LONG-TIME BEHAVIOR OF POSITIVE SOLUTIONS OF A DIFFERENTIAL EQUATION WITH STATE-DEPENDENT DELAY

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ABSTRACT. The long-time behavior of positive solutions of a differential equation with state-dependent delay \( \dot{y}(t) = -c(t)y(t - \tau(t, y(t))) \), where \( c \) is a positive coefficient, is considered. Sufficient conditions are given for the existence of positive solutions bounded from below and from above by functions of exponential type. As a consequence, criteria for the existence of positive solutions are derived and their lower bounds are given. Relationships are discussed with the existing results on the existence of positive solutions for delayed differential equations.

1. Introduction. The paper considers a differential equation

\[ \dot{y}(t) = -c(t)y(t - \tau(t, y(t))) \]

where \( t \geq t_0 \in \mathbb{R} \), with state-dependent delay represented by the function \( \tau \). It is assumed that the derivative “.” equals the usual derivative or exists (at least) from the right, \( c(t) \) is a continuous positive coefficient on the set \( I := [t_0, \infty) \) and \( \tau(t, y) \) is continuous and positive for

\[ (t, y) \in D = \{(t, y) \in I \times [0, k]\} \]

where \( k > 0 \) is a fixed number. Assume that \( t_{-1} := \inf_{(t, y) \in D} \{t - \tau(t, y)\} \) is a finite number and \( \lim_{y \to \infty}(t - \tau(t, y)) = +\infty \) for every fixed \( y \in [0, k] \). Let \( I_{-1} := [t_{-1}, \infty) \). A positive solution of (1) is defined as follows.

Definition 1.1. A function \( y: I_{-1} \to (0, k] \) is called a positive solution of (1) on \( I_{-1} \) if \( y \) is continuously differentiable on \( I_{-1} \), decreasing on \( I_{-1} \), and satisfies (1) on \( I \).

In this paper, we study the long-time behavior of positive solutions of (1). Sufficient and necessary conditions for the existence of solutions bounded from below and from above by functions of exponential type are given. We also derive criteria for the existence of positive solutions of (1) on \( I_{-1} \), giving their lower bounds.

The problem is motivated by the following criterion for the existence of a positive solution ([19, Theorem 2.1.4], for the original version in Chinese we refer to [35], a particular case of the result is given in [34]). Consider an equation

\[ \dot{y}(t) = -c(t)y(t - \tau(t)) \]

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with time-dependent continuous delay $\tau: I \to [0, \infty)$, $\tau(t) \leq t$, $t \in I$, $\lim_{t \to \infty}(t - \tau(t)) = \infty$ and continuous $c: I \to [0, \infty)$. In this case, define $t_- := \inf_{t \geq t_0}(t - \tau(t))$. A solution to (2) on $I_{-1}$ is defined as a function $y: I_{-1} \to \mathbb{R}$ continuous on $I_{-1}$, differentiable on $I$, and satisfying (2) on $I$. This solution is positive if, moreover, $y(t) > 0$ on $I_{-1}$.

**Theorem 1.2.** Equation (2) possesses a positive solution on $I_{-1}$ if and only if there exists a continuous function $\lambda$ on $I_{-1}$ such that $\lambda(t) > 0$ on $I$ and

$$\lambda(t) \geq c(t) \exp\left(\int_{t-\tau(t)}^{t} \lambda(s) \, ds\right), \quad t \in I. \quad (3)$$

Being one of the first criteria on existence of positive solutions, this result served as a motivation for various kinds of generalizations, e.g., to systems of both linear and nonlinear delayed differential equations. In part 4, a more complete discussion is given. The author contributed to the progress on this topic by a series of criteria for positivity, having their origin in the above result, and being formulated in terms of the existence of functions satisfying auxiliary inequalities generalizing inequality (3). The criteria derived are of an implicit character (with respect to unknown functions), but make it possible to get explicit criteria, e.g., in terms of inequalities for coefficients of the considered equations. In the paper, we use the following scheme: an implicit criterion is proved and then explicit results derived in terms of the coefficient $c$ of (1).

To explain the general interest in investigating positive solutions we refer, e.g., to applications of particular cases of equation (2). The number theory investigates the properties of the Dickman de Bruijn function as a positive solution of the Dickman de Bruijn equation

$$\dot{x}(t) = -\frac{1}{t}x(t-1), \quad t \geq 1$$

if $x(t) = 1$, $t \in [0, 1]$ (we refer to [28, 29] and to the reference therein). In [18, p. 226], a particular case of (2) models the amount of salt (expressed by a positive solution) in the brine in a tank diluted by fresh water and, in [30, p. 74], the same equation models water temperature regulation by a showering person. Another particular case of (2), is used in [24, p. 97]) to illustrate the fact that linear equations with delay can have (unlike linear ordinary differential equations) positive solutions decreasing for $t \to \infty$ to zero faster than an arbitrary exponential function [24, p. 97].

The paper is structured as follows. In part 2, the main result on the long-time behavior of a particular positive solution to equation (1) is proved by the method of monotone iterative sequences. An equation, more general than (1), is discussed in part 3. Part 4 analyzes the necessity of the sufficient conditions in the results of parts 2 and 3. In addition, this part formulates results on the existence of positive solutions to the equations in a simpler way. Let us note that, in the literature, various results can be found related to our investigations but mostly focused on the delayed differential equations and their systems. Therefore, part 4 includes a short review of the results obtained suggesting some open problems.

**2. Main result.** If a positive solution of equation (1) on $I_{-1}$ exists, then, by Definition 1.1, it is decreasing on $I_{-1}$. Therefore, a positive solution $y = y(t)$ of equation (1) on $I_{-1}$ can be represented in an exponential form.
where $k > 0$ is the constant used in the definition of the domain $D$ and $\lambda: I_{-1} \to [0, \infty)$ is a continuous function on $I_{-1}$ (defined by formula $\lambda(t) := -\frac{\dot{y}(t)}{y(t)}$).

Transforming equation (1) by formula (4), we get

$$-k\lambda(t) \exp \left( -\int_{t_{-1}}^{t} \lambda(s) \, ds \right) = -kc(t) \exp \left( -\int_{t_{-1}}^{t-\tau(t, \Lambda(k, \lambda)(t))} \lambda(s) \, ds \right)$$

where $t \in I$ or, after some simplification,

$$\lambda(t) = c(t) \exp \left( \int_{t-\tau(t, \Lambda(k, \lambda)(t))}^{t} \lambda(s) \, ds \right), \quad t \in I. \quad (5)$$

The formula (5) serves as a motivation for the construction of an iteration process used in the proof of the below Theorem 2.1 where, after a simple transformation of (5) into (13), an iteration process, converging to a solution of (13), is constructed.

Consider two auxiliary integro-functional inequalities

$$\lambda_1(t) \leq c(t) \exp \left( \int_{t-\tau(t, \Lambda(k, \lambda_1)(t))}^{t} \lambda_1(s) \, ds \right), \quad (6)$$

$$\lambda_2(t) \geq c(t) \exp \left( \int_{t-\tau(t, \Lambda(k, \lambda_2)(t))}^{t} \lambda_2(s) \, ds \right) \quad (7)$$

on $I$ where $\lambda_j: I_{-1} \to [0, \infty)$, $j = 1, 2$ are continuous functions on $I_{-1} \setminus \{t_0\}$. Obviously, the right-hand sides of (6), (7) are well-defined since $\Lambda(k, \lambda_i)(t) \leq k$, $i = 1, 2$. These inequalities play a key role in the below theorem whose proof uses a monotone iterative method.

**Theorem 2.1.** Let $k > 0$ be fixed. Assume the following:

(i) For an arbitrary fixed $\theta \geq t_0$, there are constants $K_1$, $K_2$ such that

$$|\tau(t, y) - \tau(t', y')| \leq K_1|t - t'| + K_2|y - y'|$$

for any $t, t' \in [t_0, \theta]$, and any $y, y' \in [0, k]$.

(ii) $\tau(t, y)$ is non-increasing with respect to $y$.

(iii) There exist functions $\lambda_j: I_{-1} \to [0, \infty)$, $j = 1, 2$ continuous on $I_{-1} \setminus \{t_0\}$, satisfying

$$\lambda_1(t) \leq \lambda_2(t), \quad t \in I_{-1}$$

and inequalities (6), (7) on $I$.

(iv) There is a Lipschitz continuous function $\varphi: [t_{-1}, t_0] \to \mathbb{R}$ satisfying $\varphi(t_0) = 0$ such that

$$\lambda_1(t) \leq c(t_0) \exp \left( \int_{t_0-\tau(t_0, \Lambda(k, \lambda_1)(t_0))}^{t_0} \lambda_1(s) \, ds \right) + \varphi(t), \quad (8)$$

$$\lambda_2(t) \geq c(t_0) \exp \left( \int_{t_0-\tau(t_0, \Lambda(k, \lambda_2)(t_0))}^{t_0} \lambda_2(s) \, ds \right) + \varphi(t) \quad (9)$$

on $[t_{-1}, t_0]$. 
Then, there exists a solution \( y = y(t) \) of (1) on \( I_{-1} \) satisfying \( y(t_{-1}) = k \) and
\[
\begin{align*}
    k \exp \left( - \int_{t_{-1}}^{t} \lambda_2(s) \, ds \right) \leq y(t) \leq k \exp \left( - \int_{t_{-1}}^{t} \lambda_1(s) \, ds \right).
\end{align*}
\] (10)

**Proof.** Substituting
\[
\lambda(t) = C(t)L(t)
\] (11)
in (5) where \( L: I_{-1} \to [0, \infty) \) and
\[
C(t) := \begin{cases} 
    c(t) & \text{if } t \in I, \\
    c(t_0) & \text{if } t \in [t_{-1}, t_0),
\end{cases}
\] (12)
we get
\[
L(t) = \exp \left( \int_{t_{-1}}^{t} \int_{t'_{-1}}^{t} C(s)L(s) \, ds \right), \quad t \in I.
\] (13)
Below we prove that the operator equation
\[
L(t) = (TL)(t)
\] (14)
where
\[
(TL)(t) := \exp \left( \int_{t_{-1}}^{t} \int_{t'_{-1}}^{t} C(s)L(s) \, ds \right), \quad t \in I
\] (15)
has a continuous solution \( L = L^*: I_{-1} \to \mathbb{R} \) satisfying
\[
0 \leq L_1(t) \leq L^*(t) \leq L_2(t), \quad t \in I_{-1}
\]
where \( L_1(t) := \lambda_1(t)/C(t), \) \( L_2(t) := \lambda_2(t)/C(t). \) Observe that, if we take \( \lambda(t) = C(t)L^*(t) \) and \( y(t) = \Lambda(k, \lambda)(t), \) \( t \in I_{-1}, \) then \( y(t) \) is the solution we seek.

From (6), (7), by substituting \( \lambda_1(t) = C(t)L_1(t), \lambda_2(t) = C(t)L_2(t), \) we get
\[
L_1(t) \leq \exp \left( \int_{t_{-1}}^{t} \int_{t'_{-1}}^{t} C(s)L_1(s) \, ds \right), \quad t \in I.
\] (16)
\[
L_2(t) \geq \exp \left( \int_{t_{-1}}^{t} \int_{t'_{-1}}^{t} C(s)L_2(s) \, ds \right)
\] (17)
on \( I. \) Obviously, inequalities (8), (9) can be written as
\[
L_1(t) \leq \exp \left( c(t_0) \int_{t_0-\tau(t_0, \Lambda(k, cL_1(t_0)))}^{t_0} L_1(s) \, ds \right) \varphi(t)(c(t_0))^{-1}, \quad t \in [t_{-1}, t_0],
\] (18)
\[
L_2(t) \geq \exp \left( c(t_0) \int_{t_0-\tau(t_0, \Lambda(k, cL_2(t_0)))}^{t_0} L_2(s) \, ds \right) \varphi(t)(c(t_0))^{-1}, \quad t \in [t_{-1}, t_0]
\] (19)
on \( [t_{-1}, t_0]. \)

For a given \( \theta > t_0, \) consider the Banach space \( \mathcal{L}(\theta) \) of continuous functions defined on \( [t_{-1}, \theta] \) with values in \( \mathbb{R}, \) equipped with a maximum norm. Further, define a normal order cone \( K(\theta) \) of the continuous functions defined on \( [t_{-1}, \theta] \) with nonnegative values in \( [t_0, \theta]. \) (For the definitions of order cone and normal cone we refer to [33, Definition 7.1. and Definition 7.6.]).
Define an operator $T_\theta: \mathcal{L}^+(\theta) \to \mathcal{L}^+(\theta)$, where $\mathcal{L}^+(\theta)$ are all functions from $\mathcal{L}(\theta)$ with nonnegative values, as

\[
(T_\theta L)(t) = \begin{cases} 
(TL)(t) & \text{if } t \in [t_0, \theta], \\
(TL)(t_0) + \varphi(t)(c(t_0))^{-1} & \text{if } t \in [t_{-1}, t_0)
\end{cases}
\]

where $L \in \mathcal{L}^+(\theta)$.

The operator $T_\theta$ is monotone increasing (by Definition 7.6 in [33]), i.e. $\nu \leq \mu$ where $\nu, \mu \in \mathcal{L}^+(\theta)$ implies (as it follows from condition (ii) and definition of $\Lambda$ by (4))

\[
T_\theta \nu \leq T_\theta \mu. \tag{21}
\]

Let

\[
\nu_\theta := L_1\lfloor_{[t_{-1}, \theta]}, \ \mu_\theta := L_2\lfloor_{[t_{-1}, \theta]}
\]

and generate two sequences of functions,

\[
\nu_\theta, T_\theta \nu_\theta, T_\theta^2 \nu_\theta, T_\theta^3 \nu_\theta, \ldots,
\]

\[
\mu_\theta, T_\theta \mu_\theta, T_\theta^2 \mu_\theta, T_\theta^3 \mu_\theta, \ldots.
\]

The first terms $\nu_\theta, \mu_\theta$ are, in general, not continuous. To overcome this circumstance, note that the domain of $T_\theta$ can be extended in the obvious way. All the subsequent terms of both sequences are continuous. Condition (ii), inequalities (16)–(19), and (21) imply the following chain of inequalities

\[
\nu_\theta \leq T_\theta \nu_\theta \leq T_\theta^2 \nu_\theta \leq T_\theta^3 \nu_\theta \leq \cdots \leq T_\theta^3 \mu_\theta \leq T_\theta^2 \mu_\theta \leq T_\theta \mu_\theta \leq \mu_\theta.
\]

Now we show that there exist limits

\[
\nu^\theta_m = \lim_{m \to \infty} (T_\theta^m \nu_\theta), \tag{22}
\]

\[
\mu^\theta_m = \lim_{m \to \infty} (T_\theta^m \mu_\theta) \tag{23}
\]

such that $\nu^\theta_m, \mu^\theta_m$ are fixed points of the operator $T_\theta$ and $\nu^\theta_m \leq \mu^\theta_m$.

We will prove that $T_\theta$ is a compact operator. Let $\mathcal{L}^*$ be a bounded subset of $K(\theta)$. Below, we will denote the bound by $M_L$. We will show that $T_\theta \mathcal{L}^*$ is a relatively compact subset of $\mathcal{L}(\theta)$. By Arzelà-Ascoli theorem it is sufficient to show that $T_\theta \mathcal{L}^*$ is bounded and equi-continuous. The boundedness is obvious as a consequence of the continuity of $\bar{C}$, $\tau$ and the boundedness of the functions $L \in \mathcal{L}^*$.

In order to prove the equi-continuity of $T_\theta \mathcal{L}^*$, we show that there is an $M$ such that

\[
|(T_\theta L)(t) - (T_\theta L)(t')| \leq M|t - t'|
\]

for any $L \in \mathcal{L}^*$ and each $t, t' \in [t_{-1}, \theta]$.

Consider three possible cases.

1) Let $t, t' \in [t_{-1}, t_0]$. Then,

\[
|(T_\theta L)(t) - (T_\theta L)(t')| = |(T_\theta L)(t_0) + \varphi(t)(c(t_0))^{-1} - (T_\theta L)(t_0) + \varphi(t')(c(t_0))^{-1}| = |\varphi(t) - \varphi(t')(c(t_0))^{-1}| \leq M_\varphi(c(t_0))^{-1}|t - t'|
\]

where $M_\varphi$ is a Lipschitz constant of $\varphi$. 
2) Let $t, t' \in I$. Then,

$$|\{T_0 L\}(t) - \{T_0 L\}(t')| = \left| \exp \left( \int_{t'}^t C(s) L(s) \, ds \right) - \exp \left( \int_{t'}^{t'} C(s) L(s) \, ds \right) \right|. $$

By Lagrange’s mean-value theorem

$$|\{T_0 L\}(t) - \{T_0 L\}(t')|$$

$$= e^{\zeta_1(t, t')} \cdot \left| \int_{t'}^t C(s) L(s) \, ds - \int_{t'}^{t'} C(s) L(s) \, ds \right|$$

where $\zeta_1(t, t')$ is a value between two values

$$\int_{t'}^t C(s) L(s) \, ds, \quad \int_{t'}^{t'} C(s) L(s) \, ds.$$ 

Due to the boundedness of $C$ (by a constant $M_C$), $L$ (by $M_L$) and $\tau$ (by a constant $M_\tau$), there exists a constant $M_1$ such that

$$\exp \left( \int_{t'}^{t'} C(s) L(s) \, ds \right) \leq \exp (M_CM_LM_\tau) \leq M_1, \quad t^* \in I.$$

Therefore,

$$|\{T_0 L\}(t) - \{T_0 L\}(t')|$$

$$\leq M_1 \left| \int_{t'}^{t'} C(s) L(s) \, ds - \int_{t}^{t'} C(s) L(s) \, ds \right|$$

$$\leq M_1 M_CM_L (2|t - t'| + |\tau(t, \Lambda(k, CL)(t)) - \tau(t', \Lambda(k, CL)(t'))|).$$

Moreover, using (i), we continue

$$|\{T_0 L\}(t) - \{T_0 L\}(t')|$$

$$\leq M_1 M_CM_L (2|t - t'| + K_1|t - t'| + K_2|\Lambda(k, CL)(t)) - \Lambda(k, CL)(t')|)$$

$$\leq M_1 M_CM_L (2 + K_1)|t - t'|$$

$$+ KM_1 M_CM_L K_2 \left| \exp \left( - \int_{t-1}^{t} C(s) L(s) \, ds \right) - \exp \left( - \int_{t-1}^{t'} C(s) L(s) \, ds \right) \right|. $$

Again, by Lagrange’s mean-value theorem, we get

$$U := \left| \exp \left( - \int_{t-1}^{t} C(s) L(s) \, ds \right) - \exp \left( - \int_{t-1}^{t'} C(s) L(s) \, ds \right) \right|$$

$$= e^{\zeta_2(t, t')} \cdot \left| \int_{t-1}^{t} C(s) L(s) \, ds - \int_{t-1}^{t'} C(s) L(s) \, ds \right|. $$
where \( \zeta_2(t, t') \) is a value between two values
\[
- \int_{t-1}^{t} C(s)L(s) \, ds, \quad - \int_{t-1}^{t'} C(s)L(s) \, ds.
\]
Obviously, \( \zeta_2(t, t') < 0 \) and \( \exp \zeta_2(t, t') < 1 \). Then,
\[
U \leq \left| \int_{t}^{t'} C(s)L(s) \, ds \right|
\]
and, by similar arguments as above, \( U \leq M_C M_L |t - t'| \). Finally, in this case,
\[
|(T_\Theta L)(t) - (T_\Theta L)(t')| \leq M_1 M_C M_L (2 + K_1 + k M_C M_L K_2) |t - t'|.
\]
3) Let \( t \in [t_{-1}, t_0] \), \( t' \in I \). In this case
\[
|(T_\Theta L)(t) - (T_\Theta L)(t')| = |(T_\Theta L)(t) - (T_\Theta L)(t_0) + (T_\Theta L)(t_0) - (T_\Theta L)(t')|
\]
\[
\leq |(T_\Theta L)(t) - (T_\Theta L)(t_0)| + |(T_\Theta L)(t_0) - (T_\Theta L)(t')|
\]
and, by 1) and 2), we have
\[
|(T_\Theta L)(t) - (T_\Theta L)(t')| \leq M |t - t'|
\]
where
\[
M := M_\phi(c(t_0))^{-1} + M_1 M_C M_L (2 + K_1 + k M_C M_L K_2).
\]
Obviously, inequality (24) holds, \( T_\Theta L^* \) is equicontinuous, and the above considerations imply that \( T_\Theta \) is compact on \( L^* \).

All conditions of Theorem 7.A in [33] are fulfilled. This means that limits (22), (23) exist, functions \( \nu_\Theta, \mu_\Theta \) are fixed points of \( T_\Theta \) and
\[
\nu_\Theta(t) \leq \nu^*_\Theta(t) \leq \mu^*_\Theta(t) \leq \mu_\Theta(t), \quad t \in [t_{-1}, \Theta]. \tag{25}
\]

The definition of the operator \( T_\Theta \) by formula (20) implies
\[
(T_\Theta \bar{L}) = (T_\Theta \bar{L})_{|[t_{-1}, \theta]}
\]
for every \( \Theta \geq \theta \) if \( \bar{L} = L_{|[t_{-1}, \Theta]} \) and \( \bar{L} = L_{|[t_{-1}, \Theta]} \) where \( L : I_{-1} \to \mathbb{R} \) is a continuous function such that \( L_1(t) \leq L(t) \leq L_2(t), \quad t \in I_{-1} \).

Therefore, repeating the above iterative process step by step with \( \Theta \geq \theta \), we conclude that instead of (25) we have
\[
\nu_\Theta(t) \leq \nu^*_\Theta(t) \leq \mu^*_\Theta(t) \leq \mu_\Theta(t), \quad t \in [t_{-1}, \Theta]
\]
and
\[
\nu^*_\Theta = \nu^*_\Theta_{|[t_{-1}, \Theta]}, \quad \mu^*_\Theta = \mu^*_\Theta_{|[t_{-1}, \Theta]}.
\]

Using, e.g., the fixed point \( \nu^*_t \), we define a continuous function
\[
L^*(t) := \begin{cases} 
\nu^*_t(t) & \text{if } t < t_0 + 1, \\
\nu^*_{t_0+1}(t) & \text{if } t \geq t_0 + 1
\end{cases}
\]
for every \( t \in I_{-1} \). Obviously,
\[
L_1(t) \leq L^*(t) \leq L_2(t), \quad t \in I_{-1}
\]
and \( L^* \) is a fixed point of the operator \( T \), defined by (14), i.e., \( L^* = TL^* \). The function
\[
y(t) := \Lambda(k, CL^*)(t), \quad t \in I_{-1}
\]
is a solution of (1). Since
\[
\Lambda(k, CL_2)(t) \leq \Lambda(k, CL^*)(t) \leq \Lambda(k, CL_1)(t), \quad t \in I_{-1},
\]
inequality (10) holds.

**Example 1.** Let equation (1) be of the form
\[ \dot{y}(t) = -\frac{1}{4r}y(t - (r - y(t))) \] (26)
where \( r > k > 0 \), \( c = 1/(4r) \) and \( \tau(t, y) = r - y > 0 \). Assumptions (i), (ii) of Theorem 2.1 hold. Set
\[ \lambda_1 = \frac{0.01}{r}, \quad \lambda_2 = \frac{1}{r}, \quad \varphi(t) \equiv 0. \]
Then \( t_{-1} = t_0 - r \),
\[ \Lambda(k, \lambda_1)(t) = k \exp \left( -\frac{0.01}{r} (t - t_0 + r) \right), \quad \Lambda(k, \lambda_2)(t) = k \exp \left( -\frac{1}{r} (t - t_0 + r) \right) \]
and inequalities (6), (7) can be written as
\[ 0.01 \leq \frac{1}{4} \exp \left( \frac{0.01}{r} \left( r - ke^{-0.01(t-t_0+r)/r} \right) \right) \]
and
\[ 1 \geq \frac{1}{4} \exp \left( \frac{1}{r} \left( r - ke^{-(t-t_0+r)/r} \right) \right). \]
For every sufficiently small \( k \), both inequalities hold since
\[ 0.01 \leq \frac{\exp(0.01)}{4} \approx 0.25, \quad 1 \geq \frac{\exp(1)}{4} \approx 0.68. \]
Inequalities (8), (9) can be written as
\[ 0.01 \leq \frac{1}{4} \exp \left( \frac{0.01}{r} \left( r - ke^{-0.01t} \right) \right), \quad 1 \geq \frac{1}{4} \exp \left( \frac{1}{r} \left( r - ke^{-t} \right) \right) \]
and, by the same argument, are valid. Then, there exists a solution \( y = y(t) \) of (26) on \([t_0 - r, \infty)\) satisfying \( y(t_0 - r) = k \) and
\[ ke^{-(t-t_0+r)/r} \leq y(t) \leq ke^{-0.01(t-t_0+r)/r}. \] (27)
Finally, we remark, that the existence of a positive solution of (26) can be expected from the fact that the equation
\[ \dot{y}(t) = -\frac{1}{4r}y(t - r), \]
obtained by ignoring the state-dependent delay, has a one-parametric family of solutions \( y(t) = k \exp(-\varepsilon t) \) where \( \varepsilon \) is a positive root of the transcendental equation
\[ 4r \varepsilon = \exp(\varepsilon r) \] (a simple proof of the existence of two positive roots is omitted).

3. **A general case.** In this part, we consider a differential equation
\[ \dot{y}(t) = -\sum_{i=1}^{n} c_i(t)y(t - \tau_i(t, y(t))) \] (28)
with state-dependent delays where \( n \geq 1 \). Although equation (28) is more general than equation (1), the result of Theorem 2.1 is not a consequence of Theorem 3.1, given below, for \( n = 1 \). As noted in the proof, the reason is that it is not possible to construct a suitable operator coinciding for \( n = 1 \) with that used in the proof of Theorem 2.1.
Let functions $c_i: I \rightarrow [0, \infty), i = 1, \ldots, n$ and $\tau_i: D \rightarrow [0, \infty), i = 1, \ldots, n$ be continuous and such that
\[
\sum_{i=1}^{n} c_i(t) > 0, \ \forall t \in I, \quad \sum_{i=1}^{n} \tau_i(t,y) > 0, \ \forall (t,y) \in D.
\]

Let us redefine $t_{-1} := \inf_{i=1,\ldots,n, (t,y) \in D} \{ t - \tau_i(t,y) \}$ and assume that $t_{-1}$ is a finite number. We omit the definition of a positive solution of (28) on $I_{-1}$ since it is similar to Definition 1.1. The following theorem holds.

**Theorem 3.1.** Let $k > 0$ be fixed. Assume the following:

(i) For an arbitrary fixed $\theta \geq t_0$, there are constants $K_{1i}$, $K_{2i}$ such that
\[
|\tau_i(t,y) - \tau_i(t',y')| \leq K_{1i}|t - t'| + K_{2i}|y - y'|
\]
for any $i = 1, \ldots, n$, $t, t' \in [t_0, \theta]$, and any $y, y' \in [0,k]$.

(ii) Functions $\tau_i(t,y), i = 1, \ldots, n$ are non-increasing with respect to $y$.

(iii) There exist functions $\lambda_j: I_{-1} \rightarrow [0, \infty), j = 1, 2$, continuous on $I_{-1} \setminus \{t_0\}$, satisfying
\[
\lambda_1(t) \leq \lambda_2(t), \ t \in I
\]
and
\[
\lambda_1(t) \leq \sum_{i=1}^{n} c_i(t) \exp \left( \int_{t - \tau_i(t,\Lambda(k,\lambda_1)(t))}^{t} \lambda_1(s) \, ds \right), \quad \lambda_2(t) \geq \sum_{i=1}^{n} c_i(t) \exp \left( \int_{t - \tau_i(t,\Lambda(k,\lambda_2)(t))}^{t} \lambda_2(s) \, ds \right)
\]
on $I$.

(iv) There exists a Lipschitz continuous function $\varphi: [t_{-1}, t_0] \rightarrow \mathbb{R}$ satisfying $\varphi(t_0) = 0$ such that
\[
\lambda_1(t) \leq \sum_{i=1}^{n} c_i(t_0) \exp \left( \int_{t_0 - \tau_i(t_0,\Lambda(k,\lambda_1)(t_0))}^{t_0} \lambda_1(s) \, ds \right) + \varphi(t),
\]
\[
\lambda_2(t) \geq \sum_{i=1}^{n} c_i(t_0) \exp \left( \int_{t_0 - \tau_i(t_0,\Lambda(k,\lambda_2)(t_0))}^{t_0} \lambda_2(s) \, ds \right) + \varphi(t)
\]
on $[t_{-1}, t_0]$.

(v) Functions $c_i(t), i = 1, \ldots, n$ are Lipschitz continuous on $I$.

Then, there exists a solution $y = y(t)$ of (28) on $I_{-1}$ satisfying $y(t_{-1}) = k$ and
\[
k \exp \left( - \int_{t_{-1}}^{t} \lambda_2(s) \, ds \right) \leq y(t) \leq k \exp \left( - \int_{t_{-1}}^{t} \lambda_1(s) \, ds \right).
\]  \hspace{1cm} (29)

**Proof.** The proof can be done along the same lines as that of Theorem 2.1 with the following modification. The auxiliary substitution (11) with $C(t)$ defined by (12) in the proof of Theorem 2.1 made it possible to define the operator equation (14) without multiplier $c(t)$ on the right-hand side of (15) because the function $c(t)$ was reduced and only appears in the integrand. Such a simplification by a suitable substitution does not exist for the considered generalization. Therefore, an analogue
\( \tilde{T} \) of the operator \( T \) (we derive this analogy as the result of transformation of equation (28) via formula (4)) is

\[
(\tilde{T}\lambda)(t) := \sum_{i=1}^{n} c_i(t) \exp \left( \int_{t-\tau_i(t,\Lambda(k,\lambda)(t))}^{t} \lambda(s) \, ds \right)
\]

(30)

and an operator equation

\[
\lambda(t) = (\tilde{T}\lambda)(t)
\]

is applied instead of the operator equation (14). Since, on the right-hand side of formula (30), the functions \( c_i(t), \, i = 1, \ldots, n \) serve as multipliers, an additional assumption \( (v) \) about their Lipschitz continuity is necessary to successfully modify the verification of the fact that the auxiliary operators are equi-continuous. Following the proof of Theorem 2.1, we conclude that the remaining parts of the proof can be simply modified. \( \Box \)

4. Concluding discussion, open problems. In this part, we discuss some further generalizations of the results derived as well as particular results related to the existence of positive solutions of the considered equations. In addition, related results are mentioned and classified.

4.1. Necessary and sufficient criteria. Assuming that \( y(t) \) is a positive solution of equation (1) on \( \mathbb{I}_{-1} \), Theorem 2.1 can be improved as follows.

**Theorem 4.1.** Let \( k > 0 \) be fixed and let the assumptions (i) and (ii) of Theorem 2.1 hold. Then, assumptions (iii) and (iv) are necessary and sufficient for the existence of a solution \( y = y(t) \) of (1) on \( \mathbb{I}_{-1} \) satisfying \( y(t_{-1}) = k \) and (10).

**Proof.** By Theorem 2.1, the conditions are sufficient. To prove the necessity, set \( \lambda_1(t) = \lambda_2(t) \equiv \lambda(t), \varphi(t) \equiv \lambda(t) - \lambda(t_0) \) (recall that, by (4), \( \lambda(t) := -\dot{y}(t)/y(t) \)) and \( k = y(t_{-1}) \). Then, assumptions (iii) and (iv) are valid since inequalities (6), (7) as well as (8), (9) become equalities. \( \Box \)

Similarly, we can improve Theorem 3.1 to:

**Theorem 4.2.** Let \( k > 0 \) be fixed and let the assumptions (i), (ii) and (v) of Theorem 3.1 hold. Then assumptions (iii) and (iv) are necessary and sufficient for the existence of a solution \( y = y(t) \) of (28) on \( \mathbb{I}_{-1} \) satisfying \( y(t_{-1}) = k \) and (29).

4.2. More on the existence of positive solutions. Theorem 4.1 involves the necessary and sufficient conditions for the existence of a solution \( y = y(t) \) of (1) on \( \mathbb{I}_{-1} \) satisfying inequalities (10). The solution \( y = y(t) \) of (1) is of course positive since it is bounded from below by a positive function. We can simplify the assumptions of Theorem 4.1 to get the following corollary.

**Corollary 1.** Let \( k > 0 \) be fixed and let the assumptions (i), (ii) of Theorem 2.1 hold. Then, the existence of a function \( \omega: \mathbb{I}_{-1} \rightarrow [0, \infty) \), continuous on \( \mathbb{I}_{-1} \setminus \{t_0\} \), satisfying

\[
\omega(t) \geq c(t) \exp \left( \int_{t-\tau(t,\Lambda(k,\omega)(t))}^{t} \omega(s) \, ds \right)
\]

(31)
on \( \mathbb{I} \) and

\[
\omega(t) \geq c(t_0) \exp \left( \int_{t_0-\tau(t_0,\Lambda(k,\omega)(t_0))}^{t_0} \omega(s) \, ds \right)
\]

(32)
Then, the existence of a function $\omega$ satisfies inequality (31).

**Theorem 2.1.** Let $y$ on $I_{-1}$ satisfying $y(t_1) = k$ and

$$k \exp \left(-\int_{t_1}^{t} \omega(s) \, ds\right) \leq y(t) \leq k.$$  \hfill (33)

**Proof.** To prove the sufficiency, let, in Theorem 2.1, $\lambda_1(t) \equiv 0$, $\lambda_2(t) = \omega(t)$, $t \in I_{-1}$ and $\varphi(t) \equiv 0$, $t \in [t_1, t_0]$. Then, the assumptions of Corollary 1 imply the validity of all assumptions of Theorem 2.1 and inequalities (10) become inequalities (33). The necessity can be proved similarly to the proof of Theorem 4.1.

The following corollary shows that inequality (32) in Corollary 1 can be omitted.

**Corollary 2.** Let $k > 0$ be fixed and assumptions (i), (ii) of Theorem 2.1 hold. Then, the existence of a function $\omega: I_{-1} \rightarrow [0, \infty)$, continuous on $I_{-1} \setminus \{t_0\}$, satisfying inequality (31) on $I$, is necessary and sufficient for the existence of a solution $y = y(t)$ of (1) on $I_{-1}$ satisfying $y(t_1) = k$ and

$$k \exp \left(-\int_{t_1}^{t} \omega^*(s) \, ds\right) \leq y(t) \leq k$$  \hfill (34)

where

$$\omega^*(t) := \left\{ \begin{array}{ll}
\omega(t) & \text{if } t \in I, \\
\omega(t_s) & \text{if } t \in [t_1, t_0]
\end{array} \right.$$  \hfill (35)

and $\omega(t_s) = \min_{t \in [t_1, t_0]} \{\omega(t)\}$.

**Proof.** To prove sufficiency, let us show that all assumptions of Corollary 1 hold with $\omega := \omega^*$. Since, obviously $\omega(t) \geq \omega^*(t)$, $t \in I_{-1}$, we have on $I$:

$$\omega^*(t) = \omega(t) \geq c(t) \exp \left(\int_{t \tau(t, \Lambda(k, \omega)(t))}^{t \tau(t, \Lambda(k, \omega^*)(t))} \omega(s) \, ds\right) \geq c(t) \exp \left(\int_{t \tau(t, \Lambda(k, \omega^*)(t))}^{t \tau(t, \Lambda(k, \omega^*)(t))} \omega^*(s) \, ds\right)$$

and inequality (31) holds for $\omega := \omega^*$. By (32), we have

$$\omega^*(t) = \omega(t_s) \geq c(t_0) \exp \left(\int_{t_0 - \tau(t_0, \Lambda(k, \omega)(t_0))}^{t_0} \omega(s) \, ds\right) \geq c(t_0) \exp \left(\int_{t_0 - \tau(t_0, \Lambda(k, \omega^*)(t_0))}^{t_0} \omega(t_s) \, ds\right) = c(t_0) \exp \left(\int_{t_0 - \tau(t_0, \Lambda(k, \omega^*)(t_0))}^{t_0} \omega^*(s) \, ds\right)$$

and (32) holds with $\omega := \omega^*$. The remaining assumptions of Corollary 1 are obviously fulfilled. Necessity is obvious, we refer to Corollary 1 and to the inequality

$$\exp \left(-\int_{t_1}^{t} \omega(s) \, ds\right) \leq \exp \left(-\int_{t_1}^{t} \omega^*(s) \, ds\right), \quad t \in I_{-1}.$$
As the proof of the following criterion for the existence of a positive solution of (28) can be done along the same lines as that of Corollary 2, it has been omitted.

**Corollary 3.** Let $k > 0$ be fixed and let the assumptions (i), (ii), (v) of Theorem 3.1 hold. Then, the existence of a function $\omega: \mathbb{I}_{-1} \to [0, \infty)$, continuous on $\mathbb{I}_{-1} \setminus \{t_0\}$, satisfying the inequality

$$\omega(t) \geq \sum_{i=1}^{n} c_i(t) \exp \left( \int_{t_{\tau_i(t,A(\omega)(t))}}^{t} \omega(s) \, ds \right)$$

on $\mathbb{I}$, is necessary and sufficient for the existence of a solution $y = y(t)$ of (28) on $\mathbb{I}_{-1}$ satisfying $y(t_{-1}) = k$ and

$$k \exp \left( - \int_{t_{-1}}^{t} \omega^*(s) \, ds \right) \leq y(t) \leq k$$

with $\omega^*$ defined by (35).

**Remark 1.** Let the assumptions of Corollary 2 be true. If $\omega(t) := c(t)e$, then the sufficient condition (31) becomes

$$\int_{t_{\tau(t,A(\omega)(t))}}^{t} c(s) \, ds \leq \frac{1}{e}, \quad t \in \mathbb{I}. \quad (36)$$

If, moreover, $c(t)$ is a constant function, i.e., $c(t) \equiv c$, then (36) can be written as

$$\tau(t, A(k,c)(t)) \leq \frac{1}{ce}, \quad t \in \mathbb{I}$$

or

$$\tau(t, k \exp (-ce)(t-t_{-1})) \leq \frac{1}{ce}, \quad t \in \mathbb{I}. \quad (37)$$

Obviously, if $\tau(t,y)$ does not depend on $y$, i.e., if $\tau(t,y) \equiv \tau(t)$, then (36) as well as (37) become classical conditions for the existence of a positive solution

$$\int_{t-\tau(t)}^{t} c(s) \, ds \leq \frac{1}{e}, \quad t \in \mathbb{I}$$

and

$$\tau(t) \leq \frac{1}{ce}, \quad t \in \mathbb{I}$$

(we refer to (3) with $\lambda(t) := c(t)e$ and $\lambda(t) := ce$).

**Example 2.** Let equation (1) be of the form

$$\dot{y}(t) = -cy(t - (r - y(t))) \quad (38)$$

where $0 < c$ and $0 < k < r < 1/(ce)$. Then, $t_{-1} = t_0 - r$, $\tau(t,y) = r - y > 0$ and condition (37) can be written as

$$\tau(t, k \exp (-ce)(t-t_{-1})) = r - ke^{-ce(t-t_{0}+r)} \leq \frac{1}{ce}, \quad t \in \mathbb{I}.$$
Consider the equation of the type (2)
\[ \dot{y}(t) = -c(t)y(t - \tau(t)) \]
where \( c \) is a positive coefficient on the set \( I \) and \( \tau \) is bounded, continuous, and positive on \( I \). Set \( L_1 := \inf_{t \in I} \{t - \tau(t)\} \). Then, as a consequence of Corollary 2, we have the following.

**Corollary 4.** Let \( \tau(t) \) be Lipschitz continuous on \( I \). Then, the existence of a function \( \omega: I_{-1} \to [0, \infty) \), continuous on \( I_{-1} \setminus \{t_0\} \) and satisfying the inequality
\[ \omega(t) \geq c(t) \exp \left( \int_{t-\tau(t)}^{t} \omega(s) \, ds \right) \]
on \( I \), is necessary and sufficient for the existence of a decreasing positive solution \( y = y(t) \) of (2) on \( I_{-1} \) satisfying \( y(t-1) = k \) and
\[ k \exp \left( -\int_{t-1}^{t} \omega^*(s) \, ds \right) \leq y(t) \leq k \]
on \( I_{-1} \) where \( \omega^*(t) \) is defined by (35).

**Remark 2.** The proof of Theorem 2.1 uses a monotone iterative technique. This is the reason why we need Lipschitz continuity of \( \tau \) in Corollary 4. If this condition is satisfied, we get the affirmation of Theorem 1.2. Moreover, we point out that, in such a case, an advantage of Corollary 4 over Theorem 1.2 is the estimation of lower bound (39) for the solution \( y(t) \).

### 4.3. Concluding remarks
Theorem 2.1 states that there exists a solution \( y = y(t) \) of (1) on \( I_{-1} \) satisfying \( y(t-1) = k \) and inequality (10). This means, that from (10) and from the relation \( \lambda(t) = -\dot{y}(t)/y(t) \), it is possible to derive
\[ k\lambda_1(t) \exp \left( -\int_{t-1}^{t} \lambda_2(s) \, ds \right) \leq -\dot{y}(t) \leq k\lambda_2(t) \exp \left( -\int_{t-1}^{t} \lambda_1(s) \, ds \right) \]
on \( I_{-1} \).

Moreover, if, in Theorem 2.1, we assume that \( \lambda_1(t) \equiv \lambda_2(t) \) on \( [t-1, t_0] \), then the initial problem for (1) represented by an initial function \( \phi(t) = \lambda_1(t) \equiv \lambda_2(t) \), \( t \in [t-1, t_0] \) defines a solution \( y = y(t) \) of (1) that behaves as described by (10) and (40). Similar remarks can be applied to the remaining results of the paper.

The long-time behavior of functional differential equations with state-dependent delays is investigated, e.g. in [6, 14, 15, 22, 27]. For an overview of results related to functional differential equations with state-dependent delays, we refer to [25].

In [6], the asymptotic behavior of solutions is studied of the functional differential equation
\[ u'(t) = -au(t - r(u)), \quad a > 0 \]
proving
\[ u(t) = e^{-at}(\xi + o(1)), \quad t \to \infty, \]
where \( \xi \) is a constant, if the state-dependent lag \( r(u) \) satisfies the condition: for some \( \delta > 0, r: [-\delta, \delta] \to [0, \tau] \) is a continuous function such that \( r(0) = 0 \) and
\[ m \in L_1[0, \infty), \quad m(t) = \sup_{|u| \leq \delta e^{-at}} r(u). \]  \( (41) \)

In [27], this result is generalized to equation
\[ y'(t) = a(t)y(t - r(t, y)) \]
where \( a : [0, \infty) \to \mathbb{R} \) is a continuous function. Conditions are given guaranteeing an asymptotic expansion of the form

\[
y(t) = e^{\int_0^t a(s) \, ds} \left[ \xi + O \left( \int_t^\infty \lambda(s) \, ds \right) \right], \quad t \to \infty
\]

with certain \( \xi \in \mathbb{R} \) and

\[
\lambda \in L_1[0, \infty)
\]

constructed by means of \( a \) and \( r \). These results are extended to systems

\[
\dot{x}(t) = A(t)x(t - r(t, x(t)))
\]

in [20] where \( A(t) \) is a square matrix continuous on \([0, \infty)\). Analyzing the assumptions of the results in [6, 27] and comparing them with our investigation, we conclude the independency of the results obtained since assumptions (41), (42) are not included and are not necessary in the formulations of our results. Similarly, we can proceed when analyzing the assumptions in [20].

There exist many investigations concerned with differential equations with delays closely connected to the results of this paper. We refer, e.g., to books [1, 2, 19, 21, 23] and papers [3]–[5], [7]–[13], [16, 17, 26, 31, 32, 34].

Various generalizations of Theorem 1.2 (or very similar results), including generalizations to linear and non-linear delayed systems can be found in [1, 2, 7, 11, 19, 21, 23].

Very sharp explicit criteria for detecting positive solutions (mostly for scalar delayed equations) are presented in [3, 5, 9, 10, 12, 16, 17, 26, 31].

Some special results on positive solutions are given (in addition to all books mentioned) in [4] where positive solutions of second-order delay differential equations with a damping term are considered, in [8] where the existence of positive solutions to delayed equations and their estimates is deduced from auxiliary systems of delayed equations and in [13] where the results on the existence of positive solutions depend on impulses.

In Corollary 4, as noted in Remark 2, we need Lipschitz continuity of \( \tau \). This is necessary for the applicability of a monotone iterative process in the proof of Theorem 2.1. Moreover, in Theorem 3.1, Lipschitz continuity of \( \tau_i(t, y) \), \( i = 1, \ldots, n \) and \( c_i(t) \), \( i = 1, \ldots, n \) is necessary. In Theorem 1.2 Lipschitz continuity of \( \tau \) is not assumed. Therefore, the following open problem arises.

**Open Problem 1.** Is it possible to improve the method used in the proof of Theorem 2.1 or is it possible to apply a different method to prove the same statement without Lipschitz continuity of \( \tau(t, y) \) (at least with respect to \( t \)). Similarly, is it possible to omit Lipschitz continuity of \( \tau_i(t, y) \), \( i = 1, \ldots, n \) (at least with respect to \( t \)) and \( c_i(t) \), \( i = 1, \ldots, n \) in Theorem 3.1?

Another improvement of the results is connected with the property in Theorem 2.1 that \( \tau(t, y) \) is non-increasing with respect to \( y \). This property seems to be very restrictive.

**Open Problem 2.** Is it possible to improve the method used in the proof of Theorem 2.1 or to derive another proof method to obtain a similar result if \( \tau(t, y) \) is non-decreasing with respect to \( y \) or even omit such an assumption? A similar problem is stated with respect to \( \tau_i(t, y) \), \( i = 1, \ldots, n \) in Theorem 3.1.
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