f. Avanzi [1] ). In this study, we focus on restricted dividend payments. For recent work on optimal dividend with bounded rate, see Azcue and Muler [4], Hojgaard and Taksar [30], Zhu [64] and the references therein.
We can summarize the contributions of this paper as follows. The optimization problem considered in this study is relevant to a diffusion process with regime switching and reflection, which leads the structure of the associated HJB equations are a coupled HJB equations with first derivative constraint on the boundary. Noting that the coupled HJB equations are different to the ones in Janecek and Sirbu [33], we cannot adopt the method in [33] directly. Using the weak dynamic programming principle (WDPP) put forth by Bouchard and Touzi [10], we prove that the value function is the unique viscosity solution to the associated coupled HJB equations with constraint. Motivated by Jiang and Pistorius [35], we prove that when the current state of the Markov chain is given, it is still optimal for the insurer in a multiple-regime market to adopt the policies in the same way as in a single-regime market. Using this result, we prove that the regularity of the value function on its domain, and the associated coupled HJB equations admit a classical solution. However, we are unable to obtain an explicit solution to the coupled HJB equations; thus, we provide a numerical method for deriving the value function and the optimal policies by the Markov chain approximating approach. We present two numerical examples to illustrate the impact of the high gain tax and the impact of regime switching on optimal policies.

The rest of this paper is organized as follows. Section 2 introduces the model and the problem. In Section 3, we prove that the value function of our control problem is the unique viscosity solution to the associated coupled HJB equations with constraint. By solving an auxiliary control problem, in Section 4, we prove that in a multiple-regime market, it is optimal to adopt the policies in the same way as in a single-regime market. We also provide the regularity of the value function. Section 5 presents a numerical approximating method for the value function and the optimal policies. Two numerical examples are given to illustrate the impact of the high gain tax and the Markov regime switching structure on the optimal policies. Section 6 provides a brief comparison of our work with several very similar state-of-art papers.

2. Models and problem. Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered probability space that satisfies the usual conditions, i.e. \(\mathcal{F}_t\) is right-continuous and \(\mathbb{P}\) is complete. \(\mathcal{F}_t\) represents the information available up to time \(t\) and all of the decisions made at time \(t\) are based on this information. Assume that the surplus process of the insurer and the return process of the financial market are both modulated by the macroeconomic condition. The dynamic of the macroeconomic condition is specified by a continuous-time, finite-state, recurrent, observable Markov chain \(\xi = \{\xi_t\}_{t \geq 0}\) with state space \(E := \{\alpha_1, \alpha_2, \ldots, \alpha_d\}\), where \(\alpha_i \in \mathbb{R}^d, i = 1, 2, \ldots d\) are unit vectors with 1 in the \(i\)th position and zero elsewhere. Let \(Q := [q_{ij}], i, j = 1, 2, \ldots, d\) be the generator of \(\xi\), then we know that \(q_{ij} \geq 0, i \neq j\) and \(\sum_{i=1}^{d} q_{ij} = 0; q_{ij} > 0, i \neq j; q_{ii} = -q_{i} < 0\). Denote by \(\tau_i\) the \(i\)th jumping time of \(X(t)\), then we have the following Lemma 2.1. The proof of the Lemma 1 can be found in Grandell [28].

**Lemma 2.1.** Suppose that \(\xi_0 = \alpha_i\), then we have holds:

\[
\mathbb{P} (\tau_1 > t) = e^{-q_{i} t}; \quad (2.1)
\]

\[
\mathbb{P} (\tau_1 \leq t, \xi(\tau_1) = \alpha_j) = (1 - e^{-q_{i} t}) \frac{q_{ij}}{q_{i}}; \quad (2.2)
\]

\[
\mathbb{P} (X(\tau_1) = \alpha_j) = \frac{q_{ij}}{q_{i}}. \quad (2.3)
\]
Assume now that the insurer pays $\lambda$ tax. Assume that the insurer has an initial maximum profit in the interval $[P_i, f_i \cdot \lambda]$ where $x > 0$ is the premium income, $c(\xi_t) > 0$ is the premium income rate at time $t$, $\sigma_1(\xi_t)W_t^1$ is the perturbation of the insurer’s surplus, $\sigma_1(\cdot)$ describes the volatility of the insurance market. Surplus process (2.5) is the diffusion approximation of the classical surplus process (see Grandell [28]). The financial market consists of a risky asset (e.g. a stock) $S_t$ and a risk free asset $B_t$. Assume that $S_t$ evolves as

$$dS_t = S_t \left( \mu(\xi_t)dt + \sigma_2(\xi_t)dW_t^2 \right),$$

where $\mu(\xi_t) > 0$ is the investment return rate, $\sigma_2(\xi_t)$ is the volatility. The price of risk-free asset evolves as

$$dB_t = r(\xi_t)B_t dt,$$

where $r(\xi_t) \geq 0$ for any $t \geq 0$ and $\mu(\cdot) \geq r(\cdot)$, $t \geq 0$. Although from practical perspective, the return of bond market shall not be zero, we can consider the “discounted process” as Gaier et al. [26], then it is reasonable for us to assume that $r(\cdot) \equiv 0$ in the rest of this paper.

**Assumption 1.** Processes $\{W^1_t\}_{t \geq 0}, \{W^2_t\}_{t \geq 0}$ and $\{\xi_t\}_{t \geq 0}$ are mutually independent.

Let $f_t$ be the amount of insurer’s wealth invested in risky asset at time $t$. Denote by $P^f_t$ the accumulated profit of the insurer with investment policy $f = \{f_t\}_{t \geq 0}$. Noting that the high gain tax is paid according to accumulated profit of the insurer, thus the initial surplus of the insurance company should not be taken into account for tax payment, this means that $P^f_0 = 0$. If there are not any other fees are imposed, then $P^f_t$ evolves as

$$\begin{cases}
  dP^f_t = f_t \frac{dS_t}{S_t} + c(\xi_t)dt + \sigma_1(\xi_t)dW_t^1, & 0 \leq t < \infty, \\
  P^f_0 = 0.
\end{cases}$$

Denote by $\{M_t, t \geq 0\}$ the maximum profit process, i.e.

$$M_t \triangleq \sup_{0 \leq s \leq t} P^f_s.$$

Assume now that the insurer pays $\lambda \Delta M_t = \lambda(M_{t+\Delta t} - M_t)$ to the management institution in the interval $[t, t + \Delta t]$, where $\lambda$ is the percentage of the high gain tax. Assume that the insurer has an initial maximum profit $i$ ($i \geq 0$), the profit of the insurer will be taxed when $P^f_t$ reaches value $i$ and will not be taxed before $P^f_t$ reaches value $i$. By taking high gain tax payment into account, we rewrite the dynamic of $P^f_t$ as

$$\begin{cases}
  dP^f_t = f_t \frac{dS_t}{S_t} + c(\xi_t)dt + \sigma_1(\xi_t)dW_t^1 - \lambda dM_t, & 0 \leq t < \infty, \\
  M^f_t = \sup_{0 \leq s \leq t} (P^f_s \vee i).
\end{cases}$$

Assume that the insurance company distributes dividend with a nonnegative dividend rate process $\{\gamma_t\}_{t \geq 0}$ either from the insurance market or from accumulated
profit. Denote by
\[
C_t \triangleq \int_0^t \gamma_s \, ds, \quad 0 \leq t < \infty, \tag{2.7}
\]
the accumulated dividend process and by \(X_t^{f,\gamma}\) the wealth process of the insurer with policy \((f, \gamma)\). Since the money market pays zero interest rate, the wealth process \(X_t^{f,\gamma}\) can be written as
\[
X_t^{f,\gamma} = x + P_t^{f,\gamma} - C_t, \quad 0 \leq t < \infty.
\]
We observe that the high gain tax is paid as soon as the current profit process into a Markovian system. As usual, the wealth has to be part of the state. We rewrite the high-watermark of the insurer’s profit as
\[
M_t^{f,\gamma} = \sup_{0 \leq s \leq t} \left( (X_s^{f,\gamma} + C_s - x) \lor 0 \right) = i + \sup_{0 \leq s \leq t} \left( (X_s^{f,\gamma} + C_s) - n \right),
\]
where \(n \triangleq x + i \geq x > 0\).

If we want to apply the HJB equation method, it is necessary to embed our state process into a Markovian system. As usual, the wealth has to be part of the state. We observe that the high gain tax is paid as soon as the current profit \(X_t^{f,\gamma} + C_t\) hits the maximum profit \(M_t^{f,\gamma} = i + \sup_{0 \leq s \leq t} \left( (X_s^{f,\gamma} + C_s) - n \right)^+\). In other words, tax is paid whenever
\[
P_t^{f,\gamma} - i = \sup_{0 \leq s \leq t} \left( (X_s^{f,\gamma} + C_s - n)^+ \right),
\]
which is the same as \(X_t^{f,\gamma} = N_t^{f,\gamma}\), where
\[
N_t^{f,\gamma} \triangleq n + \sup_{0 \leq s \leq t} \left( (X_s^{f,\gamma} + C_s) - n \right)^+ - C_t
= \sup_{0 \leq s \leq t} \left( (X_s^{f,\gamma} + C_s \lor 0) - C_t \right)
= x + M_t^{f,\gamma} - C_t \geq X_t^{f,\gamma}.
\]
We now transform our model into a two-dimensional process \((X, N)\) satisfying \(X \leq N\) and is reflected whenever \(X = N\). The controlled state process \((X, N)\) follows
\[
\begin{cases}
\begin{align}
dX_t^{f,\gamma} &= (f_t \mu(\xi_t) - \gamma_t + c(\xi_t)) \, dt + \sigma_1 \, dW_t^1 + f_t \sigma_2 \, dW_t^2 - \lambda (dM_t^{f,\gamma} + \gamma_t \, dt), \\
& \quad \text{with } X_0 = x,
\end{align}
\end{cases}
N_t^{f,\gamma} &= \sup_{0 \leq s \leq t} \left[ (X_s^{f,\gamma} + \int_0^s \gamma_u \, du) \lor 0 \right] - \int_0^t \gamma_u \, du, \quad t \geq 0.
\tag{2.8}
\]

The insurance company controls the process \((X, N)\) in (2.8) which is restricted to the domain
\[
\mathcal{G} = \{(x, n) : 0 \leq x \leq n\},
\tag{2.9}
\]
and is reflected on the diagonal \(\partial \mathcal{G} = \{(x, n) : x = n\}\) in the direction given by the vector
\[
\hat{r} \triangleq \left( -\frac{\lambda}{1} \right).
\]

We can now rewrite the dynamic of controlled state system \((\xi_t, X_t^f, N_t^f)\) as
\[
d \begin{pmatrix} \xi_t \\ X_t^f \\ N_t^f \end{pmatrix} = \begin{pmatrix} Q_{\xi_t} \, dt \\ [f_t \mu(\xi_t) + c(\xi_t) - c_t] \, dt + f_t \sigma_2(\xi_t) \, dW_t^2 + \sigma_1(\xi_t) \, dW_t^1 \\ -\gamma_t \, dt \end{pmatrix} \\
+ \begin{pmatrix} dA_t \\ -\lambda dM_t^f \\ dM_t^f \end{pmatrix},
\tag{2.10}
\]
with
\[
\int_0^t 1_{\{X_s \neq N_t\}} \, \mathrm{d}M^f_s = 0. \tag{2.11}
\]

One must also specify what kind of information is available to \( f_t, \gamma_t \) at time \( t \). We assume that the insurer is allowed to know the past history of states \( (\{\xi_s, X_s, N_s\}) \) for \( s \leq t \) when control \( (f_t, \gamma_t) \) is chosen, i.e. \( \{(f_t, \gamma_t)_{t \geq 0}\} \) is \( \mathcal{F}_t \)-adapted. Fleming and Soner [24] suggested that the Markovian nature of the problem suggests that it should suffice to consider the Markov control (c.f. Fleming and Soner [24], Page 131, also in Theorem 11.2.3 of Øksendal [42]). Inspired by this conclusion, we just focus on Markov control in the rest of this paper.

**Assumption 2.**

1. For all \( \alpha_i \in \mathcal{E} \), \( \mu(\alpha_i) > 0, \sigma_k(\alpha_i) > 0, k = 1, 2; i = 1, 2, \ldots, d. \)
2. We adopt Markov control policies, i.e.
\[
f_t \triangleq f(\xi_t, X_t^{f, \gamma}, N_t^{f, \gamma}), \gamma_t \triangleq \gamma(\xi_t, X_t^{f, \gamma}, N_t^{f, \gamma}),
\]
where \( f(\cdot, \cdot, \cdot) \) and \( \gamma(\cdot, \cdot, \cdot) \) are the determining functions of policy \( f_t \) and \( \gamma_t \) respectively.
3. \( f(\alpha_i, x, n) : E \times \mathcal{G} \to U \) is Lipschitz continuous w.r.t. \( x, n, U \) is bounded closed subset of \( \mathbb{R}^d \).
4. \( \gamma(\alpha_i, x, n) \) is Lipschitz continuous w.r.t. \( x, n \) and \( 0 \leq \gamma_t \leq X_t^{f, \gamma} \).

Under Assumption 2, the existence and uniqueness of the solution to Equation (2.8) can be obtained by a method very close to the one in Belfadli et al. [9]. On the other hand, we can also present a implicit expression to the solution of Equation (2.8) as the one in [33]. Here we omit the expression because it is not relevant to the rest of this paper.

Denote by \( \mathcal{A}(\alpha_i, x, n) \) the admissible control set for the state system (2.8) starting at point \( (\alpha_i, x, n) \). Define \( \tau^{f, \gamma} \triangleq \inf \{s \geq 0 : X_s^{f, \gamma} = 0\} \) the first time that the wealth reaches zero. In actuarial risk model, \( \tau^{f, \gamma} \) is referred to the ruin time of the insurer. For notation simplicity, denote
\[
\mu_i = \mu(\alpha_i), c_i = c(\alpha_i), \sigma_{1i} = \sigma_1(\alpha_i), \sigma_{2i} = \sigma_2(\alpha_i),
\]
and
\[
\mathbb{P}_{\alpha_i, x, n}(\cdot) = \mathbb{P}(\cdot | (\xi_0, X_0, N_0) = (\alpha_i, x, n)),
\]
\[
\mathbb{E}_{\alpha_i, x, n}(\cdot) = \mathbb{E}(\cdot | (\xi_0, X_0, N_0) = (\alpha_i, x, n))
\]
\[
\mathcal{L}^{f, \gamma} v(\alpha_i, x, n) \triangleq \sum_{j=1}^d q_{ij} v(\alpha_j, x, n) + \frac{\partial v(\alpha_i, x, n)}{\partial x} (f \mu_i + c_i - \gamma)
\]
\[
\quad + \frac{1}{2} \sigma_i^2 \frac{\partial^2 v(\alpha_i, x, n)}{\partial x^2} \left[ \sigma_1^2 + \sigma_2^2 \right] + \frac{1}{2} \sigma_i^2 \frac{\partial^2 v(\alpha_i, x, n)}{\partial n^2}, \alpha_i \in \mathcal{E},
\]
where \( v(\alpha_i, x, n) \), \( \alpha_i \in \mathcal{E} \) are supposed to belong to the domain of second order differential operator \( \mathcal{L}^{f, \gamma}(\cdot) \).

The insurer wants to seek optimal control polices to maximize the expected cumulated discount utility from dividend before ruin, i.e. the value function of the
insurer is

\[ V(\alpha_i, x, n) = \max_{(f, \gamma) \in A(\alpha_i, x, n)} J^{f, \gamma}(\alpha_i, x, n) \]

\[ = \max_{(f, \gamma) \in A(\alpha_i, x, n)} \mathbb{E}_{\alpha_i, x, n} \left[ \int_0^\tau e^{-\beta s} u(\gamma_s) ds \right], \quad (2.15) \]

where \( u(x) \) is the utility function of the insurer satisfying \( u'(x) > 0, u''(x) < 0, u'(\pm \infty) = \lim_{x \to \pm \infty} (x) = 0 \). \( \beta > 0 \) is the discount factor. In particular, we assume that utility function is a CRRA utility function, i.e.

\[ u(\gamma) = \frac{\gamma^p}{p}, \quad \gamma > 0, \quad 0 < p < 1, \]

where \( p \) is called the relative risk aversion coefficient. A pair of decision processes \((f^*, \gamma^*)_{t \geq 0}\) is said to be optimal if

\[ J^{f^*, \gamma^*}(\alpha_i, x, n) = \max_{(f, \gamma) \in A(\alpha_i, x, n)} \mathbb{E}_{\alpha_i, x, n} \left[ \int_0^\tau e^{-\beta s} u(\gamma_s) ds \right]. \quad (2.16) \]

**Remark 1.** If \( V(\alpha_i, x, n) \) is smooth enough, then by dynamic programming principle, we can prove that \( V(\alpha_i, x, n) \) is the solution to the following coupled HJB equations:

\[ \sup_{f, \gamma \in A} \left\{ -\beta v + u(\gamma) + \mathbf{L}^{f, \gamma} v(\alpha_i, x, n) \right\} = 0, \quad \text{for} \quad \alpha_i \in \mathcal{E}; 0 < x < n; \quad (2.17) \]

\[ -\lambda \frac{\partial v}{\partial x} + \frac{\partial v}{\partial n} = 0, \quad \text{for} \quad 0 < x = n; \quad (2.18) \]

\[ v(\alpha_i, x, n) = 0, \quad \text{for all} \quad x < 0, i = 1, 2, \ldots, d. \quad (2.19) \]

We firstly present an intuitive derivation of coupled HJB equations (2.17-2.19). By the generalized Itô’s Lemma and the DPP equation, we have Equation (2.17) immediately. Taking into account that \( dM \) is a singularity measure with support on the set of time \( \{ t \geq 0 : P^f_t = N^f_t \} \) (see Equation (2.11)), we have the boundary condition (2.18).

However, we cannot prove the smoothness properties of \( V(\alpha_i, x, n) \) directly. In the next section we will prove that the value function is the viscosity solution to coupled HJB equations (2.17-2.19) by the WDPP put forwarded by Bouchard and Touzi [10].

Although HJB equations (2.17—2.19) are similar to the ones in Janecek and Sîrbu[33], we cannot directly apply the method in [33]. This is due to the fact that, compare to the coefficient of \( \partial v/\partial x \) in the HJB equations of [33] (Page 797), the coefficient of \( \partial v/\partial x \) in (2.17) has a extra constant \( c_i \). This extra constant prevents us from applying the “homotheticity property” of \( u(x) \) to cope with the coupled HJB equations as [33]. In Section 4, we will prove the regularity of the viscosity solution (i.e. the value function) by solving an auxiliary problem. However, unlike [33], we are not able to provide the asymptotic behavior of the value function.

3. Value function and viscosity solution to associated coupled HJB equations. Given a locally bounded function \( \omega \) (i.e. for all \((\alpha_i, x, n)\), there exists a compact neighborhood \( U_i(x, n) \) of \((x, n)\), depending on \( \alpha_i \), such that \( \omega \) is bounded on
We define its upper-semicontinuous (u.s.c) envelope $\omega^*$ and lower-semicontinuous (l.s.c) envelope $\omega_*$ by

$$\omega^*(\alpha_i, x, n) = \limsup_{(x', n') \to (x, n)} \omega(\alpha_i, x', n'), \quad \omega_*(\alpha_i, x, n) = \liminf_{(x', n') \to (x, n)} \omega(\alpha_i, x', n').$$

Note that $\omega^*$ (resp. $\omega_*$) is the smallest (resp. largest) u.s.c function (resp. l.s.c function) above (resp. below) $\omega$.

A locally bounded function $\omega$ on $E \times G$ is l.s.c (resp. u.s.c.) if and only if $\omega = \omega^*$ (resp. $\omega = \omega_*$) on $E \times G$, and it is continuous if and only if $\omega = \omega^* = \omega_*$ on $E \times G$.

**Definition 3.1.** (Viscosity subsolution and supersolution, see Barles and Imbert [6]).

(1): An u.s.c (resp. l.s.c.) function $\omega$ is a viscosity subsolution (resp. supersolution) of Equations (2.17–2.19) if and only if, for any test function $\psi(\alpha_i, x, n) \in C^2(E, G)$, if $(\alpha_i, \bar{x}, \bar{n})$ is a global maximum point (resp. minimum point) of $\omega^* - \psi$, then

$$\beta \psi(\alpha_i, \bar{x}, \bar{n}) - \sup_{(f, \gamma) \in A(\bar{x}, \bar{n})} \{u(\gamma) + L^f \gamma \psi(\alpha_i, \bar{x}, \bar{n})\} \leq 0 \quad (\text{resp.} \geq 0),$$

$$\gamma \psi_x(\alpha_i, \bar{x}, \bar{n}) - \psi_n(\alpha_i, \bar{x}, \bar{n}) \leq 0 \quad (\text{resp.} \geq 0). \quad (3.1)$$

Here, without losing of generality, we assume that maximum (resp. minimum) of $\omega^* - \psi$ (resp. $\omega_* - \psi$) is 0.

(2): $\omega(\alpha_i, x, n)$ is a viscosity solution to Equations (2.17-2.19) if it is simultaneously a viscosity subsolution and viscosity supersolution.

To apply the WDPP, we need to prove that the value function is bounded.

**Lemma 3.2.** Suppose that $\beta$ is large enough, then $V(\alpha_i, x, n)$ is bounded, continuous and concave w.r.t. $(x, n) \in G$.

**Proof.** The proof is finished by following the standard method in Proposition 2.5 of Xu and Shreve [57], for completeness, we present the proof in Appendix A. \qed

**Remark 2.** Usually, the bounded dividend rate process satisfies

Assumption 2 (4'): $\gamma_\iota(\alpha_i, x, n)$ is Lipschitz continuous w.r.t. $x, n$ and $\gamma_\iota \in [0, \tilde{\Gamma}]$, where $\tilde{\Gamma}$ is upper bound of dividend rate.

One can easily find that the results of Lemma 3.2 still hold if replace Assumption 2(4) by Assumption 2(4').

Based on Assumptions 2(2)-(4) and Lemma 3.2, we find that the controlled process (2.10) and the value function satisfy the condition of Proposition 5.4 of Bouchard and Touzi [10], thus the value function $V(\alpha_i, x, n)$ satisfies the following WDPP, see (3.8) and (3.9) in Page 954 of Bouchard and Touzi [10].

**Proposition 1.** For every stopping time $h$, denote by $\tau^f_\iota = h \land \tau^{f, \gamma}$. Assume that Assumptions 2(2)-(4) hold true, together with Lemma 3.2 (value function is
bounded and continuous), then for every \((\alpha_i, x, n) \in \mathcal{E} \times \mathcal{G}\) we have

\[
V(\alpha_i, x, n) \leq \sup_{(f, \gamma) \in \mathcal{A}(\alpha_i, x, n)} \mathbb{E}_{\alpha_i, x, n} \left[ \int_0^{\tau_{f, \gamma}} e^{-\beta s} u(\gamma_s) ds + e^{-\beta (\tau_{f, \gamma})} V^*(\xi_{f, \gamma}, X_{f, \gamma}^{f, \gamma}, N_{f, \gamma}^{f, \gamma}) \right]
\]

\[
= \sup_{(f, \gamma) \in \mathcal{A}(\alpha_i, x, n)} \mathbb{E}_{\alpha_i, x, n} \left[ \int_0^{\tau_{f, \gamma}} e^{-\beta s} u(\gamma_s) ds + e^{-\beta (\tau_{f, \gamma})} V(\xi_{f, \gamma}, X_{f, \gamma}^{f, \gamma}, N_{f, \gamma}^{f, \gamma}) \right].
\]

\[
(3.2)
\]

For any \(\psi(\alpha_i, x, n) \in C^{2,1}(\mathcal{G}), \alpha_i \in \mathcal{E}\) such that \(V(\alpha_i, x, n) \geq \psi(\alpha_i, x, n)\), we have

\[
V(\alpha_i, x, n) \geq \sup_{(f, \gamma) \in \mathcal{A}(\alpha_i, x, n)} \mathbb{E}_{\alpha_i, x, n} \left[ \int_0^{\tau_{f, \gamma}} e^{-\beta s} u(\gamma_s) ds + e^{-\beta (\tau_{f, \gamma})} \psi(\xi_{f, \gamma}, X_{f, \gamma}^{f, \gamma}, N_{f, \gamma}^{f, \gamma}) \right]
\]

\[
= \sup_{(f, \gamma) \in \mathcal{A}(\alpha_i, x, n)} \mathbb{E}_{\alpha_i, x, n} \left[ \int_0^{\tau_{f, \gamma}} e^{-\beta s} u(\gamma_s) ds + e^{-\beta (\tau_{f, \gamma})} V(\xi_{f, \gamma}, X_{f, \gamma}^{f, \gamma}, N_{f, \gamma}^{f, \gamma}) \right].
\]

\[
(3.3)
\]

In the rest of this paper, we will frequently use an alternatively expression of the WDPP Equation (3.2), see Corollary 3.6 of Bouchard and Touzi [10]. Denote by \(\mathcal{T}_{0,\tau_{f, \gamma}}\), the set of stopping times valued in \([0, \tau_{f, \gamma}]\), then for all \(\epsilon > 0\), there exist \((f, \gamma) \in \mathcal{A}\) and \(\tilde{u} \in \mathcal{T}_{0,\tau_{f, \gamma}}\) such that

\[
V(\alpha_i, x, n) - \epsilon \leq \mathbb{E}_{\alpha_i, x, n} \left[ \int_0^{\tilde{\tau}_{f, \gamma}} e^{-\beta s} u(\gamma_s) ds + e^{-\beta \tilde{\tau}_{f, \gamma}} \psi(\xi_{f, \gamma}, X_{f, \gamma}^{f, \gamma}, N_{f, \gamma}^{f, \gamma}) \right].
\]

\[
(3.4)
\]

**Theorem 3.3.** \(V(\alpha_i, x, n), i = 1, 2, \ldots, d\) are the unique coupled viscosity solutions to Equations (2.17-2.19).

**Proof.** The proof is given in Appendix B. \(\square\)

4. Optimal policies and smoothness of the viscosity solution.

4.1. An auxiliary optimization problem. To proceed our discussion, we need to treat the following problem.

**Auxiliary Problem**

\(1\): \(\xi_0 = \alpha_i\), the surplus process of insurer is \(U_t = x + c_t t + \sigma_t W_t^3, t \geq 0\). The dynamic of risky asset is \(dS_t = S_t (\mu_t + \sigma_t dW_t^2), t \geq 0\).

\(2\): Assumption 1 holds true. Investment policy \(\{f_t, t \geq 0\}\) and dividend policy \(\{\gamma_t, t \geq 0\}\) satisfy Assumption 2.

\(3\): The controlled process is stopped whenever ruin happens or the state of \(\xi_t\) changes.

\(4\): There is a couple of beforehand settled bequest function \(w(\alpha_i, x, n), \alpha_i \in \mathcal{E}\), where \(w(\alpha_i, x, n)\), \(\alpha_i \in \mathcal{E}\) is continuous and concave w.r.t. \((x, n) \in \mathcal{G}\).

Recall that \(\tau_1\) is the first time that \(\xi_t\) leaves its initial state, say \(\alpha_i\), then \(\tau_1\) follows exponential distribution with mean \(1/q_i\). Define a performance function associated
with the Auxiliary problem as
\[
\mathcal{W}_w^f,\gamma(\alpha, x, n) = \left[ \int_0^{\tau^f,\gamma \wedge \tau_1} e^{-\beta s} u(\gamma_s) ds + 1_{(\tau_1 < \tau^f,\gamma)} e^{-\beta \tau_1} w(\xi_{\tau_1}, X^f, \gamma, N^f, \gamma) \right],
\]
and
\[
\mathcal{W}_w(\alpha, x, n) = \sup_{(f,\gamma) \in \mathcal{A}(\alpha, x, n)} \mathcal{W}_w^f,\gamma(\alpha, x, n).
\]

We introduce the following notation
\[
D^f,\gamma w(\alpha, x, n) = \partial w(\alpha, x, n) \partial x (f \mu_i + c_i - \gamma) + \frac{1}{2} \partial^2 w(\alpha, x, n) \left[ \sigma_i^2 + \sigma_i^2 f^2 \right] - \gamma \frac{\partial w(\alpha, x, n)}{\partial n},
\]
where \(w(\alpha, x, n), \alpha \in \mathcal{C}\) belongs to the domain of operator \(D\).

Similar to the discussion in Section 3, we can prove that \(\mathcal{W}_w(\alpha, x, n)\) satisfies the WDPP and is the unique viscosity solution to the following HJB equation
\[
\sup_{f,\gamma \in \mathcal{A}(\alpha, x, n)} \left\{ -\beta w + u(\gamma) + D^f,\gamma w(\alpha, x, n) \right\} = 0, \text{ for } x < n;
\]
\[
-\lambda \frac{\partial w}{\partial x} + \frac{\partial w}{\partial n} = 0, \text{ for } x = n.
\]
\[
w(\alpha, x, n) = w(\alpha, 0, n), \text{ for all } x < 0.
\]

We point out there is no regime switching structure in the Auxiliary problem and HJB equations (4.4–4.6) is not a coupled HJB equations.

**Lemma 4.1.** Suppose that Assumption 1 and Assumption 2 hold true, then \(\mathcal{W}(\alpha, x, n)\) is bounded, continuous and concave on \(G = \{ (x, n) : 0 \leq x \leq n \} \) and is the unique viscosity solution to HJB equations (4.4–4.6).

**Proof.** The proof is very similar to the one for Lemma 3.2 and Theorem 3.3, we omit it here. \(\square\)

Denote by \(w(\alpha, x, n)\) the viscosity solution to (4.4–4.6), then by Lemma 4.1, \(w(\alpha, x, n)\) has the same property as \(\mathcal{W}(\alpha, x, n)\), i.e. \(w(\alpha, x, n)\) is bounded, continuous and concave w.r.t. \((x, n) \in G\). In fact, we further have that \(w(\alpha, x, n) \in C^{2,1}(G)\), see the following Theorem 4.2.

**Theorem 4.2.** For any given \(\alpha\), \(w(\alpha, x, n) \in C^{2,1}(G)\).

**Proof.** See Appendix C. \(\square\)

We have proved that for each \(\alpha \in \mathcal{C}\), the viscosity solution to HJB Equations (4.4–4.5) is smooth enough on the domain \(G\). The feedback control is specified as
\[
\gamma^*(\alpha, x, n) = I \left( w_x(\alpha, x, n) + w_n(\alpha, x, n) \right),
\]
\[
f^*(\alpha, x, n) = -\frac{\mu_i}{\sigma_i^2} \frac{w_x(\alpha, x, n)}{w_{xx}(\alpha, x, n)},
\]
where $I \triangleq (u')^{-1}$ is the inverse of marginal utility. Moreover, we have
\[
\mathcal{M}_w(\alpha_i, x, n) = \sup_{(f, \gamma) \in \mathcal{A}(x, n)} \tilde{F}_w^{f, \gamma}(\alpha_i, x, n)
\]
\[
= \mathbb{E} \left[ \int_0^{\tau^{f, \gamma} \wedge \tau_1} e^{-\beta t} u(\gamma_s^*) ds + 1_{(\tau_1 < \tau^{f, \gamma})} e^{-\beta \tau_1} w(\xi_{\tau_1}, X_{\tau_1}^{f, \gamma}, N_{\tau_1}^{f, \gamma}) \right]. \tag{4.9}
\]
We mention that there is no regime switching in the auxiliary control problem once the current state of macroeconomic condition is given.

4.2. Optimal policies and optimal control process. In this section we will construct optimal control policies for original optimization problem (2.15) by means of the candidate optimal dividend and investment policies of the Auxiliary problem. In the Auxiliary problem, it turned out that given the initial regime $\alpha_i$, it is optimal, with respect to the performance functional $\tilde{F}_w^{f, \gamma}(\alpha_i, x, n)$, to follow strategy $\gamma_i^*, f_i^*$ before $\tau_1$. It is natural to guess that during the period between the consecutive switches of the regime, a similar strategy dependent on the regime may be optimal.

Noting that $V(\alpha_i, x, n)$ satisfies the conditions of Auxiliary problem (4), thus Theorem 4.2 still holds if we replace $w$ with $V$ in Equation (4.1). Applying WDPP to $V(\alpha_i, x, n)$, for stopping time $\tau_1$, denote by $\tau_1^{f, \gamma} = \tau_1^{f, \gamma} \wedge \tau_1$, then we have
\[
V(\alpha_i, x, n) = \mathcal{M}_w^{f, \gamma}(\alpha_i, x, n) \text{ for } x \in \mathbb{R} \text{ and } \alpha_i \in \mathcal{E}. \tag{4.10}
\]

**Definition 4.3.** Suppose that $(f^*(\alpha_i, x, n), \gamma^*(\alpha_i, x, n))$, which is specified by Equations (4.7-4.8), are the optimal control policies to Auxiliary control problem 4.1 with replacing $w(\cdot, \cdot, \cdot)$ by $V(\cdot, \cdot, \cdot)$. Define control process $\{(f_t^*, \gamma_t^*), t \geq 0\}$ as
\[
f_t^* = f^*(\xi_n, X_t^{f^*}, N_t^{f^*}), \quad \tau_n \leq t < \tau_{n+1}, \quad n = 0, 1, 2, \ldots, \tag{4.11}
\]
\[
\gamma_t^* = \gamma^*(\xi_n, X_t^{f^*}, N_t^{f^*}), \quad \tau_n \leq t < \tau_{n+1}, \quad n = 0, 1, 2, \ldots, \tag{4.12}
\]
where the functions $f^*$ and $\gamma^*$ are defined by Equations (4.7-4.8), respectively.

**Remark 3.** The control process in Definition 4.3 is determined by the HJB equation in Auxiliary control problem, which is a single regime switching control problem. The coefficients of the controlled system are changeable w.r.t. the change of condition. In the rest of this paper, we will prove that this control process is really optimal.

For later discussion convenience, we introduce the following shift operator of Markov process, for detailed introduction on this concept, readers are referred to Wentzell et al. [56]. Let
\[
(\xi, X_t^{f, \gamma}, N_t^{f, \gamma}) : (\mathcal{E} \times \mathbb{R} \times (0, \infty))^2 \mapsto (\mathcal{E} \times \mathbb{R} \times (0, \infty))^2 \tag{4.13}
\]
be the controlled canonical state process. Define the shift operators \( \theta_t : (E \times \mathbb{R} \times (0, \infty))^R \rightarrow (E \times \mathbb{R} \times (0, \infty))^R \) for \( t > 0 \) by \( (\theta_t \omega)s = \omega_{s+t} \), \( s, t \in \mathbb{R}^+ \), \( \omega \in (E \times \mathbb{R} \times (0, \infty))^R \). It is clear that \( \theta_t \in \mathcal{K} \) and \( \theta_t(\xi_t, X_t^{f, \gamma}, N_t, \gamma) = (\xi_{t+s}, X_{t+s}^{f, \gamma}, N_{t+s}^{f, \gamma}) \).

Let \( \tau_0 = 0 \), then we have

\[
\tau_{n+1} = \tau_n + \theta_{\tau_n} \tau_1, \quad n = 0, 1, 2, \ldots \quad (4.14)
\]

**Theorem 4.4.** The regime dependent optimal policy \( \{(f^*_t, \gamma^*_t), \ t \geq 0\} \) specified by Definition 4.3 is the optimal control policy.

**Proof.** It follows by setting \( w = V \) in Auxiliary control problem and using (4.2) that

\[
V(\xi_{\tau_n}, X_{\tau_n}^{f^*_n, \gamma^*_n}, N_{\tau_n}^{f^*_n, \gamma^*_n}) = \mathbb{E}V(\xi_{\tau_n}, X_{\tau_n}^{f^*_n, \gamma^*_n}, N_{\tau_n}^{f^*_n, \gamma^*_n})
\]

\[
= \mathbb{E} e^{-\beta \tau_n} V(\xi_{\tau_n}, X_{\tau_n}^{f^*_n, \gamma^*_n}, N_{\tau_n}^{f^*_n, \gamma^*_n}), \quad (4.15)
\]

From the structure of \( \{(f^*_t, \gamma^*_t), \ t \geq 0\} \) we can see that, given the initial state \( \xi_0 = \alpha_t \), the optimal strategy is \( f^*(\alpha_t, x, n), \gamma^*(\alpha_t, x, n) \) before time \( \tau_t \). Hence, if the current state of \( \xi_t \) is \( \xi_{\tau_n} \), the optimal strategy is \( f^*(\xi_{\tau_n}, x, n), \gamma^*(\xi_{\tau_n}, x, n) \) before the next jump time of \( \xi_t \). By noting that the operator \( \mathbb{E}_V^{f^*_n, \gamma^*_n} \) is defined by the path of \( (\xi, X_t^{f^*_t, \gamma^*_t}, N_t^{f^*_t, \gamma^*_t}) \), \( t \geq 0 \) up to the first transition time and using (4.15), we conclude that

\[
V(\alpha_t, x, n) = \mathbb{E}(\alpha_t, x, n) \left[ \int_0^{\tau_{n+1} \wedge \tau_n} e^{-\beta s} u(\gamma^*_s) \, ds + e^{-\beta \tau_n} V(\xi_{\tau_n}, X_{\tau_n}^{f^*_n, \gamma^*_n}, N_{\tau_n}^{f^*_n, \gamma^*_n}) \right], \quad (4.16)
\]

for \( k = 1, 2, 3, \ldots \).

Motivated by Jiang and Pistorius [35], we want to prove Equation (4.16) by mathematical induction method. It is obviously true for \( k = 1 \) (see Equation (4.15)). Suppose that Equation (4.16) holds for \( k = n \). Then

\[
V(\alpha_t, x, n) = \mathbb{E}(\alpha_t, x, n) \left[ \int_0^{\tau_{n+1} \wedge \tau_n} e^{-\beta s} u(\gamma^*_s) \, ds + e^{-\beta \tau_n} V(\xi_{\tau_n}, X_{\tau_n}^{f^*_n, \gamma^*_n}, N_{\tau_n}^{f^*_n, \gamma^*_n}) \right]
\]

\[
= \mathbb{E}(\alpha_t, x, n) \left[ \int_0^{\tau_{n+1} \wedge \tau_n} e^{-\beta s} u(\gamma^*_s) \, ds + e^{-\beta \tau_n} V(\xi_{\tau_n}, X_{\tau_n}^{f^*_n, \gamma^*_n}, N_{\tau_n}^{f^*_n, \gamma^*_n}) \right]
\]

\[
+ \mathbb{E}(\alpha_t, x, n) \left[ e^{-\beta \tau_n} V(\xi_{\tau_n}, X_{\tau_n}^{f^*_n, \gamma^*_n}, N_{\tau_n}^{f^*_n, \gamma^*_n}) 1_{(\tau_n < \tau_{n+1})} \right] \quad (4.17)
\]

\[
V(\xi_0, X_0^{f^*_0, \gamma^*_0}, N_0^{f^*_0, \gamma^*_0}) = \theta_{\tau_n} V(\xi_{\tau_n}, X_{\tau_n}^{f^*_n, \gamma^*_n}, N_{\tau_n}^{f^*_n, \gamma^*_n}) \]

by the induction hypothesis, we have that

\[
V(\xi_0, X_0^{f^*_0, \gamma^*_0}, N_0^{f^*_0, \gamma^*_0}) = \mathbb{E}(\alpha_t, x, n) \left[ \int_0^{\tau_{n+1}} e^{-\beta s} u(\gamma^*_s) \, ds + e^{-\beta \tau_n} V(\xi_{\tau_n}, X_{\tau_n}^{f^*_n, \gamma^*_n}, N_{\tau_n}^{f^*_n, \gamma^*_n}) \right] + \mathbb{E}(\alpha_t, x, n) \left[ e^{-\beta \tau_n} 1_{(\tau_n < \tau_{n+1})} V(\xi_{\tau_n}, X_{\tau_n}^{f^*_n, \gamma^*_n}, N_{\tau_n}^{f^*_n, \gamma^*_n}) \right].
\]
Thus
\[
V(\xi_{\tau_n}, X_{\tau_n}^{f^*, \gamma^*}, N_{\tau_n}^{f^*, \gamma^*}) = \theta_{\tau_n} V(\xi_0, X_0^{f^*, \gamma^*}, N_0^{f^*, \gamma^*})
\]
\[
= \mathbb{E}_{(\alpha_i, x, n)} \left[ \int_{\tau_n}^{\tau_{n+1}} e^{-\beta_s} u(\gamma^*_s(\omega_s)) ds \bigg| \mathcal{F}_{\tau_n} \right] \tag{4.19}
\]
\[+ \mathbb{E}_{(\alpha_i, x, n)} \left[ e^{-\beta_{\tau_n+1}} 1_{(\tau_{n+1} < \tau^*, \gamma^*)} V(\xi_{\tau_{n+1}}, X_{\tau_{n+1}}^{f^*, \gamma^*}, N_{\tau_{n+1}}^{f^*, \gamma^*}) \bigg| \mathcal{F}_{\tau_n} \right]. \tag{4.20}
\]

Substituting Equations (4.19–4.20) into Equations (4.17–4.18), we have
\[
V(\alpha_i, x, n) = \mathbb{E}_{(\alpha_i, x, n)} \left[ \left( \int_0^{\tau^*, \gamma^*} e^{-\beta_s} u(\gamma^*_s) ds \right) 1_{(\tau_n < \tau^*, \gamma^* < \tau_{n+1})} \right.
\]
\[+ \left( \int_0^{\tau_{n+1}} e^{-\beta_s} u(\gamma^*_s) ds \right) 1_{(\tau_{n+1} < \tau^*, \gamma^*)} \right]
\]
\[+ \mathbb{E}_{(\alpha_i, x, n)} \left[ e^{-\beta_{\tau_n+1}} V(\xi_{\tau_{n+1}}, X_{\tau_{n+1}}^{f^*, \gamma^*}, N_{\tau_{n+1}}^{f^*, \gamma^*}) 1_{(\tau_{n+1} < \tau^*, \gamma^*)} \right]
\]
\[= \mathbb{E}_{(\alpha_i, x, n)} \left[ \int_0^{\tau^*, \gamma^*} e^{-\beta_s} u(\gamma^*_s) ds \right.
\]
\[+ \left. e^{-\beta_{\tau_n+1}} 1_{(\tau_{n+1} < \tau^*, \gamma^*)} V(\xi_{\tau_{n+1}}, X_{\tau_{n+1}}^{f^*, \gamma^*}, N_{\tau_{n+1}}^{f^*, \gamma^*}) \right]. \tag{4.21}
\]

Since we have proved that \( V(\alpha_i, x, n) \) is bounded for any \( x > 0, n > x \) and \( \alpha_i \in \mathcal{C} \) and note that \( \lim_{n \to \infty} \tau_n = \infty \), letting \( n \to \infty \) in above equation, we have
\[
\lim_{n \to \infty} e^{-\beta_{\tau_n+1}} 1_{(\tau_{n+1} < \tau^*, \gamma^*)} V(\xi_{\tau_{n+1}}, X_{\tau_{n+1}}^{f^*, \gamma^*}, N_{\tau_{n+1}}^{f^*, \gamma^*}) = 0 \tag{4.22}
\]
and thus
\[
V(\alpha_i, x, n) = \mathbb{E}_{(\alpha_i, x, n)} \left[ \int_0^{\tau^*, \gamma^*} u(\gamma^*_s) ds \right]. \tag{4.23}
\]

This indicates that under the policy \((f^*, \gamma^*)\), the performance function is really the value function and the proof is completed. \( \Box \)

**Remark 4.** In fact, we known that optimal control policy \(\{(f^*_t, \gamma^*_t), \ t \geq 0\}\) specified by Equations (4.7–4.8) is obtained under the system that regime does not change. Theorem 4.4 asserts that, as long as the current state of the Markov chain is given, it is optimal for an insurer in a multiple-regime market to adopt the policies in the same way as in a single-regime market. Thus, if one wants to find the optimal control policy by numerical method, once the present state of Markov chain is known, it is necessary to find the numerical optimal policy in a single regime system. Jang and Kim [34] studied optimal investment and reinsurance problem under a regime switching model by numerical method. The numerical results therein show that an insurer with poor financial prudence might be myopic so that the optimal strategies are similar to those of the single-regime model. This result coincides with Theorem 4.4. However, Jang and Kim [34] also found that optimal insurance companies that
Theorem 4.5. For any given \( \alpha_i \in \mathcal{E} \), \( V(\alpha_i, x, n) \in C^{2,1}(\mathcal{G}) \), thus control problem (2.15) under model (2.10) admits a classical solution.

**Proof.** Let \( V_0(\alpha_i, x, n) = \mathcal{M}_w(\alpha_i, x, n) \), where \( \mathcal{M}_w(\alpha_i, x, n) \) is the value function to the Auxiliary problem. By Theorem 4.2 we know that \( V_0(\alpha_i, x, n) \in C^{2,1}(\mathcal{G}) \) for all \( \alpha_i \in \mathcal{E} \). Define the following iterative procedure

\[
V_k(\alpha_i, x, n) = \mathfrak{F}^{f^*, \gamma^*}_{V_{k-1}}(\alpha_i, x, n)
\]

where the Process \( (f^*, \gamma^*) \) is specified by Equations (4.11-4.12). We observe that the iterative function procedure \( V_k(\alpha_i, x, n), k = 0, 1, 2, \ldots \) is bounded on \( \mathcal{E} \times \mathcal{G} \) and Theorem 4.4 shows that \( V(\alpha_i, x, n) = \lim_{k \to \infty} \mathfrak{F}^{f^*, \gamma^*}_{V_{k-1}}(\alpha_i, x, n) \). Note that \( V_0(\alpha_i, x, n) \in C^{2,1}(\mathcal{G}) \), by inductive method, we have \( V_k(\alpha_i, x, n) \in C^{2,1}(\mathcal{G}) \) for all \( k \geq 1 \). This indicates that \( V(\alpha_i, x, n) \) is \( C^{2,1}(\mathcal{G}_M) \), where \( \mathcal{G}_M \) is the arbitrary compact set of \( \mathcal{G} \) defined in Equation (5.2) below. By Stone-Weierstrass Theorem (c.f. Brinkhuis and Tikhomirov [11]), such property can be extended to \( \mathcal{G} \). 

The following verification theorem is standard result for stochastic control with smooth value functions, which states that the a classical coupled solutions to HJB equations is really the optimal control. The proof can be finished by an direct application of the method in Fleming and Pang [23], we omit it here.

**Theorem 4.6. (Verification Theorem) Suppose that there exists a function \( \omega : \mathcal{E} \times \mathcal{G} \to \mathbb{R}^+ \) such that**

1. \( \omega(\alpha_i, \cdot, \cdot) \in C^{2,1}(\mathcal{G}) \) for each \( \alpha_i \in \mathcal{E} \),
2. \( \omega \) satisfies the polynomial growth condition: that is for some positive constants \( p \) and \( K \), we have

\[
|\omega(\alpha_i, x, n)| \leq K(1 + x^p + n^p) \tag{4.25}
\]

for any \( (x, n) \in \mathcal{G} \) and \( \alpha_i \in \mathcal{E} \),
3. \( \omega \) satisfies the coupled HJB equation

\[
\begin{align*}
\sup_{f, \gamma \in \mathcal{A}} \left\{ -\beta v + u(\gamma) + \mathcal{L}_f \omega(\alpha_i, x, n) \right\} &= 0 \quad \text{for } \alpha_i \in \mathcal{E}; 0 < x < n;
-\lambda \frac{\partial}{\partial x} v + \frac{\partial}{\partial n} \omega \leq 0, \quad \text{for } 0 < x = n;
\omega(\alpha, x, n) &= 0, \quad \text{for } x < 0.
\end{align*}
\tag{4.26}
\]

Then \( \omega(\alpha_i, x, n) \geq J^{f^*, \gamma^*}(\alpha_i, x, n) \) for any initial condition \( (\alpha_i, x, n) \in \mathcal{E} \times \mathcal{G} \) and any admissible feedback control \( f^*, \gamma^* \) is an admissible
feedback control such that
\[
\begin{cases}
-\beta \omega + u(\gamma^*) + \mathcal{L}^{f^*} \cdot \omega(\alpha_i, x, n) = 0 & \text{for } \alpha_i \in \mathcal{E}; 0 < x < n; \\
-\lambda \frac{\omega}{n^2} + \frac{\partial \omega}{\partial n} = 0, & \text{for } 0 < x = n; \\
\omega(\alpha_i, x, n) = 0, & \text{for all } x < 0,
\end{cases}
\]
then \(\omega(\alpha_i, x, n) = V(\alpha_i, x, n), \alpha_i \in \mathcal{E}\) and \((f^*, \gamma^*)\) is optimal control.

5. Numerical schemes. Although we have provided the uniqueness and regularity of the coupled viscosity solutions to HJB equations (2.17–2.19), there are no explicit solutions available. It is practical relevant to construct numerical scheme for the value function and optimal policies. The method here mostly relies on finite-difference method and weak convergence theory of Markov chains. A comprehensive introduction to this method can be found in Kushner and Dupuis [39]. The main method of our approximating method is to divide the domain \(\mathcal{G}\) into two parts: the interior of \(\mathcal{G}\) and the boundary set of \(\mathcal{G}\). In the interior of \(\mathcal{G}\), our approximating method is very close to Song et al. [51]. When at the boundary of \(\mathcal{G}\), our method highly relies on the Chapter 11 of Kushner and Dupuis [39]. Recall that the domain of the solution to the associated HJB equations is
\[
\mathcal{G} = \{(x, n) : 0 \leq x \leq n\},
\]
which is not a compact domain in \(\mathbb{R}^2\). To proceed discussion, we should confine the problem to the exit a bounded domain. For any chosen constant \(M\) (which is large enough), define
\[
\mathcal{G}_M \doteq \{(x, n) : 0 \leq x \leq n \leq M, \tau_M \doteq \min\{\tau, M\}\}.
\]
\(\tau_M\) is the first exit time of process \(X_t^{(f, \gamma)}\) from domain \(\mathcal{G}_M\). The construction of the Markov chain approximation scheme mainly relies on modifications on the boundary of \(\partial \mathcal{G}_M \doteq \{(x, n) : 0 \leq x = n \leq M\}\). Denote the stopping domain \(\partial \mathcal{G}_{M2} \doteq \{(x, n) : x \leq 0\}\), once approximating Markov chain run into \(\mathcal{G}_{M2}\), the procedure should be stopped since the ruin has already occurred. Denote by \(Dom(\mathcal{G}) = \{(x, n) : 0 < x < n\}\).

5.1. Approximating Markov Chain in \(Dom(\mathcal{G})\). In this subsection, we construct a discrete-time, finite-state, controlled Markov chain to approximate the controlled diffusion processes with regime switching. The approximating Markov chain is locally consistent with controlled state system (2.10), so that the weak limit of the Markov chain satisfies (2.10). The method in this subsection is closely related to the one in Song et al. [51]. We only need to check the constructed approximating schedule satisfies the required properties and then the convergence results can be guaranteed by [51]. Let \(h > 0\) be a discretization parameter. The approximating Markov chain is locally consistent with (1), so that the weak limit of the Markov chain satisfies (1). Let \(h > 0\) be the discretization step size. Define
\[
\mathcal{G}_M^h = \{\{i, j\} : x = ih, n = jh, 0 \leq i \leq j \leq \left\lfloor \frac{M}{h} \right\rfloor - 1\},
\]
where \(\lfloor \cdot \rfloor\) is the rounding-down function. One should note the construction of Markov chain is confined in the domain of \(\mathcal{G}_M\), and we should reconsider our approximations at the boundary of \(\mathcal{G}_M\). Let \(\{(h^n_i, y^n_h, \xi^n_h)\}\) be a controlled discrete-time Markov chain on a discrete state space \(\mathcal{G}_M \times \mathcal{E}\) with transition probabilities from a state \((i_1h, j_1h, i) \in \mathcal{G}_M \times \mathcal{E}\) to another state \((i_2h, j_2h, l) \in \mathcal{G}_M \times \mathcal{E}\), denoted by \(P^h((i_1h, j_1h, i), (i_2h, j_2h, l)|u^n_h)\) for a control policy \(u^n_h \doteq (f^n_h, \gamma^n_h) \in \mathcal{A}^h\). In order to
approximate the continuous-time controlled state process \((X_t^{f,\gamma}, N_t^{f,\gamma})\), we need to use an appropriate continuous-time interpolation (see [39]). Suppose that we have an interpolation interval \(\Delta t_n^h = \Delta t_n^h (\zeta_n^h, \eta_n^h, \xi_n^h, u_n^h) > 0, n \geq 1\) on \(G_M \times \mathbb{E} \times A^h\). Define the interpolated time \(t_n^h = \sum_{k=1}^{n-1} \Delta t_k^h\). Hence, the piecewise constant interpolations, denoted by \((\zeta_n^h, \eta_n^h, \xi_n^h, u_n^h)\), are naturally defined as

\[
\zeta_n^h = \zeta_n^h, \eta_n^h = \eta_n^h, \xi_n^h = \xi_n^h, u_n^h = u_n^h \quad \text{for} \quad t \in [\Delta t_n^h, \Delta t_{n+1}^h). \tag{5.4}
\]

**Remark 5.** We need to verify that the approximating Markov chain discussed satisfies local consistency. Let \(P_h ((i_1 h, j_1 h, i), (i_2 h, j_2 h, l)| u_n^h)\) for \((i_1 h, j_1 h, i), (i_2 h, j_2 h, l)\), \(u_n^h\) be a collection of well defined transition probabilities for the Markov Chain \(\zeta_n^h = \zeta_n^h, \eta_n^h = \eta_n^h, \xi_n^h = \xi_n^h\) under the control policy \(u_n^h\), which approximates to \((X_t^{f,\gamma}, N_t^{f,\gamma})\) and certain control trajectory \(u_t\). Define the difference

\[
\Delta (\zeta_n^h, \eta_n^h, \xi_n^h, u_n^h) = (\zeta_n^h + \zeta_n^h, \eta_n^h + \eta_n^h, \xi_n^h + \xi_n^h) - (\zeta_n^h, \eta_n^h, \xi_n^h).
\]

Assume that

\[
\inf_{i,j,u} \Delta t_n^h(i, j, u) > 0 \quad \text{and} \quad \lim_{h \to 0} \sup_{i,j,u} \Delta t_n^h(i, j, u) = 0. \tag{5.6}
\]

Let \(E_n^{u,h}|_{i,j,\alpha}, V_n^{u,h}|_{i,j,\alpha,n}, \text{ and } P_n^{u,h}|_{i,j,\alpha,n}\) denote the conditional expectation, variance and marginal probability given \(\{s_n^h = x = i h, \eta_n^h = n = j h, \xi_n^h = \alpha, u_n^h = (j_n^h, \gamma_k^h)\}\) respectively. Then, \(\epsilon_n = o(\Delta t^h(x, n, \alpha))\), the sequence \((\zeta_n^h, \eta_n^h, \xi_n^h)\) satisfies

\[
E_n^{u,h}|_{i,j,\alpha,n} \Delta (\zeta_n^h, \eta_n^h, \xi_n^h, u_n^h) = (\mu^h_n + c_i - \gamma_n^h, 0, 0) \Delta t_n^h + \epsilon_n, \tag{5.7}
\]

\[
V_n^{u,h}|_{i,j,\alpha,n} \Delta (\zeta_n^h, \eta_n^h, \xi_n^h, u_n^h) = \text{diag} \left( \frac{1}{2} (\sigma_{11}^h + \sigma_{12}^h), 0, 0 \right) \Delta t_n^h + \epsilon_n, \tag{5.8}
\]

\[
P_n^{u,h}|_{i,j,\alpha,n} \{\zeta_n^h = \alpha_i\} = \Delta t^h(x, n, \alpha_k) \delta_{l, \alpha} + \epsilon_n \quad \text{for} \quad l \neq \alpha, \tag{5.9}
\]

\[
P_n^{u,h}|_{i,j,\alpha,n} \{\zeta_n^h = \alpha_i\} = \Delta t^h(x, n, \alpha_k) (1 + q_{\alpha_i}) + \epsilon_n, \tag{5.10}
\]

\[
\sup_{i,j,\alpha,n} \Delta (\zeta_n^h, \eta_n^h, \xi_n^h) \to 0 \quad \text{as} \quad h \to 0. \tag{5.11}
\]

One may check that such approximating Markov chain satisfies the condition of local consistent in [51].

Suppose we have the approximating Markov chain discussed above, then we can obtain approximation of the cost function. Let \(\tau_M^h\) denote the first time that the controlled approximating Markov Chain \(\{(\zeta_n^h, \eta_n^h), n \geq 1\}\) leaves the meshed domain of \(G_M^h\). Then the approximating performance function and value function are

\[
W^h(x, n, \alpha_i, u_n^h) = E_{x,n,\alpha_i}^u \left[ \sum_{n=0}^{\tau_M^h} e^{-\beta t_n^h u(\gamma_n^h)} \Delta t_n^h \right] = E_{x,n,\alpha_i}^u \left[ \int_0^{\tau_M^h} e^{-\beta t_s^h u(\gamma_s^h)} ds \right], \tag{5.12}
\]

and

\[
V^h(x, n, \alpha_i) = \sup_{u_n^h} W^h(x, n, \alpha_i, u_n^h) \tag{5.13}
\]

respectively. Practically, we can compute \(V^h(x, n, \alpha_i)\) by solving the corresponding dynamic programming equation using iteration method. That is, for \((x, n, \alpha_i) \in \)
with boundary conditions \( V^h(0, n, \alpha_i) = 0 \). One may find that Assumptions (A1)-(A5) in [51] are naturally satisfied in our formulation, thus Theorem 7 of [51] holds, we have the following convergence results on the approximating Markov chain.

**Proposition 2.** Assume that conditions (1)-(3) hold, then

\[
V^h(x, n, \alpha_i) \to V(x, n, \iota) \quad \text{for all } (x, n, \iota) \in \mathcal{G}_M \times \mathcal{E}, \quad h \to 0.
\]

Here we approximate them by finite-difference method, discrete \( V_x, V_{xx}, V_n \) as

\[
V_x \to \frac{1}{2h} [V^h(x + h, n, \alpha_i) - V^h(x - h, n, \alpha_i)], \quad (5.16)
\]

\[
V_n \to \frac{1}{2h} [V^h(x, n + h, \alpha_i) - V^h(x, n - h, \alpha_i)], \quad (5.17)
\]

\[
V_{xx} \to \frac{1}{h^2} [V^h(x + h, n, \alpha_i) + V^h(x - h, n, \alpha_i) - 2V^h(x, n, \alpha_i)].
\]

The aforementioned results enable us to obtain the partial derivatives numerically, together with Equation (4.7) and Equation (4.8), we can compute the optimal investment amount and optimal dividend numerically when \( (x, n) \in \mathcal{G}_M \).

5.2. Locally consistent approximation on the boundary \( \partial \mathcal{G}_{M1} \). We have completed the construction of approximating Markov chain in the interior of \( \mathcal{G}_M \), which is defined on the mesh \( G^h_M \). The aim of the reflection or constraint is to keep the process in the set \( G \), if it ever attempts to leave it. We will construct the “reflecting boundary” for the approximating chain. It is not difficult in programming. One should note that \( \mathcal{G}_{M1} \) is a line contained in the first quadrant in \( \mathbb{R}^2 \), we should extend the boundary to its “neighborhood” with radius \( h \). Without loss of generality, we set \( \partial \mathcal{G}^h_M = \{(x - h, x - h, \alpha_i), \ h > 0 \text{ and } h \text{ small enough}\} \supset \mathcal{G}_{M1} \).

The transition probabilities at the states in \( \partial \mathcal{G}^h_M \) are chosen so as to calibrate the behavior of the reflection for (2.8) with direction \( \gamma \). Thus, we may define \( \mathbb{P}(x, n, \alpha_i) \) as follows. Define the approximating Markov chain as \( (\zeta^{h/k}, \eta^{h/k}, \xi^{h/k}) \), where \( h \) is the radius of \( \partial \mathcal{G}^h_M \) and \( k \) is the mesh size of \( \partial \mathcal{G}^h_M \). For any selected \( (x, n, \alpha_i) \in \partial \mathcal{G}^h_M \).

The transition probability is

\[
\mathbb{P}((i, j, \alpha_i) \to (i, j + 1, \alpha_i)) = 0, \mathbb{P}((i, j, \alpha_i) \to (i, j - 1, \alpha_i)) = 0,
\]

\[
\mathbb{P}((i, j, \alpha_i) \to (i + 1, j, \alpha_i)) = \lambda \Delta_i^h, \quad \mathbb{P}((i, j, \alpha_i) \to (i - 1, j + 1, \alpha_i)) = (1 - \lambda \Delta_i^h).
\]

The probability comes from the fact the increasing of \( N_{t, \gamma}^f \) relies on the increasing of \( X_t^f \). Linear boundary condition (2.18) implies Equations (5.18-5.19).

5.3. Numerical examples. In this subsection, by the numerical methods presented in previous subsection, we want to illustrate: (1) the impact of tax and premium income on insurer’s optimal decisions; (2) the impact of high gain tax and regime state on the value function and control policies.

**Example 1.** In this example, we illustrate the impact of tax and the premium income on optimal decisions. Merton [41] studied the optimal consumption and investment problem of an financial agent without any high gain tax, Janecek and
Sirbu [33] reconsidered this problem when the financial agent is subject to the high-watermark fee, we focus this problem on the assumption that the decision maker is an insurer. Table 1 contains all parameters involved in preceding mentioned three cases, where we the high-watermark $n = 10$ and the current wealth $x$ varies from 1 to 10. The decisions of the agent (or insurer) are affected by ratio of $x$ to $n$. Figure 1 illustrates the optimal dividend/investment ratio and Figure 2 illustrates the optimal investment amount, under three different cases respectively.

Merton [41] showed that the optimal investment and consumption are constants proportion to current wealth, determined by the coefficients of financial market. Thus the optimal amount of investment or consumption is a linear function w.r.t. current wealth of agent. If the financial agent is subject to high gain tax, Figure 1 shows that when $x$ approaches to $n$, the optimal investment amount of the financial agent is less than that in [41]. However, the optimal dividend amount of insurer is greater than that in Merton’s model. This is because that the insurer has a positive increment on wealth with rate $c(ξ_t)$ when $x$ approaches to $n$. It will subject to tax once the current wealth goes beyond $n$, thus the rational decision should be consume more to avoid tax. In Figure 2, one may find there is no apparently difference between the investment amount for the three decision makers when $x$ is far less than $n$. When $x$ is close to $n$, the investment of the insurer is larger than that in Merton’s model for avoiding tax.

<table>
<thead>
<tr>
<th>(Investor,Parameter)</th>
<th>$μ_1$</th>
<th>$σ_1$</th>
<th>$μ_2$</th>
<th>$σ_2$</th>
<th>$p$</th>
<th>$β$</th>
<th>$λ$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Merton</td>
<td>0</td>
<td>0</td>
<td>0.06</td>
<td>0.3</td>
<td>0.33</td>
<td>0.15</td>
<td>0</td>
<td>–</td>
</tr>
<tr>
<td>Financial Agent</td>
<td>0</td>
<td>0</td>
<td>0.06</td>
<td>0.3</td>
<td>0.33</td>
<td>0.15</td>
<td>0.2</td>
<td>10</td>
</tr>
<tr>
<td>Insurer</td>
<td>0.4</td>
<td>0.5</td>
<td>0.06</td>
<td>0.3</td>
<td>0.33</td>
<td>0.15</td>
<td>0.2</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 1: Parameters setting in Example 1

Example 2. In this example we want to understand the impact of switching state on the optimal policies. We suppose that there are only two states of the external condition: the bull state $α_1$ and the bear state $α_2$, the $Q$ matrix of the Markov
The following Table 2 lists corresponding parameters w.r.t. different regime state. In the bull market, the market price of the risky asset is \((\mu_2 - 0)/\sigma_2 = 0.2\) and the corresponding one in bear market is 0.1. The "market price" of insurer under the bull market and bear market are 0.8 and 0.6, respectively. One may observe that, the market price of both insurer and the risky asset has advantage under the bear market, such parameter selection may not be realistic, however, it is easy for our numerical analysis. Numerical results (see Figure 3 and Figure 4) show that under the bull market, the decision makers are apt to invest and distribute dividend more than the ones in the bear market. It is noteworthy that in the bear market, the financial agent even take more conservative consumption (i.e. less consumption amount) than the one in Merton’s case. This indicates that, due to the possibility of being taxed, the financial agent is apt to choose less consumption. However, in this example, the insurer has premium income with relative lower “market price”, the insurer is apt to consume more than ordinary in both bear and bull conditions.

<table>
<thead>
<tr>
<th>(State, Parameter)</th>
<th>( \mu_1 )</th>
<th>( \sigma_1 )</th>
<th>( \mu_2 )</th>
<th>( \sigma_2 )</th>
<th>( p )</th>
<th>( \beta )</th>
<th>( \lambda )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bull</td>
<td>0.4</td>
<td>0.5</td>
<td>0.06</td>
<td>0.3</td>
<td>0.3</td>
<td>0.05</td>
<td>0.2</td>
<td>10</td>
</tr>
<tr>
<td>Bear</td>
<td>0.3</td>
<td>0.5</td>
<td>0.03</td>
<td>0.3</td>
<td>0.3</td>
<td>0.05</td>
<td>0.2</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 2: Parameters setting in Example 2
characterization result to check optimality even when the optimal value function is not differentiable. The Cramér–Lundberg model with investment chance is a type of jump diffusion model, which results in an integro operator in the associated HJB equation. It is very difficult to prove that the regularity of the viscosity solution. In this study, we prove the regularity of the viscosity solutions in a diffusion framework with regime switching, but we cannot provide explicit expressions for optimal strategies. Zheng [63] studied optimal investment and consumption with high gain tax under a multidimensional jump diffusion framework and proved that the value function is the viscosity solution to the associated HJB equation. The goal of the decision maker in Zheng [63] is to maximize the cumulated discounted expected dividend from utility. By Perron’s method and the “form reduction” skill, Zheng [63] proved the smoothness of the value function. However, the “form reduction” skill is not suitable for the problem in this paper, as the insurance business products are an extra term in the associated HJB equation. Thus, this study uses a different method.

In terms of the mathematical framework, the control problem in this study is similar to Song and Zhu [50] to some extent. Song and Zhu [50] considered a class of singular control problems with state constraints and regime switching. The controlled dynamic is given by a regime-switching diffusion confined in the unbounded domain $S = \mathbb{R}_+^n$ and the objective is to maximize the total expected discounted
rewards from exerting singular control. The rewards can be regarded as dividend in the insurance business. Using the WDPP and an appropriate exponential transformation, they showed that the value function is the unique constrained viscosity solution of the system of coupled HJB equations with gradient constraints. The framework of [50] covers a wide range of control problems that arise in finance and other fields. The framework of this paper differs from theirs on the extra stochastic income from the insurance business. Another difference between our study and [50] is in the gradient constraints. In [50], the gradient constraints is on \( S = \mathbb{R}^n \), but it is on the boundary \( \{ x, n | x = n; x > 0, n > 0 \} \) in this paper. As the framework in this study is more specific, we can prove the regularity of the viscosity solution.

Jin and Yin [36] developed numerical approximation schemes for finding the optimal investment and dividend payment policy to maximize the total discounted dividend paid out until the lifetime of ruin. Jin et al. [37] developed numerical approximation schemes to find optimal dividend and capital injections in a Markov modulated market. The controlling terms in this study are the same as in [36] and the controlled systems are all with regime switching. Our work differs from [37] on the high tax payment and we prove the regularity of the viscosity using the fixed point method. The numerical method applied in this study is also the Markov chain approximation for the stochastic control problem, which is as same as [37, 36, 50]. In fact, when designing the approximation procedure in our numerical examples, our primary inspirations were the above papers. Besides the Markov chain approximation method, Barth et al. [7] provide a numerical method by a finite element approximation and a time-marching scheme. As we are not familiar with this method, we do not adopt it in this study.

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Appendix A: Proof of Lemma 3.2

Proof. We firstly prove that \( V(\alpha_i, x, n) \) is bounded. Revisit the definition of \( V(\alpha_i, x, n) \), suppose that at time \( t \) the wealth process of insurer is \( X_{t}^{f, \gamma} \), then obviously \( 0 \leq \gamma_t \leq X_{t}^{f, \gamma} \), otherwise ruin occur immediately, which can not be the optimal policy for insurer. Thus, one can see that

\[
V(\alpha_i, x, n) \leq E\left[ \int_{0}^{\tau_{f,0}} e^{-\beta t} u(X_{t}^{f,0}) dt \right], \tag{A.1}
\]

where \( X_{t}^{f,0}, 0 \leq t \leq \tau_{f,0} \) is the wealth process under policy \( f \) and \( \gamma_t \equiv 0 \). So, a policy that maximizes the ruin time \( \tau \) also maximizes \( E\left[ \int_{0}^{\tau} e^{-\beta t} U(X_{t}^{f,0}) dt \right] \). Zhang and Siu [61] proved that a piecewise constant investment policy maximizes this amount. If the insurer adopts the piecewise constant investment policy, if the current state of \( \xi_t = \alpha_i \) is known, then the wealth process of insurer is

\[
X_{t}^{C_{t,0}} = x + c_i t + \sigma_{1_1} W_{t}^{1} + A_{t}^{*}(\mu_{t} t + \sigma_{2_1} W_{t}^{2}), \tau_{t} \leq t \leq \tau_{t+1}, \tag{A.2}
\]
where \( \tau_i, \tau_{i+1} \) denote the two consecutive jump time of \( \xi_t \). For any \( t \in [\tau_i, \tau_{i+1}) \), \( f_t = A_t^i \) is the optimal constant investment policy during time epoch \([\tau_i, \tau_{i+1})\).

Protter [45] (Theorem 67, Page 342) shows that

\[
E \left[ \sup_{0 \leq s \leq t} |X^C_s,0| \right] \leq \Gamma e^{\rho t}(1 + x^p), \tag{A.3}
\]

where \( l > 1, \Gamma \) and \( \rho \) are constants depend on coefficients involved in the wealth process. Secondly, we prove that \( V(\alpha_i, x, n) \) is concave v.r.t. \((x, n) \in \mathcal{G}\). Noting the \( n \) is the boundary of domain \( \mathcal{G} \), and \( V(\alpha_i, x, n) \) is obviously increasing w.r.t. \( n \). We just need to prove the \( V(\alpha_i, x, n) \) is concave w.r.t. \( x \in [0, n) \). For any given \( \alpha_i \) and \( 0 \leq x_1 < x_2 < n \), consider \((f_1, \gamma_1) \in \mathcal{A}(\alpha_i, x_1, n), (f_2, \gamma_2) \in \mathcal{A}(\alpha_i, x_2, n)\), denote by \( X_t^{f_1,\gamma_1}, X_t^{f_2,\gamma_2} \) and \( \tau^{f_1,\gamma_1}, \tau^{f_2,\gamma_2} \) the corresponding wealth processes and ruin times. Then

\[
J^{f_1,\gamma_1}(\alpha_i, x_1, n) = E_{\alpha_i, x_1, n} \left[ \int_0^{\tau^{f_1,\gamma_1}} e^{-\beta s} u(\gamma_1 s) ds \right],
\]

\[
J^{f_2,\gamma_2}(\alpha_i, x_2, n) = E_{\alpha_i, x_2, n} \left[ \int_0^{\tau^{f_2,\gamma_2}} e^{-\beta s} u(\gamma_2 s) ds \right]. \tag{A.5}
\]

For any \( t \in [0, 1], \) consider a wealth process with the dynamic of Equation (2.8) under the control policy \((\tilde{f}, \tilde{\gamma}) = (tf_1 + (1-t)f_2, t\gamma_1 + (1-t)\gamma_2)\), starting with \( \tilde{x} = tx_1 + (1-t)x_2 \). Denote by \( X_t^{\tilde{f},\tilde{\gamma}} \) and \( \tau^{\tilde{f},\tilde{\gamma}} \) the corresponding wealth process and ruin time respectively. Noting that Equation (2.8) is a controlled linear SDE, thus we have

\[
X_t^{\tilde{f},\tilde{\gamma}} = tX_t^{f_1,\gamma_1} + (1-t)X_t^{f_2,\gamma_2}. \tag{A.6}
\]

Note the fact that

\[
\inf_{0 \leq s \leq t} (tX_s^{f_1,\gamma_1} + (1-t)X_s^{f_2,\gamma_2}) \geq t \inf_{0 \leq s \leq t} X_s^{f_1,\gamma_1} + (1-t) \inf_{0 \leq s \leq t} X_s^{f_2,\gamma_2}, \tag{A.7}
\]

we have

\[
\left\{ \tau^{\tilde{f},\tilde{\gamma}} \leq t \right\} \subseteq \left\{ \inf_{0 \leq s \leq t} X_s^{\tilde{f},\tilde{\gamma}} \leq 0 \right\}
\]

\[
\subseteq \left\{ \inf_{0 \leq s \leq t} X_s^{f_1,\gamma_1} + (1-t) \inf_{0 \leq s \leq t} X_s^{f_2,\gamma_2} \leq 0 \right\}
\]

\[
\subseteq \left\{ \inf_{0 \leq s \leq t} X_s^{f_1,\gamma_1} \leq 0 \right\} \cup \left\{ \inf_{0 \leq s \leq t} X_s^{f_2,\gamma_2} \leq 0 \right\}
\]

\[
= \left\{ \tau^{f_1,\gamma_1} \land \tau^{f_2,\gamma_2} \leq t \right\}. \tag{A.8}
\]

By Theorem 3.2.1 of Rolski et al. [48], we can regard that \( \tau^{\tilde{f},\tilde{\gamma}} \geq \tau^{f_1,\gamma_1} \land \tau^{f_2,\gamma_2} \) a.s.. It is easy to see that \( \tau^{\tilde{f},\tilde{\gamma}} \leq \tau^{f_1,\gamma_1} \lor \tau^{f_2,\gamma_2} \) now we design the construct a pair of
control policy as
\[
(f_t, \gamma_t) = (\tilde{f}_t, \tilde{\gamma}_t)_{0 \leq t \leq \tau_{f_1, \gamma_1} + \tau_{f_2, \gamma_2}} + (f_{2t}, \gamma_{2t})_{\tau_{f_2, \gamma_2} \leq t \leq \tau_{f_2, \gamma_2}}. \tag{A.9}
\]
It is easy to check that this control policy satisfies Assumption 2 and thus is one admissible control. The concavity of \( u(\cdot) \) implies that
\[
u u(\gamma_{1s}) + (1 - \nu)(\gamma_{2s}) \leq u(\nu \gamma_{1s} + (1 - \nu)u(\gamma_{2s})) , \tag{A.10}
\]
thus
\[
\begin{align*}
\iota J_{f_1, \gamma_1}(\alpha_i, x_1, n) + (1 - \iota)J_{f_2, \gamma_2}(\alpha_i, x_2, n) & = E_{\alpha_i, i \mathbf{x}_1, n} \left[ \int_0^{\tau_{f_1, \gamma_1} + \tau_{f_2, \gamma_2}} e^{-\beta s} \nu u(\gamma_{1s}) d\tau \right] \\
& + E_{\alpha_i, (1 - \iota) \mathbf{x}_2, n} \left[ \int_0^{\tau_{f_1, \gamma_1} + \tau_{f_2, \gamma_2}} e^{-\beta s} (1 - \iota)u(\gamma_{2s}) d\tau \right] \\
& + E_{\alpha_i, \mathbf{x}_1, n} \left[ \int_{\tau_{f_1, \gamma_1}}^{\tau_{f_2, \gamma_2}} e^{-\beta s} u(\gamma_{1s})_{\tau_{f_2, \gamma_2} \leq t \leq \tau_{f_1, \gamma_1}} d\tau \right] \\
& + E_{\alpha_i, (1 - \iota) \mathbf{x}_2, n} \left[ \int_{\tau_{f_1, \gamma_1}}^{\tau_{f_2, \gamma_2}} e^{-\beta s} (1 - \iota)u(\gamma_{2s})_{\tau_{f_2, \gamma_2} \leq t \leq \tau_{f_2, \gamma_2}} d\tau \right] \\
& \leq J_{f, \gamma}(\alpha_i, \iota \mathbf{x}_1 + (1 - \iota)\mathbf{x}_2, n) \\
& \leq \sup_{(f, \gamma) \in \mathcal{A}(\alpha_i, \iota \mathbf{x}_1 + (1 - \iota)\mathbf{x}_2, n)} J_{f, \gamma}(\alpha_i, \iota \mathbf{x}_1 + (1 - \iota)\mathbf{x}_2, n) \\
& = V(\alpha_i, \iota \mathbf{x}_1 + (1 - \iota)\mathbf{x}_2, n). \tag{A.11}
\end{align*}
\]
Taking supremum on the left hand side of previous inequality \( f_1, \gamma_1 \) and \( f_2, \gamma_2 \) yields \( V(\alpha_i, x_1, n) + (1 - \iota)V(\alpha_i, x_2, n) \leq V(\alpha_i, x_1 + (1 - \iota)x_2, n) \). This proves that \( V(\alpha_i, x, n) \) is concave \( x \) for all \( 0 \leq x \leq n \). With similar arguments, we can prove that \( V(\alpha_i, x, n) \) is concave \( n \). The continuity of \( V(\alpha_i, x, n) \) is a natural result of its convexity on \( \mathcal{G} \).

Appendix B: Proof of Theorem 3.3

Proof. Let us first prove that \( V \) is a viscosity supersolution. Let \((\bar{x}, \bar{n}) \in \mathcal{G}\) and let \( \varphi(\alpha_i, x, n) \in C^{1,1}(\mathcal{G}) \), \( i \in \mathcal{E} \) be test functions such that
\[
0 = (V_* - \varphi)(\alpha_i, \bar{x}, \bar{n}) = \min_{(x, n) \in \mathcal{G}} (V_* - \varphi)(\alpha_i, x, n). \tag{A.12}
\]
By Lemma 4.1, we know that \( V(\alpha_i, \bar{x}, \bar{n}) = V_*(\alpha_i, \bar{x}, \bar{n}) \), thus there exists a sequence \((x_m, n_m)_{n \geq 1} \subset \mathcal{G}\) such that
\[
(x_m, n_m) \to (\bar{x}, \bar{n}) \text{ and } V(\alpha_i, x_m, n_m) \to V(\alpha_i, \bar{x}, \bar{n}),
\]
when \( m \) goes to infinity. By the continuity of \( \varphi \) and by (A.12) we also have that
\[
\xi_m := V(\alpha_i, x_m, n_m) - \varphi(\alpha_i, x_m, n_m) \to 0, \quad m \to \infty.
\]
Let
\[
(f_m, \gamma_m) = (f(\alpha_i, x_m, n_m), \gamma(\alpha_i, x_m, n_m)) \in \mathcal{A}(\alpha_i, x_m, n_m)
\]
and denote by \((\xi_s, X^m_{l, s}, N^m_{r, s})\) the associated controlled process. Let \(\tau^1_m\) and \(\tau^2_m\) be the stopping times defined by

\[
\tau^1_m = \inf \{0 < s \leq \tau : |X^m_{l, s}(x_m, n_m) - x_m| \geq \eta > 0\}, \quad (A.13)
\]

\[
\tau^2_m = \inf \{0 < s \leq \tau : |N^m_{r, s}(x_m, n_m) - n_m| \geq \eta > 0\}, \quad (A.14)
\]

where \(\eta > 0\) is arbitrary chosen positive constant, \(X^m_{l, s}(x_m, n_m), N^m_{r, s}(x_m, n_m)\) are wealth processes and reflect boundary process starting with \((x_m, n_m)\). One should note that due to the property of diffusion process and the fact that \(\tau^1_m > 0, \tau^2_m > 0\) a.s., \(\tau_m := \tau^1_m \wedge \tau^2_m\). Let \((h_m)\) be a strictly positive sequence such that

\[
h_m \to 0 \quad \text{and} \quad \frac{\zeta_m}{h_m} \to 0,
\]

when \(m\) goes to infinity. We apply WDPP equation (3.3) for \(v(x_m, n_m)\) to \(\vartheta_m := \tau_m \wedge h_m\) and obtain

\[
V(\alpha_i, x_m, n_m) \geq E_{\alpha_i, x_m, n_m} \left[ \int_0^{\vartheta_m} e^{-\beta s} u(\gamma_s) ds + e^{-\beta \vartheta_m} V(\alpha_i, X^m_{\vartheta_m}, N^m_{\vartheta_m}) \right]. \quad (A.15)
\]

Equation (A.12) implies that \(V = V_s \geq \varphi\) for any \(\alpha_i \in \mathcal{C}\). Thus

\[
\varphi(\alpha_i, x_m, n_m) + \zeta_m \geq E_{\alpha_i, x_m, n_m} \left[ \int_0^{\vartheta_m} e^{-\beta s} u(\gamma_s) ds + e^{-\beta \vartheta_m} \varphi(\alpha_i, X^m_{\vartheta_m}, N^m_{\vartheta_m}) \right]. \quad (A.16)
\]

Applying Itô’s formula to \(e^{-\beta s} \varphi(\alpha_i, X^m_{l, s}, N^m_{r, s})\) between 0 and \(\vartheta_m\), we obtain

\[
\frac{\zeta_m}{h_m} + E_{\alpha_i, x_m, n_m} \left[ \frac{1}{h_m} \int_0^{\vartheta_m} (\beta \varphi - u(\gamma_s) - \mathcal{L}^m_{l, s} \varphi)(\alpha_s, X^m_{l, s}, N^m_{r, s}) ds \right] + E_{\alpha_i, x_m, n_m} \left[ \frac{1}{h_m} \int_0^{\vartheta_m} (\lambda \varphi_x - \varphi_n)(\alpha_s, X^m_{l, s}, N^m_{r, s}) dM_s \right] \geq 0. \quad (A.17)
\]

Since the random variable inside the expectation in (A.17) is bounded by a constant independent of \(m\). Letting \(\eta \to 0\) and then letting \(m \to \infty\) yields

\[
\beta \varphi(\alpha_i, \bar{x}, \bar{n}) - u(\gamma) - \mathcal{L}^m_{l, s} \varphi(\alpha_i, \bar{x}, \bar{n}) \geq 0, \quad (A.18)
\]

\[
\lambda \varphi_x(\alpha_i, \bar{x}, \bar{n}) - \varphi_n(\alpha_i, \bar{x}, \bar{n}) \geq 0.
\]

Here we applied Assumption 2, thus \(f(\alpha_i, x_m, n_m) \to f(\alpha_i, \bar{x}, \bar{n}), \gamma(\alpha_i, x_m, n_m) \to \gamma(\alpha_i, \bar{x}, \bar{n})\). Because of the arbitrariness of \((f_m, \gamma_m) \in \mathcal{A}(\alpha_i, x_m, n_m)\), we have (3.1), which claims that the value function is a supersolution to associated HJB equation.

We next to prove that \(V\) is a viscosity subsolution. Let \((\bar{x}, \bar{n}) \in \mathcal{G}\) and let \(\psi(\alpha_i, x, n) \in C^{2,1}(\mathcal{G})\), \(i \in \mathcal{C}\) test functions such that

\[
0 = (V^* - \psi)(\alpha_i, \bar{x}, \bar{n}) = (V - \psi)(\alpha_i, \bar{x}, \bar{n}) = \max_{(x, n) \in \mathcal{G}} (V - \psi)(\alpha_i, x, n). \quad (A.19)
\]

We will show the result by contradiction. Assume on the contrary that

\[
\beta \psi(\alpha_i, \bar{x}, \bar{n}) - \sup_{(f, \gamma) \in \mathcal{A}(x, n)} \{u(\gamma) + \mathcal{L}^m_{l, s} \psi(\alpha_i, \bar{x}, \bar{n})\} > 0,
\]

\[
\lambda \psi_x(\alpha_i, \bar{x}, \bar{n}) - \psi_n(\alpha_i, \bar{x}, \bar{n}) > 0.
\]
There exist $\eta > 0$ and $\varepsilon > 0$ such that
\[
\beta \psi(\alpha_i, x', n') - \sup_{(f, \gamma) \in \mathcal{A}(x,n)} \left\{ u(\gamma) + \mathcal{L}^{f, \gamma} \psi(\alpha_i, x', n') \right\} \geq \varepsilon, \tag{A.20}
\]
\[
\lambda \psi_x(\alpha_i, x', n') - \psi_n(\alpha_i, x', n') \geq \varepsilon,
\]
for all $(x', n') \in B(\bar{x}, \bar{n}, \eta) = \{(x', n') \in (\mathcal{G} : |\bar{x} - x'|^2 + |\bar{n} - n'|^2 < \eta)\}$. By definition of $V(\alpha_i, \bar{x}, \bar{n})$, there exists a sequence $(x_m, n_m)$ taking values in $B(\bar{n}, \eta)$ such that
\[
(x_m, n_m) \to (\bar{x}, \bar{n}) \text{ and } V(\alpha_i, x_m, n_m) \to V(\alpha_i, \bar{x}, \bar{n}), \quad m \to 0.
\]
By continuity of $\psi$ and using (A.19), we also find that
\[
\zeta_m := V(\alpha_i, x_m, n_m) - \psi(\alpha_i, x_m, n_m) \to 0, \quad m \to 0.
\]
Let $(h_m)$ be a strictly positive sequence such that
\[
h_m \to 0 \text{ and } \frac{\zeta_m}{h_m} \to 0.
\]
Then, according to WDPP equation (3.4) and using (A.19), there is an $(f_m, \gamma_m) = (f(\alpha_i, x_m, n_m), \gamma(\alpha_i, x_m, n_m)) \in \mathcal{A}(\alpha_i, x_m, n_m)$ such that
\[
\psi(\alpha_i, x_m, n_m) + \zeta_m - \frac{\varepsilon h_m}{2} \leq \mathbb{E}_{\alpha_i, x_m, n_m} \left[ \int_0^{\vartheta_m} e^{-\beta s} u(\gamma_m) ds + e^{-\beta \vartheta_m} \psi(\alpha_i, X_{\vartheta_m}^{f_m, \gamma_m}, N_{\vartheta_m}^{f_m, \gamma_m}) \right],
\]
where
\[
\vartheta_m = \tau_m' \land h_m, \tau_m' = \tau_m^3 \land \tau_m^4,
\]
\[
\tau_m^3 = \inf \left\{ 0 \leq s \leq \tau : |X_s^{f, \gamma}(x_m, n_m) - x_m| \geq \eta' \right\},
\]
\[
\tau_m^4 = \inf \left\{ 0 \leq s \leq \tau : |N_s^{f, \gamma}(x_m, n_m) - n_m| \geq \eta' \right\},
\]
and $0 < \eta' < \eta$.

Since $(x_m, n_m)$ converges to $(\bar{x}, \bar{n})$, we can always assume that
\[
B(x_m, n_m, \eta') \subset B(\bar{x}, \bar{n}, \eta) \text{ for all } 0 \leq s \leq \vartheta_m \leq \tau.
\]

By Itô’s formula, we have
\[
0 \geq \frac{\zeta_m}{h_m} - \frac{\varepsilon}{2} + \mathbb{E}_{\alpha_i, x_m, n_m} \left[ \frac{1}{h_m} \int_0^{\vartheta_m} \mathcal{L} v(\xi_s, X_{s+}^{f_m, \gamma_m}, N_{s+}^{f_m, \gamma_m}) ds \right] + \mathbb{E}_{\alpha_i, x_m, n_m} \left[ \frac{1}{h_m} \int_0^{\vartheta_m} (\lambda \psi_x - \psi_n) (\alpha_i, X_{s+}^{f_m, \gamma_m}, N_{s+}^{f_m, \gamma_m}) dM_s \right], \tag{A.21}
\]
with $\mathcal{L} v(\alpha_i, x, n) = \beta v(\alpha_i, x, n) - u(\gamma) - \mathcal{L}^{f, \gamma} \psi(\alpha_i, x, n)$. Note that the stochastic integral term is cancelled by taking expectations due to the integrand is bounded. Moreover, note that,
\[
\mathcal{L} v(\xi_s, X_{s+}^{f_m, \gamma_m}, N_{s+}^{f_m, \gamma_m}) \geq \beta v(\alpha_i, x_m, n_m) - \sup_{(f, \gamma) \in \mathcal{A}(\alpha_i, x_m, n_m)} \left\{ u(\gamma_m) + \mathcal{L}^{f, \gamma_m} \psi(\alpha_i, x_m, n_m) \right\} \geq \varepsilon, \tag{A.22}
\]
by (A.20) and (A.21), it follows that

\[ 0 \geq \frac{c_m}{h_m} - \varepsilon \left( \frac{1}{2} - \frac{1}{h_m} \mathbb{E}_{\alpha, x_m, n_m}[\vartheta_m] \right). \]  

(A.23)

Since

\[
\lim_{h_m \downarrow 0^+} \mathbb{E}_{\alpha, x_m, n_m} \left[ \sup_{s \in (0, h_m]} |X^{f_m, \gamma_m}_s - x_m|^2 \right] = 0,
\]

\[
\lim_{h_m \downarrow 0^+} \mathbb{E}_{\alpha, x, n} \left[ \sup_{s \in (0, h_m]} |N^{f_m, \gamma_m}_s - n_m|^2 \right] = 0.
\]

By Chebyshev’s inequality, we deduce that when \( h_m \) goes to zero (i.e. \( m \) goes to infinity),

\[
P_{\alpha, x_m, n_m}[\tau'_m \leq h_m] \leq P_{\alpha, x_m, n_m} \left[ \sup_{s \in (0, h_m]} |X^{f_m, \gamma_m}_s - x_m| \geq \eta \right] \times P_{\alpha, x, n} \left[ \sup_{s \in (0, h_m]} |N^{f_m, \gamma_m}_s - n_m| \geq \eta \right]
\]

\[
\leq \frac{\mathbb{E}_{\alpha, x_m, n_m} \left[ \sup_{s \in (0, h_m]} |X^{f_m, \gamma_m}_s - x_m|^2 \right]}{\eta^2} \times \frac{\mathbb{E}_{\alpha, x, n} \left[ \sup_{s \in (0, h_m]} |N^{f_m, \gamma_m}_s - n_m|^2 \right]}{\eta^2} \to 0.
\]

Since \( \mathbb{E}_{\alpha, x_m, n_m}[\vartheta_t] = \int_{\tau'_m > h_m} h_m d\mathbb{P} + \int_{\tau'_m \leq h_m} (\tau'_m) d\mathbb{P} \), we have

\[
h_m P_{\alpha, x_m, n_m}(\tau'_m > h_m) = h_m P_{\alpha, x_m, n_m}(\tau'_m > h_m)
\]

\[
= \int_{\tau'_m > h_m} h_m d\mathbb{P} \leq \mathbb{E}_{\alpha, x_m, n_m}[\vartheta_t]
\]

\[
\leq \int_{\tau'_m > h_m} h_m d\mathbb{P} + \int_{\tau'_m \leq h_m} h_m d\mathbb{P} = h_m.
\]

This leads to

\[
P_{\alpha, x_m, n_m}[\tau'_m > h_m] \leq \frac{1}{h_m} \mathbb{E}_{\alpha, x_m, n_m}[\vartheta_t] \leq 1
\]

and implies that \( \frac{1}{h_m} \mathbb{E}_{\alpha, x_m, n_m}[\vartheta_t] \) converges to 1 when \( h_m \) goes to zero. We thus get the desired contradiction by letting \( m \) go to infinity in (A.23). This proves that \( V \) is the viscosity subsolution to the associated coupled HJB equations.

It remains to prove that the uniqueness of the viscosity solution of HJB equation. The method closely related to Section 7 of Crandall et al. [18], where the uniqueness of viscosity solution to a kind of second order PDE with boundary conditions are derived. However, we can not apply the result there directly since the domain of the PDE there is defined on open set \( \Xi \) with compact closure. To proceed our discussion, we should make slightly adjustment on our original problem. Define the following \( \epsilon \)-optimal control problem. Suppose that the state system is still given
by Equation (2.8), for any $0 < x < n$, choose small enough $\epsilon$ such that $1/\epsilon \gg n$. Define $\tau_{l_0}^{f}\gamma = \inf\{t \geq 0 : M_t^{f,\gamma} = 1/\epsilon\}$ and

$$V^*(\alpha_i, x, n) = \max_{f, \gamma \in A} \mathbb{E}_{\alpha_i, x, n} \left[ \int_{0}^{\tau_{l_0}^{f}\gamma \wedge \tau_{l_0}^{f}\gamma} u(\gamma_s)ds \right]. \quad (A.24)$$

Obviously, we have

$$\lim_{\epsilon \downarrow 0} V^*(\alpha_i, x, n) = V(\alpha_i, x, n). \quad (A.25)$$

We can find the HJB equation associated with $V^*(\alpha_i, x, n)$ is specified by

$$\sup_{f, \gamma \in A} \left\{ -\beta v + u(\gamma) + \mathcal{L}^{f,\gamma}v(\alpha_i, x, n) \right\} = 0 \text{ for } \alpha_i \in \mathcal{E}; 0 < x < n;$$

$$-\lambda \frac{\partial v}{\partial x} + \frac{\partial v}{\partial n} = 0, \text{ for } 0 < x = n;$$

$$v(\alpha_i, x, n) = 0, \text{ for all } 0 < n = \frac{1}{\epsilon};$$

$$v(\alpha_i, x, n) = 0, \text{ for all } x < 0.$$

(A.26)

Similar to the proof that $V(\alpha_i, x, n)$ is the viscosity solution to HJB equation (A.20), one can also find that $V^*(\alpha_i, x, n)$ is also the viscosity solution of HJB equation (A.26). To proceed our discussion, denote by $\bar{a} = (\alpha_i, x, n), \bar{u} = (f, \gamma), \mathcal{D}v = (\partial v/\partial x, \partial v/\partial n), \mathcal{D}^2v = (\partial^2 v/\partial x^2, \partial^2 v/\partial n^2), \mathcal{F} = \{(\alpha_i, x, n) : 0 < x < n < 1/\epsilon\}$. Then Equation (A.26) can be reformulated as

$$F(\bar{a}, \bar{u}, \mathcal{D}v, \mathcal{D}^2v) = 0, \bar{a} \in \mathcal{F}, \quad (A.27)$$

with boundary condition

$$B(\bar{a}, \bar{u}, \mathcal{D}v) = \left\{ \begin{array}{ll}
-\lambda \frac{\partial v}{\partial x} + \frac{\partial v}{\partial n} = 0, & \text{for } 0 < x = n; \\
v(\alpha_i, x, n) = 0, & \text{for all } 0 < n = 1/\epsilon; \\
v(\alpha_i, x, n) = 0, & \text{for all } x < 0.
\end{array} \right. \quad (A.28)$$

One can check that PDE (A.27) and boundary condition (A.28) satisfy conditions (7.11)-(7.17) in Crandall et al. [18], by means of Theorem 7.5 of Crandall et al. [18] we have that if $\omega^*\epsilon$ is the subsolution to PDE (A.27) with boundary condition (A.28) and $\omega^*_s$ is the corresponding supersolution, then we have

$$\omega^*\epsilon \leq \omega^*_s, \quad (A.29)$$

which asserts that $V^*(\alpha_i, x, n)$ is the unique viscosity solution to HJB equation (A.26). Note that the boundary condition

$$v(\alpha_i, x, n) = 0, \text{ for all } 0 < n = 1/\epsilon \quad (A.30)$$

vanishes as $\epsilon \rightarrow 0$. The uniqueness of viscosity solution is derived by $\omega^* = V^* \geq V_s \leq \omega_s \leq \omega^*$. \hfill \Box

Appendix C: Proof of Theorem 4.2

Proof. Since $w(\alpha_i, x, n)$ is bounded, continuous and concave w.r.t. $(x, n) \in \mathcal{G}, w_x(\alpha_i, x, n)$ and $w_x(\alpha_i, x, n)$ exist and $w_x(\alpha_i, x, n) \leq w_x(\alpha_i, x, n)$. Noting that $w(\alpha_i, x, n)$ is increasing w.r.t. $x \in (0, n)$ thus we have $0 \leq w_x(\alpha_i, x, n) \leq w_x(\alpha_i, x, n)$. Suppose that there exists $0 < x_0 < n$ such that $w_x(\alpha_i, x_0, n) < w_x(\alpha_i, x_0, n)$, then there exists $a > 0$ such that $w_x(\alpha_i, x_0, n) \leq a \leq w_x(\alpha_i, x_0, n)$. Recall that $w(\alpha_i, x, n)$ is the viscosity supsolution to HJB equation (4.4–4.6), for chosen $x_0$ and a, define a twice continuously differential function

$$\varphi_1(\alpha_i, x, n) = w(\alpha_i, x_0, n) + a(x - x_0) - \frac{1}{2}b(x - x_0)^2$$
as the test function, where $b$ large enough such that $\varphi_1(\alpha_i, x, n) \leq w(\alpha_i, x_0, n)$. Then by the definition of viscosity solution, we have $w(\alpha_i, x_0, n) \geq \varphi(x_0)$ and

$$\beta w(\alpha_i, x_0, n) - \sup_{(f,\gamma)\in A(\alpha_i, x_0, n)} \left\{ u(\gamma) - \gamma a + (f \mu_i + c_i)a + \frac{1}{2} \left[ \sigma_{ii}^2 f^2 + \sigma_{1i}^2 \right] (-b) \right\} = \beta w(\alpha_i, x_0, n) - \bar{u}(a) - \frac{2\sigma_{ii}^2 b + a^2 \mu_i^2}{2\sigma_{ii}^2 b} \geq 0, \quad (A.31)$$

where $\bar{u}(a) = \sup_{0 < \gamma < x_0} [u(\gamma) - \gamma a]$. However, letting $b \to \infty$ in (A.31) yields

$$\beta w(\alpha_i, x_0, n) - \bar{u}(a) - \frac{\sigma_{ii}^2}{\sigma_{ii}^2} \geq 0. \quad (A.32)$$

Particularly, we have $\beta w(\alpha_i, x_0, n) - \bar{u}(w_x(\alpha_i, x_{0+}, n)) - \frac{\sigma_{ii}^2}{\sigma_{ii}^2} \geq 0$, i.e.

$$\bar{u}(w_x(\alpha_i, x_{0+}, n)) \leq \beta w(\alpha_i, x_0, n) - \frac{\sigma_{ii}^2}{\sigma_{ii}^2}, \quad (A.33)$$

On the other hand, since $w(\alpha_i, x, n)$ is also the viscosity subsolution to HJB equations (4.4–4.6), similar to previous argument, consider the following test function

$$\varphi_2(\alpha_i, x, n) = w(\alpha_i, x_0, n) + a(x - x_0) + \frac{1}{2} b(x - x_0)^2, \quad (A.34)$$

where $b$ is large enough such that $w(\alpha_i, x, n) \leq \varphi_2(x)$. We can similarly find that

$$\beta w(\alpha_i, x_0, n) - \bar{u}(a) - \frac{2\sigma_{ii}^2 b + a^2 \mu_i^2}{2\sigma_{ii}^2 b} \leq 0. \quad (A.35)$$

Letting $b \to \infty$ yields $\beta w(\alpha_i, x_0, n) - \bar{u}(a) - \frac{\sigma_{ii}^2}{\sigma_{ii}^2} \leq 0$. Particularly, we have $\beta w(\alpha_i, x_0, n) - \bar{u}(w_x(\alpha_i, x_{0-}, n)) - \frac{\sigma_{ii}^2}{\sigma_{ii}^2} \geq 0$, i.e.

$$\bar{u}(w_x(\alpha_i, x_{0-}, n)) \geq \beta w(\alpha_i, x_0, n) - \frac{\sigma_{ii}^2}{\sigma_{ii}^2}. \quad (A.36)$$

Noting that $\bar{u}(a)$ is decreasing w.r.t. $a$, Equation (A.33) and Equation (A.36) implies that $w_x(\alpha_i, x_0, n) \leq w_x(\alpha_i, x_{0+}, n)$ and thus $w_x(\alpha_i, x_0, n) = w_x(\alpha_i, x_{0+}, n)$. This proves that $w_x(\alpha_i, x, n)$ is continuous on $(0, n)$ w.r.t. $x$.

The rest of this proof is similar to the one for Theorem 5 of [20]. We have proved that $w_x(\alpha_i, x, n)$ is continuous and decreasing w.r.t. $x$, thus $w_{xx}(\alpha_i, x, n)$ exists almost everywhere in $(0, n)$. Let $\Lambda_0 \subset (0, n)$ be the set of point such that $w_{xx}(\alpha_i, x, n)$ exists. Note that $w_{xx}(\alpha_i, x, n)$ is a bounded variation function on $\Lambda_0$, thus it can be expressed as the difference of two monotone increasing function. Without of generality, we assume now that $w_{xx}(\alpha_i, x, n)$ is decreasing, then the point such that $w_{xx}(\alpha_i, x_+, n) < w_{xx}(\alpha_i, x_-, n)$ is countable (see Chapter 1 of Chung [16]). Thus we can find $x_1, x_2 \in (0, n)$, $0 < x_1 < x_2 < n$ such that $w_{xx}(\alpha_i, x, n)$ is continuous in the very small neighbourhood of $x_1$ and $x_2$. Consider the Dirichlet problem (one should note now the boundary condition holds)

$$\sup_{f,\gamma\in A} \left\{ -\beta W + u(\gamma) + D^f \gamma w(\alpha_i, x, n) \right\} = 0, \quad (A.37)$$

$$W(\alpha_i, x_1, n) = w(\alpha_i, x_1, n), \quad W(\alpha_i, x_2, n) = w(\alpha_i, x_2, n), \quad 0 < x_1 < x_2. \quad (A.38)$$

Based on previous analysis, we know that this Dirichlet problem have solution in the small neighbourhood of $x_1$ and $x_2$. By classical results for Dirichlet problem (c.f. Krylov [38]) provides the existence and uniqueness of a smooth $C^2$ solution (w.r.t. $x$).
$W$ to Equations (A.37–A.38) on $(x_1, x_2)$. In particular, this smooth function $W$ is a viscosity solution to Equation (A.37–A.38) on $(x_1, x_2)$. From comparison results for viscosity solutions (see Appendix B) for linear PDEs in bounded domain, we deduce that $w = W$ on $(x_1, x_2)$. Since $x_1$ and $x_2$ are arbitrary, this proves that $w$ is $C^2$ on $(0, n)$ w.r.t. $x$, and satisfies the HJB equations (4.4-4.6) in classical sense. □

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