Abstract. This paper deals with the quasilinear Keller–Segel system
\begin{align*}
    u_t &= \nabla \cdot (D(u) \nabla u) - \nabla \cdot (S(u) \nabla v), \quad x \in \Omega, \quad t > 0, \\
    v_t &= \Delta v - v + u, \quad x \in \Omega, \quad t > 0
\end{align*}

in \( \Omega = \mathbb{R}^N \) or in a smoothly bounded domain \( \Omega \subset \mathbb{R}^N \), with nonnegative initial data \( u_0 \in L^1(\Omega) \cap L^\infty(\Omega) \), and \( v_0 \in L^1(\Omega) \cap W^{1,\infty}(\Omega) \); in the case that \( \Omega \) is bounded, it is supplemented with homogeneous Neumann boundary condition. The diffusivity \( D(u) \) and the sensitivity \( S(u) \) are assumed to fulfill \( D(u) \geq u^{m-1} \) \((m \geq 1)\) and \( S(u) \leq u^{q-1} \) \((q \geq 2)\), respectively. This paper derives uniform-in-time boundedness of nonnegative solutions to the system when \( q < m + \frac{2}{N} \). In the case \( \Omega = \mathbb{R}^N \) the result says boundedness which was not attained in a previous paper (J. Differential Equations 2012; 252:1421-1440). The proof is based on the maximal Sobolev regularity for the second equation. This also simplifies a previous proof given by Tao–Winkler (J. Differential Equations 2012; 252:692-715) in the case of bounded domains.

1. Introduction and results. The purpose of this paper is to establish global-in-time existence and uniform-in-time boundedness of solutions to a quasilinear fully parabolic Keller–Segel system via the maximal Sobolev regularity for parabolic equations. More specifically, we consider the following parabolic system:
\begin{align}
    \frac{\partial u}{\partial t} &= \nabla \cdot (D(u) \nabla u) - \nabla \cdot (S(u) \nabla v), \quad x \in \Omega, \quad t > 0, \\
    \frac{\partial v}{\partial t} &= \Delta v - v + u, \quad x \in \Omega, \quad t > 0, \\
    u(x,0) &= u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega.
\end{align}

2010 Mathematics Subject Classification. Primary: 35K51; Secondary: 35B35.

Key words and phrases. Quasilinear degenerate Keller–Segel systems, boundedness.

The first and second authors are supported by Grant-in-Aid for Young Scientists Research (B) (No. 15K17578) and Scientific Research (C) (No. 16K05182), JSPS, respectively.

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where $\Omega = \mathbb{R}^N$ or $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N \in \mathbb{N}$. In the case that $\Omega \neq \mathbb{R}^N$, we always supplement with the no-flux boundary condition

$$(D(u) \nabla u - S(u) \nabla v) \cdot \nu = 0, \quad \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0,$$

where $\frac{\partial}{\partial \nu}$ denotes differentiation with respect to the outward normal of $\partial \Omega$. The initial data $(u_0, v_0)$ is assumed to be a pair of functions satisfying

$$u_0 \geq 0, \quad u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),$$

$$v_0 \geq 0, \quad v_0 \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N).$$

We also assume that the diffusivity function $D$ and the sensitivity function $S$ fulfill the following conditions:

$$D \in C([0, \infty)) \cap C^2((0, \infty)), \quad S \in C([0, \infty)) \cap C^2((0, \infty)),$$

$$D(\sigma) \geq k_1 \sigma^{m-1} \quad (\sigma \geq 0) \quad \text{with some } m \geq 1, \quad k_1 > 0,$$

$$0 \leq S(\sigma) \leq k_2 \sigma^{q-1} \quad (\sigma \geq 0) \quad \text{with some } q \geq 2, \quad k_2 > 0 \quad \text{(and hence } S(0) = 0).$$

The Keller–Segel system (1) with the simplest choices $D(u) \equiv 1$ and $S(u) = u$ is proposed by Keller and Segel [17] in 1970. The system describes a part of the life cycle of cellular slime molds with chemotaxis. In more detail, slime molds move towards higher concentration of the chemical substance when they plunge into hunger. Such chemotactic effect is originally described by $-\nabla \cdot (u \nabla v)$, while the diffusion effect of slime molds is represented by $\Delta u$, where we denote by $u(x, t)$ the density of the cell population and by $v(x, t)$ the concentration of the signal substance at place $x$ and time $t$. A number of variations of the original Keller–Segel system are proposed and studied (see Hillen and Painter [8], Bellomo, Bellouquid, Tao and Winkler [2]). A quasilinear system such as (1) was proposed by Painter and Hillen [21]. In particular, we emphasize that the diffusivity $D(u)$ in (1) is allowed to be “degenerate”, e.g., $\Delta u^m$ with some $m > 1$. This means that the mathematical analysis of (1) will be more delicate because of the lack of regularity of solutions.

From a mathematical point of view, it is a meaningful question whether the solutions of (1) remain bounded or blow up. This depends on the relation between the powers $m$ and $q$ in the conditions (5) and (6).

We first review known results in the case of non-degenerate diffusion such that $D(0) > 0$.

Tao and Winkler [25] proved that there exists a unique global-in-time bounded classical solution of the Neumann problem (1) on bounded convex domains under the condition $q < m + \frac{2}{N}$. The result was extended to the Neumann problem on bounded non-convex domains by the joint paper with Seki [12]. As to the case $q = 2$, Senba and Suzuki [22] proved boundedness of solutions to (1) under some additional conditions for the initial data. In contrast, Winkler [27] showed that the Neumann problem with non-degenerate diffusion admits a smooth solution which blows up either in finite or infinite time if $q > m + \frac{2}{N}$. Moreover, Winkler [28] and Cieślak and Stinner [5] proved that the solution blows up in finite time when $q > m + \frac{2}{N}$. This means that the case $q = m + \frac{2}{N}$ is critical.

We next observe previous works in the case of degenerate diffusion such that $D(0) = 0$. 

$$D(0) > 0.$$
The Cauchy problem (1) on $\mathbb{R}^N$ with $D(u) = u^{m-1}$ ($m > 1$) and $S(u) = u^{q-1}$ ($q \geq 2$) was firstly studied by Sugiyama and Kunii [24] in which existence of global-in-time weak solutions was shown when $q \leq m$. After that, the condition $q \leq m$ for global existence was extended to $q < m + \frac{2}{N}$ by [13]. Unfortunately, both [24] and [13] assert only global existence. As to the Neumann problem on bounded domains, boundedness of solutions was proved by [12]. However, boundedness in the Cauchy problem on the whole space $\mathbb{R}^N$ is still open. On the other hand, when $q \geq m + \frac{2}{N}$ and the initial data $(u_0, v_0)$ is small in some sense, existence of global-in-time weak solutions was proved by [14, 15] and boundedness was established by [9]; whereas, if $q > m + \frac{2}{N}$, then blow-up (either in finite or infinite) was studied by [11, 16].

In summary, boundedness and blow-up in finite time in the Cauchy problem (1) on the whole space $\mathbb{R}^N$ are still open. The purpose of this paper is to establish boundedness in the case $q < m + \frac{2}{N}$. We give our motivation more precisely. The analysis of this case was essentially made in our previous paper [13] and Tao and Winkler’s paper [25]. As explained above, in [13] global-in-time existence of weak solutions to the Cauchy problem (1) on $\mathbb{R}^N$ was established; however, uniform-in-time boundedness was not obtained. On the other hand, in [25] both global existence and boundedness of solutions to the Neumann problem (1) in bounded domains were established. Moreover, the proofs in [13] and [25] are essentially different. The key idea in [13] is to apply the maximal Sobolev regularity for parabolic equations and to deal with the system as a single equation. This simplifies the proof; however, the weak point is not being able to yield uniform-in-time boundedness. Thus, in this paper, we would like to establish uniform-in-time boundedness by a simple way via the maximal Sobolev regularity.

Taking account of degeneracy of diffusion, we define weak solutions to (1).

**Definition 1.1.** A pair $(u, v)$ of nonnegative functions defined a.e. on $\Omega \times (0, \infty)$ is called a global weak solution of (1) if

(a) $u \in L^\infty(0, T; L^p(\Omega))$ ($\forall p \in [1, \infty]$), $\int_0^T \int_\Omega D(\sigma) \, d\sigma \in L^2(0, T; H^1(\Omega))$ for all $T > 0$;

(b) $v \in L^\infty(0, T; W^{1, \infty}(\Omega)) \cap L^2(0, T; L^2(\Omega))$ for all $T > 0$;

(c) for every $\varphi \in C^\infty_c(\Omega \times [0, \infty))$,

$$
\int_0^\infty \int_\Omega \left( \nabla \left( \int_0^u D(\sigma) \, d\sigma \right) \cdot \nabla \varphi - S(u) \nabla v \cdot \nabla \varphi - u \varphi_t \right) \, dx \, dt = \int_\Omega u_0(x) \varphi(x, 0) \, dx,
$$

$$
\int_0^\infty \int_\Omega \left( \nabla v \cdot \nabla \varphi + v \varphi - u \varphi - v \varphi_t \right) \, dx \, dt = \int_\Omega v_0(x) \varphi(x, 0) \, dx.
$$

We now state the main result in this paper.

**Theorem 1.2.** Let $N \in \mathbb{N}$, $m \geq 1$ and $q \geq 2$. Assume that $D$ and $S$ satisfy the conditions (4), (5) and (6) with

$$
q < m + \frac{2}{N}.
$$

Then for any initial data $(u_0, v_0)$ fulfilling (2) and (3), there exists a global weak solution $(u, v)$ of (1) such that $u$ is bounded in the following sense:

$$
\text{ess-sup}_{t \in (0, \infty)} \|u(t)\|_{L^\infty(\Omega)} \leq K_\infty,
$$

where $K_\infty$ is a positive constant.

**Remark 1.** Uniqueness of weak solutions to (1) was recently studied by Miura and Sugiyama [20] and Kim and Lee [18].
The proof of Theorem 1.2 will be divided into the following three steps.
(1) Construction of local-in-time approximate solutions.
We will construct local-in-time approximate solutions \((u_\varepsilon, v_\varepsilon)\) of (1), where the diffusion term is approximated by non-degenerate type of
\[\nabla \cdot (D(u_\varepsilon + \varepsilon)\nabla u_\varepsilon),\]
\(\varepsilon > 0\). Since the maximal existence time depends on \(\varepsilon\), we need to evaluate it from below uniformly in \(\varepsilon\).
(2) Estimates for approximate solutions.
In order to extend local-in-time approximate solutions \((u_\varepsilon, v_\varepsilon)\) globally in time we need the \(L^\infty\)-estimate for \(u_\varepsilon\) which will be derived from the \(L^r\)-estimate:
\[\|u_\varepsilon(t)\|_{L^\infty(\Omega)} \leq K_r < \infty \text{ for each } r \in [1, \infty).\]
(3) Passage to the limit.
Letting \(\varepsilon \to 0\), we can show that \((u, v) := \lim_{\varepsilon \to 0} (u_\varepsilon, v_\varepsilon)\) is the desired solution.

The cornerstone of the above strategy is the \(L^r\)-estimate for \(u_\varepsilon\) in (2). We derive the \(L^r\)-estimate from standard testing arguments for the first equation in (1). One of the difficulties is how to estimate the approximate chemotaxis term
\[-\nabla \cdot (\frac{u_\varepsilon}{u_\varepsilon + \varepsilon} S(u_\varepsilon + \varepsilon) \nabla v_\varepsilon),\]
in particular, \(\Delta v_\varepsilon\). To overcome the difficulty we turn our eyes to the maximal Sobolev regularity for parabolic equations. This roughly means that \(\Delta v_\varepsilon\) is estimated by \(u_\varepsilon\) in the norm of \(L^p(t_0, T; L^p(\Omega))\). As a consequence, we can establish the \(L^r\)-estimates for \(u_\varepsilon\) in the same way as in the single parabolic equation
\[\frac{\partial u_\varepsilon}{\partial t} = \nabla \cdot (D(u_\varepsilon + \varepsilon)\nabla u_\varepsilon) + \frac{u_\varepsilon}{u_\varepsilon + \varepsilon} S(u_\varepsilon + \varepsilon) u_\varepsilon \in \Omega \times (0, \infty)\]
by virtue of some inequalities, e.g., Hölder’s inequality, Young’s inequality, the Gagliardo–Nirenberg inequality and the mass conservation law
\[\|u_\varepsilon(t)\|_{L^1(\Omega)} = \|u_\varepsilon(0)\|_{L^1(\Omega)}, \quad t \geq 0.\]

Remark 2. The above idea via the maximal Sobolev regularity was already used by [22]. We also employed it in the previous works [13], [14] and [15], however, these have not yet used effectively some decaying effects in the equations. We also note that Cao [3] developed this idea in a three-dimensional chemotaxis-haptotaxis model.

This paper is organized as follows. In the following section we will collect basic and useful inequalities in deriving estimates for solutions. Section 3 is devoted to approximate problems with some estimates; in particular, the main purpose is to establish \(L^r\)-estimate for solutions. In Section 4 we will discuss passage to the limit from approximate problems to (1).

2. Preliminaries. In this section we collect some fundamental estimates for solutions of the following problem for inhomogeneous linear heat equations:
\[
\begin{align*}
\frac{\partial z}{\partial t} &= \Delta z - az + w \quad \text{on } \Omega \times (0, T), \\
z(x, 0) &= z_0(x), \quad x \in \Omega,
\end{align*}
\]
where \(a > 0\) is a constant, with \(\frac{\partial z}{\partial n} = 0\) on \(\partial \Omega\) when \(\Omega\) is bounded.
Lemma 2.1. Let $p \in [1, \infty]$. Let $z_0 \in L^p(\Omega)$ and let $w \in L^1(0, T; L^p(\Omega))$. Then there exists a unique solution $z \in C([0, T]; L^p(\Omega))$ to (9) represented by

$$z(t) = e^{-at} e^{t\Delta} z_0 + \int_0^t e^{-a(t-s)} e^{(t-s)\Delta} w(s) \, ds,$$

where $\{e^{t\Delta}\}_{t \geq 0}$ denotes the heat semigroup on $\mathbb{R}^N$ or the Neumann heat semigroup on the bounded domain $\Omega$. Moreover, the following estimates hold.

(I) $L^p$-$L^q$ estimates.

Let $1 \leq q \leq p \leq \infty$ such that $\frac{1}{q} - \frac{1}{p} < \frac{1}{N}$. Assume that $z_0 \in W^{1,p}(\Omega)$, $w \in L^\infty(0, T; L^2(\Omega))$. Then the solution $z$ belongs to $C([0, T]; W^{1,p}(\Omega))$ with the following estimates:

$$\|z(t)\|_{L^p(\Omega)} \leq e^{-at} \|z_0\|_{L^p(\Omega)} + c_{p,q} \|w\|_{L^\infty(0, T; L^q(\Omega))}, \quad t \in [0, T],$$

$$\|\nabla z(t)\|_{L^q(\Omega)} \leq e^{-at} \|\nabla z_0\|_{L^q(\Omega)} + c'_{p,q} \|w\|_{L^\infty(0, T; L^q(\Omega))}, \quad t \in [0, T],$$

where $c_{p,q} := a^{1-\frac{N}{q}} (\frac{1}{q} - \frac{1}{p}) \Gamma(1 - \frac{N}{q} (\frac{1}{q} - \frac{1}{p})) > 0$, $c'_{p,q} := a^{\frac{1}{2} - \frac{N}{q}} (\frac{1}{q} - \frac{1}{p}) \Gamma(\frac{1}{2} - \frac{N}{2} (\frac{1}{q} - \frac{1}{p})) > 0$, and $\Gamma$ is the Gamma function.

(II) Maximal Sobolev regularity.

Let $p \in (1, \infty)$ and take $z_0 \in W^{2,p}(\Omega)$ (with $\frac{\partial z_0}{\partial \nu} = 0$ on $\partial \Omega$ if $\Omega$ is bounded). If $w \in L^p(0, T; L^p(\Omega))$, then the solution $z$ belongs to $L^p(0, T; W^{2,p}(\Omega))$ and satisfies

$$\int_0^T \|\Delta z(t)\|_{L^p(\Omega)}^p \, dt \leq C_{MR} \left( \|\Delta z_0\|_{L^p(\Omega)}^p + \int_0^T \|w(t)\|_{L^p(\Omega)}^p \, dt \right),$$

where $C_{MR} = C_{MR}(p) > 0$ is a constant independent of $T$. Moreover, if $t_0 \in (0, T)$ and $z(t_0) \in W^{2,p}(\Omega)$ (with $\frac{\partial z(t_0)}{\partial \nu} = 0$ on $\partial \Omega$ if $\Omega$ is bounded), then

$$\int_{t_0}^T \|\Delta z(t)\|_{L^p(\Omega)}^p \, dt \leq C_{MR} \left( \|\Delta z(t_0)\|_{L^p(\Omega)}^p + \int_{t_0}^T \|w(t)\|_{L^p(\Omega)}^p \, dt \right).$$

Proof. Existence of solutions to (9) with the representation formula and $L^p$-$L^q$ estimates are known in the semigroup theory. In order to prove (10) and (11) it suffices to estimate $\int_0^t e^{-a(t-s)} (t-s)^{-\frac{N}{q}} (\frac{1}{q} - \frac{1}{p}) ds$ and $\int_0^t e^{-a(t-s)} (t-s)^{-\frac{N}{q}} (\frac{1}{q} - \frac{1}{p}) ds$, respectively. For the proof of (12) we can refer Ladyženskaja et al. [19, Chapter III] and Cannarsa and Vespri [4] if $\Omega = \mathbb{R}^N$; Hieber and Prüss [7, Theorem 3.1] and Weidemaier [26, Theorem 3.2] if $\Omega$ is bounded. Finally, uniqueness of solutions to (9) yields (13).

Remark 3. Because $G[w](t) := \int_0^T e^{-a(t-s)} e^{(t-s)\Delta} w \, dt$ is a solution of (9) with $z(x, 0) = z_0(x) = 0$, we can see it fulfills

$$\int_0^T \|\Delta G[w](t)\|_{L^p(\Omega)}^p \, dt \leq C_{MR} \int_0^T \|w(t)\|_{L^p(\Omega)}^p \, dt.$$

Remark 4. We have to mention that the regularity of $z_0$ in (12) is absolutely reduced. More precisely, if $z_0 \in B^{2(1-\frac{1}{p})}_{p,p} := (L^p(\Omega), D(\Delta))_{1-\frac{1}{p},p}$, then

$$\int_0^T \left( \|z(t)\|_{L^p(\Omega)}^p + \|\Delta z(t)\|_{L^p(\Omega)}^p \right) \, dt \leq C_{MR} \left( \|z_0\|_{B^{2(1-\frac{1}{p})}_{p,p}}^p + \int_0^T \|w(t)\|_{L^p(\Omega)}^p \, dt \right).$$

The next lemma gives a variant of the Gagliardo–Nirenberg inequality in which we consider the Lebesgue space with exponents less than one (see [6, Lemma 2.1]).
Lemma 2.2 (Gagliardo–Nirenberg type inequality). Let $2^*$ be the Sobolev critical exponent, i.e., $2^* := \infty$ if $N = 1, 2$ and $2^* := \left(\frac{1}{2} - \frac{1}{N}\right)^{-1}$ if $N \geq 3$. Assume that $r \in (0, 2^*)$, $p \in (0, r)$ and $q > 0$. Then there exists some constant $C_{GN} = C_{GN}(N, p, q, r, \Omega) > 0$ such that

$$
\|f\|_{L^r(\Omega)} \leq C_{GN} \left(\|\nabla f\|_{L^2(\Omega)} + \delta_\Omega \|f\|_{L^q(\Omega)}\right)^\theta \|f\|^{1-\theta}_{L^p(\Omega)} \quad \forall f \in H^1(\Omega),
$$

where the power $\theta$ is determined as

$$
\theta = \frac{1}{N} - \frac{2}{2} + \frac{1}{p} \in (0, 1),
$$

and $\|f\|_{L^q(\Omega)} := \left(\int_\Omega f^q(x) \, dx\right)^{\frac{1}{q}}$ for all $q > 0$, the constant $\delta_\Omega$ is given by $\delta_\Omega = 0$ when $\Omega = \mathbb{R}^N$, and $\delta_\Omega = 1$ when $\Omega$ is a bounded domain.

3. Approximate problems with some estimates. In this section we consider the following non-degenerate parabolic system with $\varepsilon > 0$, which is regarded as an approximate problem of (1):

$$
\begin{align*}
\frac{\partial u_\varepsilon}{\partial t} &= \nabla \cdot (\nabla u_\varepsilon + \varepsilon \nabla v_\varepsilon) - \nabla \cdot \left(\frac{u_\varepsilon}{u_\varepsilon + \varepsilon} S(u_\varepsilon + \varepsilon) \nabla v_\varepsilon\right), \quad x \in \Omega, \ t > 0, \\
\frac{\partial v_\varepsilon}{\partial t} &= \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, \quad x \in \Omega, \ t > 0, \\
u_\varepsilon(x, 0) &= u_{0\varepsilon}(x), \quad v_\varepsilon(x, 0) = v_{0\varepsilon}(x), \quad x \in \Omega.
\end{align*}
$$

In the case that $\Omega$ is bounded, the system is supplemented with homogeneous Neumann boundary condition. The initial data $(u_{0\varepsilon}, v_{0\varepsilon})$ is defined as

$$
u_{0\varepsilon} := [\zeta_\varepsilon (\rho_\varepsilon * \underline{u_0})]|_{\Omega}, \quad v_{0\varepsilon} := [\zeta_\varepsilon (\rho_\varepsilon * \underline{v_0})]|_{\Omega},$$

where $\underline{u_0}$ and $\underline{v_0}$ denote the zero extensions of $u_0$ and $v_0$ on $\mathbb{R}^N$ and $\rho_\varepsilon$ is the mollifier such that $0 \leq \rho_\varepsilon \in C_\infty^c(\mathbb{R}^N)$, $\text{supp} \rho_\varepsilon \subset \overline{B(0, \varepsilon)}$, $\int_{\mathbb{R}^N} \rho_\varepsilon(x) \, dx = 1$, and $\zeta_\varepsilon$ is the cut-off function defined as $\zeta_\varepsilon(x) := \zeta(\varepsilon x)$, where $\zeta$ is a fixed function in $C_\infty^c(\mathbb{R}^N)$ such that $0 \leq \zeta \leq 1$, $\zeta(x) = 1$ $(|x| \leq 1)$, $\zeta(x) = 0$ $(|x| \geq 2)$.

We first give local-in-time existence of solutions to (15). Using the standard technique for non-degenerate parabolic systems, we have the maximal existence time $T_{\max}(\varepsilon)$ depending on the approximate parameter $\varepsilon > 0$, and $T_{\max}(\varepsilon)$ possibly tends to 0 as $\varepsilon \to 0$. The following proposition gives that the interval for local existence actually does not depend on $\varepsilon > 0$ by considering “lower-estimate” for $T_{\max}(\varepsilon)$. The proof is based on [16, Lemma 2.4]. We write a full-new-proof here, because [16] deals with only the case of bounded domains and some conditions do not work.

Lemma 3.1 (Local existence of solutions to approximate problems). Let $\varepsilon \in (0, 1)$, $N \in \mathbb{N}$, $m \geq 1$ and $q \geq 2$. Then there exists $T_{\max}(\varepsilon) \in (0, \infty]$ such that (15) has a unique nonnegative solution $(u_\varepsilon, v_\varepsilon)$ on $[0, T_{\max}(\varepsilon))$ satisfying

$$
\begin{align*}
u_\varepsilon &\in C^1([0, T_{\max}(\varepsilon)); L^p(\Omega)) \cap C([0, T_{\max}(\varepsilon)); W^{2, p}(\Omega)), \\
v_\varepsilon &\in C^1([0, T_{\max}(\varepsilon)); L^p(\Omega)) \cap C([0, T_{\max}(\varepsilon)); W^{2, p}(\Omega))
\end{align*}
$$

for any $\rho > N$, with

$$
\|u_\varepsilon(t)\|_{L^1(\Omega)} = \|u_{0\varepsilon}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)} \quad (\forall t \in [0, T_{\max}(\varepsilon))),
$$

$$
\|v_\varepsilon(t)\|_{L^1(\Omega)} \leq \|v_{0\varepsilon}\|_{L^1(\Omega)} \quad (\forall t \in [0, T_{\max}(\varepsilon))).$$
and,

\[
either \quad T_{\text{max}}(\varepsilon) = \infty \quad \text{or} \quad \limsup_{t \to T_{\text{max}}(\varepsilon)} \left( \|u_\varepsilon(t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(t)\|_{L^\infty(\Omega)} \right) = \infty. \quad (19)
\]

Moreover, for each sufficiently large \(p\) there exists \(T_p > 0\), independent of \(\varepsilon\), such that \(T_{\text{max}}(\varepsilon) \geq T_p\) and

\[
\|u_\varepsilon(t)\|_{L^p(\Omega)}^p \leq \|u_0\|_{L^p(\Omega)}^p + 1 \quad (\forall t \in [0, T_p]).
\]

**Proof.** In the case \(\Omega = \mathbb{R}^N\) the local existence was already proved in [24, Proposition 8, Lemmas 11 and 12] by linearizing the first equation in (15) and applying the theory for analytic semigroups and using the contraction mapping principle. For the rigorous proof of the mass conservation law in the case \(\Omega = \mathbb{R}^N\) see the joint paper with Maeda [10]. In the case for bounded domains, the standard parabolic regularity theory and the fixed point argument (see e.g., Amann [1, Theorems II.1.2.1 and II.1.2.2]) guarantee the local existence. We therefore see that there exists \(T_{\text{max}}(\varepsilon) \in (0, \infty]\) such that (15) has a unique nonnegative solution \((u_\varepsilon, v_\varepsilon)\) on \([0, T_{\text{max}}(\varepsilon)]\) satisfying (16)–(19). We next find the uniform lower estimate for \(T_{\text{max}}(\varepsilon)\) on \(\varepsilon\). Without loss of generality we can assume that \(T_{\text{max}}(\varepsilon) < \infty\). Indeed, if \(T_{\text{max}}(\varepsilon) = \infty\), then we can take and use any finite positive number \(T_{\text{max}}(\varepsilon)\) instead of \(T_{\text{max}}(\varepsilon)\) in the following argument. We first let \(p \in (N, \infty)\) be large (if need, see the conditions in [25, Lemma A.1], [9, Theorem 2.2]) and set

\[
t_{p, \varepsilon} := \sup \{ t \in (0, T_{\text{max}}(\varepsilon)) \mid \|u_\varepsilon(\tau)\|_{L^p(\Omega)}^p \leq \|u_0\|_{L^p(\Omega)}^p + 1, \forall \tau \in (0, t) \}.
\]

Then noting that \(u_\varepsilon \in C([0, T_{\text{max}}(\varepsilon)]; L^p(\Omega))\) by (16), we have \(t_{p, \varepsilon} \in (0, T_{\text{max}}(\varepsilon))\) and \(\|u_\varepsilon(t_{p, \varepsilon})\|_{L^p(\Omega)}^p = \|u_0\|_{L^p(\Omega)}^p + 1\). Because if \(t_\varepsilon = T_{\text{max}}(\varepsilon)\), then the solution has \(L^\infty\)-bounds (by Lemma A.1 in [25], Theorem 2.2 in [9]) and be extended after \(T_{\text{max}}(\varepsilon)\). This contradicts the definition of the maximal existence time. Here, multiplying the first equation in (15) by \(u_\varepsilon^{p-1}\) and integrating it over \(\Omega\) give that for all \(t \in (0, t_{p, \varepsilon})\),

\[
\frac{1}{p} \frac{d}{dt} \|u_\varepsilon(t)\|_{L^p(\Omega)}^p = - \int_\Omega D(u_\varepsilon + \varepsilon) \nabla u_\varepsilon \cdot \nabla u_\varepsilon^{p-1} + \int_\Omega \left( \frac{u_\varepsilon}{u_\varepsilon + \varepsilon} S(u_\varepsilon + \varepsilon) \nabla v_\varepsilon \right) \cdot \nabla u_\varepsilon^{p-1}
\]

\[
= -(p - 1) \int_\Omega D(u_\varepsilon + \varepsilon) u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 + (p - 1) \int_\Omega \frac{u_\varepsilon^{p-1}}{u_\varepsilon + \varepsilon} S(u_\varepsilon + \varepsilon) \nabla u_\varepsilon \cdot \nabla v_\varepsilon
\]

\[
\leq - \frac{(p - 1)}{2} \int_\Omega D(u_\varepsilon + \varepsilon) u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 + \frac{(p - 1)}{2} \int_\Omega \frac{u_\varepsilon^p |S(u_\varepsilon + \varepsilon)|^2}{(u_\varepsilon + \varepsilon)^2} |\nabla v_\varepsilon|^2.
\]

We next take \(p \in (1, \infty)\) such as \(p > 2q + m + 3\). By virtue of the positivity of \(u_\varepsilon\) and the conditions (5) and (6) it follows that

\[
-D(u_\varepsilon + \varepsilon) u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 \leq -k_1 (u_\varepsilon + \varepsilon)^{m-1} u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2
\]

\[
\leq -k_1 u_\varepsilon^{p+m-3} |\nabla u_\varepsilon|^2
\]
as well as
\[
\frac{u_p^p [S(u_\varepsilon + \varepsilon)^2]}{(u_\varepsilon + \varepsilon)^2 D(u_\varepsilon + \varepsilon)} \leq \frac{k_2}{k_1} u_p^p (u_\varepsilon + \varepsilon)^{2q - m - 3}
\]
\[
\leq \frac{k_2}{k_1} \left( \frac{2^{q - m - 3} (u_p^{p+2q-m-3} + \varepsilon^{2q-m-3} u_p^p)}{u_p^{p+2q-m-3}} \right)
\]
when \(2q - m - 3 > 0\),

\[
\leq \frac{k_2}{k_1} \left( \frac{2^{q - m - 3} (u_p^{p+2q-m-3} + \varepsilon^{2q-m-3} u_p^p)}{u_p^{p+2q-m-3}} \right)
\]
when \(2q - m - 3 \leq 0\).

Here we proceed the proof only in the case that \(2q - m - 3 \leq 0\), because the other case can be proved by a small modification. Then we deduce from (11) that
\[
\|\nabla u_\varepsilon(t)\|_{L^\infty(\Omega)} \leq e^{-t} \|\nabla u_0\|_{L^\infty(\Omega)} + c_{\infty, p}^\prime \|u_\varepsilon\|_{L^\infty(0, t_{p, \varepsilon}; L^p(\Omega))} \leq M_v, \quad t \in (0, t_{p, \varepsilon})
\]
with some positive constant \(M_v\) which depends on \(\|u_0\|_{W^{1, \infty}(\Omega)}\) and \(\|u_0\|_{L^p(\Omega)}\) in view of the definition of \(t_{p, \varepsilon}\). Therefore we have
\[
\frac{1}{p} \frac{d}{dt} \|u_\varepsilon(t)\|_{L^p(\Omega)} \leq -\frac{k_1 (p - 1)}{2} \int_\Omega u_p^{p+2q-m-3} |\nabla u_\varepsilon|^2 + \alpha (p - 1) \int_\Omega u_p^{p+2q-m-3}
\]
\[
= -\frac{2k_1 (p - 1)}{(p + m - 1)^2} \|\nabla u_\varepsilon^{p+m-1} \|^2_{L^2(\Omega)} + \alpha (p - 1) \int_\Omega u_p^{p+2q-m-3}
\]
(21)
for all \(t \in (0, t_{p, \varepsilon})\), where \(\alpha := \frac{2^{q - m - 3} k_2^2 M_v^2}{2k_1}\). Let \(p > \max\{2m + 6 - 4q, 2\}\). Then the Gagliardo–Nirenberg type inequality (Lemma 2.2) gives
\[
\int_\Omega u_p^{p+2q-m-3} = \int_\Omega \frac{u_p^{p+m-1}}{2^{(p+2q-m-3)} (p+m-1)} \frac{2^{(p+2q-m-3)} (p+m-1)}{L^{p+m-1}(\Omega)}
\]
\[
= \|u_\varepsilon^{p+2q-m-3}(t)\|_{L^2(\Omega)} + \delta_\Omega \|u_\varepsilon^{p+2q-m-3}(t)\|_{L^{p+m-1}(\Omega)}^\theta
\]
\[
\leq c_1 \left[ \left( \frac{\|\nabla u_\varepsilon^{p+2q-m-3}(t)\|_{L^2(\Omega)} + \delta_\Omega \|u_\varepsilon^{p+2q-m-3}(t)\|_{L^{p+m-1}(\Omega)}^\theta \right) \right]
\]
\[
\times \|u_\varepsilon^{p+2q-m-3}(t)\|^{1-\theta}_{L^{p+m-1}(\Omega)} \frac{2^{(p+2q-m-3)} (p+m-1)}{2^{(p+2q-m-3)} (p+m-1)}
\]
\[
\leq c_1 \left[ \left( \frac{\|\nabla u_\varepsilon^{p+2q-m-3}(t)\|_{L^2(\Omega)} + \delta_\Omega \|u_\varepsilon^{p+2q-m-3}(t)\|_{L^{p+m-1}(\Omega)}^\theta \right) \right]
\]
\[
\times \|u_\varepsilon^{p+2q-m-3}(t)\|^{1-\theta}_{L^{p+m-1}(\Omega)} \frac{2^{(p+2q-m-3)} (p+m-1)}{2^{(p+2q-m-3)} (p+m-1)}
\]
for some constant \(c_1 > 0\), where \(\theta := \frac{p+m-1}{2(p+2q-m-3)} - \frac{p+m-1}{2(p+2q-m-3)} (\frac{2p+1}{2} + \frac{1}{N})^{-1}\).

Now, we know \(\|u_0\|_{L^1(\Omega)} \leq \|\nabla u_\varepsilon\|_{L^1(\Omega)}\) and
\[
\|u_\varepsilon(t)\|_{L^2(\Omega)} \leq \|u_\varepsilon(t)\|_{L^1(\Omega)} \leq \|u_\varepsilon(t)\|_{L^p(\Omega)} \leq \|u_0\|_{L^p(\Omega)} + 1 \right)^{\frac{p-m}{p}}
\]
for \(t \in [0, t_{p, \varepsilon})\) by Hölder’s inequality and the definition of \(t_{p, \varepsilon}\), and these imply that there exist \(c_2, c_3 > 0\) which is independent of \(\varepsilon\) fulfilling
\[
\int_\Omega u_p^{p+2q-m-3} \leq c_2 \|\nabla u_\varepsilon^{p+2q-m-3}(t)\|_{L^2(\Omega)} + \delta_\Omega c_3, \quad t \in (0, t_{p, \varepsilon}).
\]
Let us take \( p > \max\{2N(q - m - 1), 2m + 6 - 4q, 2\} \). This condition means
\[
\theta \cdot \frac{2(p + 2q - m - 3)}{p + m - 1} < 2 \Leftrightarrow 2N(q - m - 1) < p,
\]
and we can use Young’s inequality to have that for all \( t \in (0, t_{p, \varepsilon}) \),
\[
\alpha (p - 1) \int_{\Omega} u_{\varepsilon}^{p+2q-m-3} \leq \alpha (p - 1) \left( c_2 \| \nabla u_{\varepsilon} \|_{L^2(\Omega)}^{p+m-1} + \delta_\Omega c_3 \right)
\]
\[
= (p - 1) \left[ \frac{2k_1}{(p+m-1)^2} \| \nabla u_{\varepsilon} \|_{L^2(\Omega)}^{p+m-1} \right] \frac{\theta (p+2q-m-3)}{p+m-1} \left[ c_2 \left( \frac{2k_1}{(p+m-1)^2} \right) - \theta (p+2q-m-3) \right] + \delta_\Omega (p - 1) \alpha c_3
\]
\[
\leq \frac{2k_1(p - 1)}{(p + m - 1)^2} \| \nabla u_{\varepsilon} \|_{L^2(\Omega)}^{p+m-1} + c_4,
\]
with some constant \( c_4 > 0 \) depending upon \( p \) but independent of \( t \) and \( \varepsilon \). Therefore, combining the above estimate (22) with (21), we obtain
\[
\frac{1}{p} \frac{d}{dt} \| u_{\varepsilon}(t) \|_{L^p(\Omega)}^p \leq c_4
\]
for all \( t \in (0, t_{p, \varepsilon}) \). Integrating this inequality over \( [0, t_{p, \varepsilon}] \) yields
\[
\| u_{\varepsilon}(t_{p, \varepsilon}) \|_{L^p(\Omega)}^p \leq \| u_{0\varepsilon} \|_{L^p(\Omega)}^p + (pc_4) t_{p, \varepsilon},
\]
where the definition of \( t_{p, \varepsilon} \) implies that the left-hand side is equal to \( \| u_{0\varepsilon} \|_{L^p(\Omega)}^p + 1 \) for large \( p \). Consequently, we find for large \( p \) which is at least bigger than \( \max\{N, 2N(q - m - 1), 2m + 6 - 4q, 2\} \) that
\[
t_{p, \varepsilon} \geq T_p := \frac{1}{pc_4}
\]
and \( T_p \) is \( \varepsilon \)-independent. Therefore \( T_{\max}(\varepsilon) \geq t_{p, \varepsilon} \geq T_p \) and (20) holds.

The next proposition gives boundedness of \( u_{\varepsilon}(t) \) uniformly in \([0, T_{\max}(\varepsilon))\) in the norm of \( L^r(\Omega) \).

**Proposition 3.2 (\( L^r \)-estimate).** Let \((u_{\varepsilon}, v_{\varepsilon})\) be a solution of (15) on \([0, T_{\max}(\varepsilon))\). Then
\[
\| u_{\varepsilon}(t) \|_{L^r(\Omega)} \leq K_r \quad (\forall r \in [1, \infty))
\]
for all \( t \in [0, T_{\max}(\varepsilon)) \), where \( K_r > 0 \) is a constant which depends on \( \| u_0 \|_{L^1(\Omega)} \), \( \| u_0 \|_{L^r(\Omega)} \), \( \| v_0 \|_{W^{1, \infty}(\Omega)} \), \( k_1, k_2, m, q, N \) but does not depend on \( t \) and \( \varepsilon \).

**Proof.** Let \( r \in (1, \infty) \). In view of (18) and (20) we know that there exists \( t_r \in (0, T_{\max}(\varepsilon)) \) such that
\[
\| u_{\varepsilon}(t) \|_{L^r(\Omega)} \leq K_{1,r}, \quad t \in [0, t_r]
\]
for some positive constant \( K_{1,r} \) which is independent of \( \varepsilon \). From Lemma 2.1, \( v_{\varepsilon}(t) \) is written as
\[
v_{\varepsilon}(t) = e^{-t}e^{\int_0^t \Delta v_{0\varepsilon} + G[u_{\varepsilon}](t)} dt, \quad t \in [0, t_r],
\]
where \( G[u_{\varepsilon}](t) := \int_0^t e^{-t-s}e^{(t-s)\Delta u_{\varepsilon}} ds \). Hence
\[
\| \Delta v_{\varepsilon}(t) \|_{L^r(\Omega)} \leq e^{-t}t^{\frac{1}{2}} \| \nabla v_{0\varepsilon} \|_{L^r(\Omega)} + \| \Delta G[u_{\varepsilon}](t) \|_{L^r(\Omega)},
\]
\[
\leq e^{-t}t^{\frac{1}{2}} \| v_0 \|_{W^{1, r}(\Omega)} + \| \Delta G[u_{\varepsilon}](t) \|_{L^r(\Omega)}, \quad t \in (0, t_r).
\]
In light of the maximal Sobolev regularity (14) together with (23) we know that
\[
\int_0^t \| \Delta G[u_\varepsilon](t) \|_{L^r(\Omega)}^r \, dt \leq C_{MR} \int_0^t \| u_\varepsilon(t) \|_{L^r(\Omega)}^r \, dt \leq C_{MR} K_{1,r}^r t_r
\]
for all \( \varepsilon > 0 \). In particular, Fatou’s lemma implies
\[
\int_0^t \liminf_{\varepsilon \to 0^+} \| \Delta G[u_\varepsilon](t) \|_{L^r(\Omega)}^r \, dt \leq C_{MR} K_{1,r}^r t_r.
\]
Therefore there exists \( t_0 \in (0, t_r) \) such that
\[
\liminf_{\varepsilon \to 0^+} \| \Delta G[u_\varepsilon](t_0) \|_{L^r(\Omega)}^r \leq C_{MR} K_{1,r}^r,
\]
because if this does not holds for almost all \( t_0 \in (0, t_r) \) then its integration over \( (0, t_r) \) contradicts (25). Choosing a subsequence going to zero which is denoted again by \( \{ \varepsilon \} \), we can assume that
\[
\| \Delta G[u_\varepsilon](t_0) \|_{L^r(\Omega)}^r \leq C_{MR} K_{1,r}^r + 1
\]
for all \( \varepsilon > 0 \). Applying this estimate to (24) with \( t = t_0 \) yields
\[
\| \Delta u_\varepsilon(t_0) \|_{L^r(\Omega)}^r \leq e^{-t_0} t_0^{\frac{1}{2}} \| v_0 \|_{W^{1,r}(\Omega)}^r + (C_{MR} K_{1,r}^r + 1)^{\frac{1}{2}}.
\]
Moreover, from (16) and (17) we have that
\[
u_\varepsilon(t_0), v_\varepsilon(t_0) \in W^{2,\rho}(\Omega), \quad \frac{\partial u_\varepsilon(t_0)}{\partial \nu} = \frac{\partial v_\varepsilon(t_0)}{\partial \nu} = 0 \text{ on } \partial \Omega,
\]
with \( \rho > N \). In what follows we shall show that
\[
\| u_\varepsilon(t) \|_{L^r(\Omega)} \leq K_{2,r}, \quad t \in [t_0, T_{\max}(\varepsilon))
\]
for some positive constant \( K_{2,r} \) which is independent of \( \varepsilon \). For simplicity we use the notation \((u, v)\) instead of \((u_\varepsilon, v_\varepsilon)\). We multiply the first equation in (15) by \( u^{r-1} \) and obtain that for all \( t \in (0, T_{\max}(\varepsilon)) \),
\[
\frac{1}{r} \frac{d}{dt} \| u(t) \|_{L^r(\Omega)}^r = - \int_\Omega D(u + \varepsilon) \nabla u \cdot \nabla u^{r-1} + \int_\Omega \left( \frac{u}{u + \varepsilon} S(u + \varepsilon) \nabla v \right) \cdot \nabla u^{r-1}
\]

\[
= -(r - 1) \int_\Omega D(u + \varepsilon) u^{r-2} |\nabla u|^2 + (r - 1) \int_\Omega \frac{u^{r-1}}{u + \varepsilon} S(u + \varepsilon) \nabla u \cdot \nabla v
\]

\[
= -(r - 1) \int_\Omega D(u + \varepsilon) u^{r-2} |\nabla u|^2 + (r - 1) \int_\Omega \nabla \left( \int_0^u \frac{\sigma^{r-1}}{\sigma + \varepsilon} S(\sigma + \varepsilon) \, d\sigma \right) \cdot \nabla v
\]

\[
= -(r - 1) \int_\Omega D(u + \varepsilon) u^{r-2} |\nabla u|^2 - (r - 1) \int_\Omega \left( \int_0^u \frac{\sigma^{r-1}}{\sigma + \varepsilon} S(\sigma + \varepsilon) \, d\sigma \right) \Delta v.
\]

Here the condition (5) yields that
\[
-D(u + \varepsilon) u^{r-2} |\nabla u|^2 \leq -k_1 (u + \varepsilon)^{m-1} u^{r-2} |\nabla u|^2 \leq -k_1 u^{m+r-3} |\nabla u|^2.
\]

We note that
\[
(s + \varepsilon)^{-2} \leq \begin{cases} 2q^{-2}(s^q-2 + \varepsilon^{q-2}) & \text{when } q > 2, \\ s^q-2 & \text{when } q \in [1, 2] \end{cases}
\]
in order to estimate the second term on the right-hand side of (28). In the following we consider only the case \( q > 2 \). Using (6) and the above inequality yields
\[
\int_0^u \frac{\sigma^{r-1}}{\sigma + \varepsilon} S(\sigma + \varepsilon) \, d\sigma \leq k_2 \int_0^u \sigma^{r-1}(\sigma + \varepsilon)^{q-2} \, d\sigma \\
\leq 2^{q-2} k_2 \int_0^u \sigma^{r-1} (\sigma^{q-2} + \varepsilon^{q-2}) \, d\sigma \\
= 2^{q-2} k_2 \left( \frac{1}{r + q - 2} u^{r+q-2} + \frac{\varepsilon^{q-2}}{r} u^r \right).
\]
Let \( r_1 > 1 \) such that \( \frac{4r_1(r_1-1)}{(r_1+1)m-1} \geq 1 \) and take \( r > r_1 \). Then Young’s inequality gives
\[
\frac{d}{dt} \| u(t) \|_{L^p(\Omega)} \\
\leq -k_1 r (r-1) \int_\Omega u^{m+r-3} |\nabla u|^2 \\
+ 2^{q-2} k_2 r (r-1) \int_\Omega \left( \frac{1}{r + q - 2} u^{r+q-2} + \frac{\varepsilon^{q-2}}{r} u^r \right) |\Delta v| \\
= -\frac{4k_1 r (r-1)}{(r + m - 1)^2} \| \nabla u^{\frac{r}{r+m-1}} (t) \|_{L^2(\Omega)}^2 \\
+ \frac{2^{q-2} k_2 r (r-1)}{r + q - 2} \int_\Omega u^{r+q-2} |\Delta v| + (2\varepsilon)^{q-2} k_2 (r-1) \int_\Omega u^r |\Delta v| \\
\leq -k_1 \| \nabla u^{\frac{r}{r+m-1}} (t) \|_{L^2(\Omega)}^2 + 2^{q-2} k_2 r \| u(t) \|_{L^{r+q-1}(\Omega)}^{r+q-1} + \| \Delta v(t) \|_{L^{r+q-1}(\Omega)}^{r+q-1} \\
+ (2\varepsilon)^{q-2} k_2 r \| u(t) \|_{L^{r+1}(\Omega)}^{r+1} + \| \Delta v(t) \|_{L^{r+1}(\Omega)}^{r+1} \\
=: -I + J + J_\varepsilon
\]
for all \( t \in (0, T_{\text{max}}(\varepsilon)) \). We next estimate \( \| u(t) \|_{L^{r+q-1}(\Omega)}^{r+q-1} \) and \( \| u(t) \|_{L^{r+1}(\Omega)}^{r+1} \). Lemma 2.2 and (18) give that for any \( p \in (1, \infty) \) and \( r > p(1 - \frac{2}{n}) - m + 1 \) there exists a constant \( c_1 > 0 \) fulfilling
\[
\| u(t) \|_{L^p(\Omega)}^p \\
\leq c_1 \left( \| \nabla u^{\frac{r}{2}} (t) \|_{L^2(\Omega)} + \delta \| u^{\frac{r}{2}} (t) \|_{L^\frac{2p}{r-m-1}(\Omega)} \right)^\theta \\
\times \| u^{\frac{r}{2}} (t) \|_{L^\frac{2p}{r+m-1}(\Omega)}^{1-\theta} \\
\leq c_1 \left( \| \nabla u^{\frac{r}{2}} (t) \|_{L^2(\Omega)} + \delta \| u^{\frac{r}{2}} (t) \|_{L^1(\Omega)} \right)^\theta \| u^{\frac{r}{2}} (t) \|_{L^1(\Omega)}^{1-\theta} \| u_0 \|_{L^1(\Omega)}^{(r+m-1)(1-\theta)},
\]
where \( \theta := \frac{r+m-1}{2} - \frac{r+m-1}{2p} \left( \frac{r+m-1}{2} - \frac{1}{2} + \frac{1}{N} \right)^{-1} \). To reduce the argument we consider only the case \( \Omega = \mathbb{R}^N \) and then \( \delta = 0 \), because we can easily modify the proof in the case of bounded domains. We will apply this estimate with \( p = r+q-1 \) and \( p = r+1 \). Here we note that the exponents satisfy
\[
\theta \cdot \left. \frac{2p}{r+m-1} \right|_{p=r+q-1} < 2 \iff (2 <) q < m + \frac{2}{N},
\]
\[
\theta \cdot \left. \frac{2p}{r+m-1} \right|_{p=r+1} < 2 \iff 2 < m + \frac{2}{N}.
\]
Let us take \( q < m + \frac{2}{N} \) and \( r > \max\{r_1, \frac{N}{2}(q - m) - q + 1\} \) and set \( \theta_q := \theta_{p=r+q-1} \). Then (30) (with \( \delta_0 = 0 \)) together with Young’s inequality gives

\[
2^{q-2}k_2r\|u(t)\|_{L^{r+q-1}(\Omega)}^{r+q-1} \leq 2^{q-2}k_2c_1r\left(\|\nabla u\|_{L^{2r}(\Omega)}^{\frac{r+m-1}{2}}\|u_0\|_{L^2(\Omega)}^{\frac{r+m-1}{2}}(1-\theta_q)\right)^{\frac{2(r+q-1)}{r+m-1}} + \frac{1}{4}\mathcal{I} + \left(\frac{1}{N}\right)^{\frac{q(r+q-1)}{r+m-1}}2^{q-2}k_2c_1r\|u_0\|_{L^1(\Omega)}(1-\theta_q)\right)^{\frac{r+m-1}{r+m-1-q(q+1)}}.
\] (31)

Since

\[
\frac{\theta_q(r+q-1)}{r+m-1} = O(1),
\]

\[
\frac{r+m-1}{r+m-1-\theta_q(r+q-1)} = \frac{r+m-2+\frac{2}{N}}{m-q+\frac{2}{N}} = O(r),
\]

\[
(r+q-1)(1-\theta_q) = \frac{2}{N}r + (m - 2 + \frac{2}{N})(q-1) -(m-1)(q-2) = O(1)
\]

as \( r \to \infty \), we can find \( c_2, c_3 \) such that

\[
\left(\frac{1}{N}\right)^{\frac{q(r+q-1)}{r+m-1}}2^{q-2}k_2c_1r\|u_0\|_{L^1(\Omega)}(1-\theta_q)\right)^{\frac{r+m-1}{r+m-1-q(q+1)}} \leq (c_2r)^{c_3r}
\] (32)

for all \( r > r_2 \) with some \( r_2 \geq 1 \). Take \( r > \max\{\frac{N}{2}(q - m) - q + 1, r_1, r_2\} \). Then (31) and (32) yield

\[
\mathcal{J} \leq \frac{1}{4}\mathcal{I} + (c_2r)^{c_3r} + 2^{q-2}k_2r\|\Delta u(t)\|_{L^{r+q-1}(\Omega)}^{r+q-1}.
\] (33)

The same argument can be applied to \( \mathcal{J}_\varepsilon \), i.e.,

\[
\mathcal{J}_\varepsilon \leq \frac{1}{4}\mathcal{I} + \left(\frac{1}{N}\right)^{\frac{q(r+q-1)}{r+m-1}}(2\varepsilon)^{q-2}k_2r\|u_0\|_{L^1(\Omega)}(1-\theta_q)\right)^{\frac{r+m-1}{r+m-1-q(q+1)}}
\]

\[
+ (2\varepsilon)^{q-2}k_2r\|\Delta u(t)\|_{L^{r+1}(\Omega)}^{r+1} + (2\varepsilon)^{q-2}k_2r\|\Delta u(t)\|_{L^{r+1}(\Omega)}^{r+1}
\] (34)

for all \( r > \max\{\frac{N}{2}(2m - 1), r_1, r_3\} \) with some \( r_3 \geq 1 \) and positive constants \( c_2', c_3' \). From (29), (33) and (34) we obtain the following estimate

\[
\frac{d}{dt}\|u(t)\|_{L^r(\Omega)} \leq -\frac{1}{2}\mathcal{I} + (c_2r)^{c_3r} + (c_2r)^{c_3r}
\]

\[
+ 2^{q-2}k_2r\|\Delta u(t)\|_{L^{r+q-1}(\Omega)}^{r+q-1} + (2\varepsilon)^{q-2}k_2r\|\Delta u(t)\|_{L^{r+1}(\Omega)}^{r+1}
\] (35)

for \( r \geq \max\{\frac{N}{2}(q - m) - q + 1, \frac{N}{2}(2 - m) - 1, r_1, r_2, r_3\} \) and for \( t \in (0, T_{\text{max}}(\varepsilon)) \).

We now notice that Lemma 2.2 with \( \delta_0 = 0 \) gives for \( r > (1 - \frac{2}{N}) - m + 1 \), i.e., for \( r > \frac{N}{2}(-m + 1) \),

\[
\|u(t)\|_{L^r(\Omega)} \leq c_4\left(\|\nabla u\|_{L^2(\Omega)}^{\frac{r+m-1}{2}}\|u_0\|_{L^1(\Omega)}^{\frac{r+m-1}{2}}(1-\theta)\right)^{\frac{2}{r+m-1}}
\]

\[
\leq \|u(t)\|_{L^r(\Omega)} \leq c_4\|\nabla u\|_{L^2(\Omega)}^{\frac{r+m-1}{2}}\|u_0\|_{L^1(\Omega)}^{\frac{r+1}{2}}(1-\theta)
\]

where 

\( c_4 \) is a positive constant.
\[ \Rightarrow (c_4 \|u_0\|_{L^1(\Omega)}^{r(1-\theta)})^{-1} \|u(t)\|_{L^r(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)}^{\frac{2p}{2p+m}} \]

\[ \Rightarrow (c_4 \|u_0\|_{L^1(\Omega)}^{r(1-\theta)})^{-1} \|u(t)\|_{L^r(\Omega)}^{\frac{r+m-1}{r}} \leq \|\nabla u\|_{L^2(\Omega)}^{\frac{2p}{2p+m}} \]

\[ \Rightarrow -\frac{1}{4} \mathcal{I} = -\frac{k_1}{4} \|\nabla u\|_{L^2(\Omega)}^{\frac{2p}{2p+m}} \leq -\frac{k_1}{4} (c_4 \|u_0\|_{L^1(\Omega)}^{r(1-\theta)})^{\frac{r+m-1}{r}} \|u(t)\|_{L^r(\Omega)}^{\frac{r+m-1}{r}} \]

with some constant \( c_4 > 0 \), where \( \theta = (\frac{r+m-1}{2} - \frac{r+m-1}{2}) (\frac{r+m-1}{2} - \frac{1}{N})^{-1} \). Noting that

\[ r < \frac{r+m-1}{\theta} \]

because \( 1 - \frac{2}{N} < m \), we deduce from Young’s inequality that

\[ -\frac{1}{4} \mathcal{I} \leq -\|u(t)\|_{L^r(\Omega)} + \left( \frac{4}{k_1} \right)^{\frac{r+m-1}{r}} c_4 \|u_0\|_{L^1(\Omega)}^{r(1-\theta)} \]

Take \( r_1 \) such that there are \( c_3', c_3'' > 0 \) fulfilling \( \left( \left( \frac{4}{k_1} \right)^{\frac{r+m-1}{r}} c_4 \|u_0\|_{L^1(\Omega)}^{r(1-\theta)} \right)^{\frac{r+m-1}{r}} \leq c_2^{\frac{r+m-1}{r}} \) for all \( r > r_1 \). Set \( r \geq \max \left\{ \frac{2N}{N} (q - m) - q + 1, \frac{N}{2} (2 - m) - 1, r_1, r_2, r_3, r_4 \right\} \). Then it follows from (35) that for all \( t \in (0, T_{\text{max}}(\varepsilon)) \),

\[ \frac{d}{dt} \|u(t)\|_{L^r(\Omega)}^r \leq -\|u(t)\|_{L^r(\Omega)}^r + \left( \frac{r}{k_2} \right)^{\frac{r+m-1}{r}} + \frac{1}{4} \mathcal{I} \]

where \( \bar{c}_2 := \max\{c_1, c_2, c'_2, c''_2\} \). Let \( t_0 \in (0, t_\ast) \) as in (26) and let \( t \in (t_0, T_{\text{max}}(\varepsilon)) \). Then the above ordinary differential inequality for \( \|u(t)\|_{L^r(\Omega)}^r \) yields

\[ \|u(t)\|_{L^r(\Omega)}^r \leq e^{t_{\ast} - t} \|u(t_0)\|_{L^r(\Omega)}^r + (\bar{c}_2 r)^{\bar{c}_3 r} (1 - e^{t_{\ast} - t}) - \frac{1}{4} \int_{t_0}^{t} e^{-\tau r} \mathcal{I} \, d\tau \]

\[ + 2q^{-2} k_2 r \int_{t_0}^{t} e^{s-t} \|\Delta v(s)\|_{L^{r+q-1}(\Omega)}^{r+q-1} \, ds \]

\[ + (2\varepsilon)^{q-2} k_2 r \int_{t_0}^{t} e^{s-t} \|\Delta v(s)\|_{L^{r+q+1}(\Omega)}^{r+q+1} \, ds \]

\[ \leq e^{t_{\ast} - t} \|u(t_0)\|_{L^r(\Omega)}^r + (\bar{c}_2 r)^{\bar{c}_3 r} (1 - e^{t_{\ast} - t}) - \frac{1}{4} \int_{t_0}^{t} e^{-\tau r} \mathcal{I} \, d\tau \]

\[ + 2q^{-2} k_2 r \int_{t_0}^{t} \|\Delta e^{t_{\ast} - \tau} v(s)\|_{L^{r+q-1}(\Omega)}^{r+q-1} \, ds \]

\[ + (2\varepsilon)^{q-2} k_2 r \int_{t_0}^{t} \|\Delta e^{t_{\ast} - \tau} v(s)\|_{L^{r+q+1}(\Omega)}^{r+q+1} \, ds. \] (36)

We next estimate the last two terms. Let us set

\[ \bar{u}(x, s) := e^{\frac{s-t}{r}} u(x, s), \quad \bar{v}(x, s) := e^{\frac{s-t}{p}} v(x, s) \quad (p > 1). \]

Then the second equation in (15) gives

\[ \frac{\partial \bar{v}}{\partial s} = e^{\frac{s-t}{p}} \left( \frac{\partial v}{\partial s} + \frac{1}{p} v \right) = e^{\frac{s-t}{r}} \left( \Delta v - v + u + \frac{1}{p} v \right) = \Delta \bar{v} - \left( 1 - \frac{1}{p} \right) \bar{v} + \bar{u}. \]

Applying the maximal Sobolev regularity (Lemma 2.1, (II)) to this equation gives

\[ \int_{t_0}^{t} \|\Delta \bar{v}(s)\|^p_{L^r(\Omega)} \, ds \leq C_{\text{MR}}^{(p)} \left( \|\Delta \bar{v}(t_0)\|^p_{L^r(\Omega)} + \int_{t_0}^{t} \|\bar{u}(s)\|^p_{L^r(\Omega)} \, ds \right). \] (37)
Then (37) with \( p = r + q - 1 \) (or \( p = r + 1 \)) and the argument (31)-(32) imply
\[
2^{q-2} k_r \int_{t_0}^t \| \Delta e^{\frac{r+q-1}{r+1} v(s)} \|_{L^{r+q-1}(\Omega)}^r ds
\]
\[
= 2^{q-2} k_r \int_{t_0}^t \| \Delta \hat{v}(s) \|_{L^{r+q-1}(\Omega)}^r ds
\]
\[
\leq 2^{q-2} k_r C_{\text{MR}}^{(r+q-1)} \left( \| \Delta \hat{v}(t_0) \|_{L^{r+q-1}(\Omega)}^{r+q-1} + \int_{t_0}^t \| \hat{u}(s) \|_{L^{r+q-1}(\Omega)}^{r+q-1} ds \right)
\]
\[
= 2^{q-2} k_r C_{\text{MR}}^{(r+q-1)} \left( \| \Delta \hat{v}(t_0) \|_{L^{r+q-1}(\Omega)}^{r+q-1} + \int_{t_0}^t \| e^{\frac{r+q-1}{r+1} u(s)} \|_{L^{r+q-1}(\Omega)}^{r+q-1} ds \right)
\]
\[
= 2^{q-2} k_r C_{\text{MR}}^{(r+q-1)} \left( \| \Delta \hat{v}(t_0) \|_{L^{r+q-1}(\Omega)}^{r+q-1} + \int_{t_0}^t e^{r-1} \| u(s) \|_{L^{r+q-1}(\Omega)}^{r+q-1} ds \right)
\]
\[
\leq 2^{q-2} k_r C_{\text{MR}}^{(r+q-1)} \| \Delta \hat{v}(t_0) \|_{L^{r+q-1}(\Omega)}^{r+q-1} + \int_{t_0}^t e^{r-1} \left[ \frac{1}{8} I + (\epsilon^{q-2} C_{\text{MR}}^{(r+q-1)})^{-1} \right] ds.
\]

and also
\[
(2\epsilon)^{q-2} k_r \int_{t_0}^t \| \Delta e^{\frac{r+q-1}{r+1} v(s)} \|_{L^{r+1}(\Omega)}^{r+1} ds
\]
\[
\leq (2\epsilon)^{q-2} k_r C_{\text{MR}}^{(r+1)} \| \Delta \hat{v}(t_0) \|_{L^{r+1}(\Omega)}^{r+1} + \int_{t_0}^t e^{r-1} \left[ \frac{1}{8} I + (\epsilon^{q-2} C_{\text{MR}}^{(r+1)})^{-1} \right] ds.
\]

Connecting (36) and the above two estimates, we deduce
\[
\| u(t) \|_{L^r(\Omega)}^r \leq e^{r-1} \| u(t_0) \|_{L^r(\Omega)}^r + (\epsilon^{q-2} C_{\text{MR}}^{-1}) \left( 1 - e^{r-1} \right)
\]
\[
+ 2^{q-2} k_r C_{\text{MR}}^{(r+q-1)} \| \Delta \hat{v}(t_0) \|_{L^{r+q-1}(\Omega)}^{r+q-1} + \int_{t_0}^t e^{r-1} (\epsilon^{q-2} C_{\text{MR}}^{(r+q-1)})^{-1} ds
\]
\[
+ (2\epsilon)^{q-2} k_r C_{\text{MR}}^{(r+1)} \| \Delta \hat{v}(t_0) \|_{L^{r+1}(\Omega)}^{r+1} + \int_{t_0}^t e^{r-1} (\epsilon^{q-2} C_{\text{MR}}^{(r+1)})^{-1} ds
\]
\[
\leq \| u(t_0) \|_{L^r(\Omega)}^r + \left[ \epsilon^{q-2} C_{\text{MR}}^{-1} + (\epsilon^{q-2} C_{\text{MR}}^{-1}) \right] \left[ (\epsilon^{q-2} C_{\text{MR}}^{-1})^{-1} + (\epsilon^{q-2} C_{\text{MR}}^{-1})^{-1} \right]
\]
\[
+ 2^{q-2} k_r C_{\text{MR}}^{(r+q-1)} \| \Delta \hat{v}(t_0) \|_{L^{r+q-1}(\Omega)}^{r+q-1} + (2\epsilon)^{q-2} k_r C_{\text{MR}}^{(r+1)} \| \Delta \hat{v}(t_0) \|_{L^{r+1}(\Omega)}^{r+1}.
\]

Here (23) asserts that
\[
\| u(t_0) \|_{L^r(\Omega)}^r \leq K_{1,r},
\]
and (26) guarantees that
\[
\| \Delta \hat{v}(t_0) \|_{L^q(\Omega)}^p = \| e^{\frac{r+q-1}{r+1} \Delta v(t_0)} \|_{L^p(\Omega)} \leq \| \Delta v(t_0) \|_{L^p(\Omega)} \leq K_{1,p} \quad (p = r + q - 1, r + 1)
\]
with some constant \( K_{1,p} \), independent of \( \epsilon \). Altogether, (38) means that (27) holds with \( r \geq \max \{ \frac{N}{2} (q - m) - q + 1, \frac{N}{2} (2 - m) - 1, r_1, r_2, r_3, r_4 \} \). This completes the proof.

Once we have \( L^r \)-boundedness, we can achieve the following boundedness results by means of a Moser-type iteration argument.
Lemma 3.3 \((L^\infty\text{-estimate})\). Let \((u_\varepsilon, v_\varepsilon)\) be a solution of \((15)\) on \([0, T_{\text{max}}(\varepsilon))\). Then
\[
\|u_\varepsilon(t)\|_{L^\infty(\Omega)} \leq K_\infty, \tag{39}
\|v_\varepsilon(t)\|_{W^{1,\infty}(\Omega)} \leq L_\infty \tag{40}
\]
for all \(t \in [0, T_{\text{max}}(\varepsilon))\), where \(K_\infty, L_\infty > 0\) are constants which do not depend on \(\varepsilon, t\). Consequently, \(T_{\text{max}}(\varepsilon) = \infty\).

Proof. Since Proposition 3.2 holds for suitable large \(r\), \((39)\) is established by means of iteration arguments in \([9]\) and \([25, \text{Appendix}]\). This boundedness and the \(L^p-L^q\) estimate (Lemma 2.1, \((I)\)) ensure \((40)\). \(\square\)

We conclude this section with a regularity estimate for \(\int_0^{u_\varepsilon} D(\sigma + \varepsilon) \, d\sigma\).

Lemma 3.4. Let \((u_\varepsilon, v_\varepsilon)\) be a solution of \((15)\) on \([0, \infty)\). Then
\[
\int_0^t \| \nabla \int_0^{u_\varepsilon(s)} D(\sigma + \varepsilon) \, d\sigma \|^2_{L^2(\Omega)} \, ds \leq K_D(t + 1) \tag{41}
\]
for all \(t \in [0, \infty)\), where \(K_D > 0\) is a constant which does not depend on \(\varepsilon, t\).

Proof. Multiplying the first equation in \((15)\) by \(\int_0^{u_\varepsilon} D(\sigma + \varepsilon) \, d\sigma\) and integrating it over \(\Omega\), we see from \((39)\) that for all \(t \in (0, \infty)\),
\[
\frac{d}{dt} \int_\Omega \int_0^{u_\varepsilon} D(\sigma + \varepsilon) \, d\sigma + \int_\Omega D(u_\varepsilon + \varepsilon) \nabla u_\varepsilon \cdot \nabla \left( \int_0^{u_\varepsilon} D(\sigma + \varepsilon) \, d\sigma \right) = \int_\Omega \frac{u_\varepsilon}{u_\varepsilon + \varepsilon} S(u_\varepsilon + \varepsilon) \nabla v_\varepsilon \cdot \nabla \left( \int_0^{u_\varepsilon} D(\sigma + \varepsilon) \, d\sigma \right)
\leq c_1 \| \nabla v_\varepsilon(t) \|_{L^2(\Omega)} \| \nabla \left( \int_0^{u_\varepsilon(t)} D(\sigma + \varepsilon) \, d\sigma \right) \|_{L^2(\Omega)}
\]
for some constant \(c_1 > 0\). Rewriting the second term on the left-hand side as
\[
\int_\Omega D(u_\varepsilon + \varepsilon) \nabla u_\varepsilon \cdot \nabla \left( \int_0^{u_\varepsilon} D(\sigma + \varepsilon) \, d\sigma \right) = \| \nabla \int_0^{u_\varepsilon(t)} D(\sigma + \varepsilon) \, d\sigma \|^2_{L^2(\Omega)},
\]
we have
\[
\frac{d}{dt} \int_\Omega \int_0^{u_\varepsilon} D(\sigma + \varepsilon) \, d\sigma + \frac{1}{2} \| \nabla \int_0^{u_\varepsilon(t)} D(\sigma + \varepsilon) \, d\sigma \|^2_{L^2(\Omega)} \leq \frac{c_1^2}{2} \| \nabla v_\varepsilon(t) \|^2_{L^2(\Omega)}
\]
for all \(t \in (0, \infty)\). Integrating this inequality and using \((45)\) (see below) imply
\[
\int_\Omega \int_0^{u_\varepsilon(t)} D(\sigma + \varepsilon) \, d\sigma + \frac{1}{2} \int_0^t \| \nabla \int_0^{u_\varepsilon(s)} D(\sigma + \varepsilon) \, d\sigma \|^2_{L^2(\Omega)} \, ds \leq K_D(t + 1)
\]
for all \(t \in [0, \infty)\) with some \(K_D > 0\). This proves \((41)\). \(\square\)

4. Passage to the limit. Proof of Theorem 1.2. In the previous section we constructed a family of global-in-time approximate solutions \((u_\varepsilon, v_\varepsilon)\). In this section we will discuss passage to the limit as \(\varepsilon \to 0\) to obtain the desired global-in-time bounded weak solution of \((1)\). To this end we need one more lemma which will be used in applying the Lions–Aubin compactness theorem.
Lemma 4.1. Let \((u_\varepsilon, v_\varepsilon)\) be a solution of (15) on \([0, \infty)\). Then for all \(r \geq \max\{-2q + 3m + 2, (m + 1), 2(-q + m + 2), 4(-q + m + 1)\},
\[
\| \int_0^T \int_\Omega D^2(\sigma + \varepsilon)\sigma^\varepsilon \, d\sigma \|_{L^2(0,T;L^2(\Omega))} \leq C_1T,
\]
\[
\| \frac{d}{dt} \int_0^T \int_\Omega D^2(\sigma + \varepsilon)\sigma^\varepsilon \, d\sigma \|_{L^1(0,T;L^2(\Omega))} \leq C_2 T \quad \text{with some } k > \frac{N}{2}
\] for all \(T > 1\), where \(C_1, C_2 > 0\) are constants which do not depend on \(\varepsilon\).

Proof. First, by virtue of Proposition 3.2, (39) and the continuity of \(D\) in (4) we see that for all \(r \geq 2\) and \(T \in (0, \infty)\),
\[
\| \int_0^T \int_\Omega D^2(\sigma + \varepsilon)\sigma^\varepsilon \, d\sigma \|_{L^2(0,T;L^2(\Omega))} \leq \left( \max_{\sigma \in [0,K_{\infty} + 1]} D^2(\sigma) \right)^{1/2} \left( \int_0^T \int_\Omega \sigma^\varepsilon \, d\sigma \right)^{1/2} dx \, ds \leq \frac{4}{r^2} \left( \max_{\sigma \in [0,K_{\infty} + 1]} D^2(\sigma) \right)^{1/2} K_r^{1/2} T.
\]
Next we may invoke (28) in Proposition 3.2, the conditions (5) and (6) to see that for all \(r \geq 2\) and \(t \in (0, \infty)\),
\[
\frac{1}{r} \left\{ -\frac{1}{2} \int_\Omega D(u_\varepsilon + \varepsilon)u_\varepsilon^{-2}\|\nabla u_\varepsilon\|^2 \right\} = -\frac{r - 1}{2} \int_\Omega D(u_\varepsilon + \varepsilon)u_\varepsilon^{-2}\|\nabla u_\varepsilon\|^2 + (r - 1) \int_\Omega \frac{u_\varepsilon^{-1}}{u_\varepsilon + \varepsilon} \lambda(u_\varepsilon + \varepsilon) \nabla u_\varepsilon \cdot \nabla u_\varepsilon \leq -\frac{k_1(r - 1)}{2} \int_\Omega (u_\varepsilon + \varepsilon)^{m-1} u_\varepsilon^{-2}\|\nabla u_\varepsilon\|^2 + k_2(r - 1) \int_\Omega u_\varepsilon^{r-1}(u_\varepsilon + \varepsilon)^{q-2}\|\nabla u_\varepsilon \cdot \nabla v_\varepsilon\| \leq \frac{k_2^2(r - 1)}{k_1} \int_\Omega u_\varepsilon^r(u_\varepsilon + \varepsilon)^{2q-m-3}\|\nabla v_\varepsilon\|^2,
\]
where we used Young’s inequality in the last line. Combining (11) with \(p = q = 2\) and Proposition 3.2 with \(r = 2\), we have
\[
\|\nabla v_\varepsilon(t)\|_{L^2(\Omega)} \leq e^{-t} \|\nabla v_0\|_{L^2(\Omega)} + \sqrt{\pi} K_2 \leq c_1, \quad t \in (0, \infty)
\] for some constant \(c_1 > 0\). Hence (39) gives that for all \(r \geq \max\{-2q + m + 2, 3\},
\[
\frac{k_2^2}{k_1} \int_\Omega u_\varepsilon^r(u + \varepsilon)^{2q-m-3}\|\nabla v_\varepsilon\|^2 \leq \frac{k_2^2}{k_1} (K_\infty + 1)^{r+2q-m-3} c_1^2,
\]
and so, plugging this estimate into (44) and integrating it over \((0, T)\) yield
\[
\frac{1}{r} \|u_\varepsilon(T)\|^r_{L^r(\Omega)} + \frac{r - 1}{2} \int_0^T \int_\Omega D(u_\varepsilon + \varepsilon)u_\varepsilon^{-2}\|\nabla u_\varepsilon\|^2 \leq \frac{1}{r} \|u_0\|^r_{L^r(\Omega)} + \frac{(r - 1)k_2^2}{k_1}(K_\infty + 1)^{r+2q-m-3} c_1^2 T
\]
for all $T \in (0, \infty)$. Noting that
\[ \int_{\Omega} D(u_s + \varepsilon)u_s^{r-2}|\nabla u_s|^2 = \left\| \nabla \int_0^{u_s} D^{\frac{1}{2}}(\sigma + \varepsilon)\sigma^{\frac{r-2}{2}} d\sigma \right\|^2_{L^2(\Omega)}, \tag{46} \]
we therefore have the estimate (42).

Before proving (43), we note from (42) and $T \geq 1$ that for all $\gamma \geq \max\{-q + m + 1, \frac{m+1}{2}\}$ there exist constants $c_2, c_3 > 0$ satisfying
\[ \|\nabla u_s\|^2_{L^2(0,T;L^2(\Omega))} \leq c_2 T, \tag{47} \]
\[ \|\nabla v_s\|^2_{L^2(0,T;L^2(\Omega))} \leq c_3 \sqrt{T} \leq c_3 T. \tag{48} \]

Indeed, the condition (5) implies that
\[ D(u_s + \varepsilon)u_s^{r-2}|\nabla u_s|^2 \geq k_1 u_s^{m+r-3}|\nabla u_s|^2 = \frac{4k_1}{(m + r - 1)^2} |\nabla u_s|^{m+r-1}, \]
which combined with (42) and (46) yield (47). Moreover, (48) directly follows from (45).

We now prove (43). Let $r \geq \max\{m + 1, 4\}$. Let $\varphi \in H^k(\Omega)$ with some $k > \frac{N}{2}$. Then it follows from the first equation in (15) that
\[
\int_{\Omega} \frac{d}{dt} \int_0^{u_s} \frac{\partial}{\partial t} \left( T(u_s + \varepsilon)u_s^{r-2} \varphi \right) = \int_{\Omega} \left( D^{\frac{1}{2}}(u_s + \varepsilon)u_s^{r-2} \nabla u_s \right) \cdot \left( D(u_s + \varepsilon)\nabla u_s - \frac{u_s}{u_s + \varepsilon} S(u_s + \varepsilon)\nabla v_s \right) \varphi
\[
\leq c_1 \left( \int_{\Omega} \left| D^{\frac{1}{2}}(u_s + \varepsilon)u_s^{r-2} \varphi \right|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} \left| D(u_s + \varepsilon)\nabla u_s \right|^2 \right)^{\frac{1}{2}} \varphi
\[
\leq c_4 \left( u_s^{r-2} \left| \nabla u_s \right|^2 \varphi + u_s^{r-4} \left| \nabla \varphi \right|^2 \right)
\]
for some constant $c_4 > 0$ and a similar computation with the condition (5) implies
\[
\leq c_5 \left( \int_{\Omega} \left| D^{\frac{1}{2}}(u_s + \varepsilon)u_s^{r-2} \varphi \right|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} \left| D(u_s + \varepsilon)\nabla u_s \right|^2 \right)^{\frac{1}{2}} \varphi
\[
+ D^{\frac{1}{2}}(u_s + \varepsilon)u_s^{r-2} \left| \nabla \varphi \right|^2 \left| \nabla v_s \right|^2 \varphi
\[
\leq c_6 \left( u_s^{r-2} \left| \nabla u_s \right|^2 \varphi + u_s^{r-4} \left| \nabla \varphi \right|^2 \right)
\]
for some constants $c_5, c_6 > 0$. Here we note that for all $\alpha, \beta > 0$,
\[
\int_{\Omega} \left| \nabla u_s \right|^2 \varphi \leq \frac{4}{(\alpha + 2)^2} \left( \int_{\Omega} \left| D^{\frac{1}{2}}(t) \right|^2 L^{2}(\Omega) \right) \left( \int_{\Omega} \left| \nabla u_s \right|^2 \varphi \right)
\[
\leq c_7 \left( \int_{\Omega} \left| D^{\frac{1}{2}}(t) \right|^2 L^{2}(\Omega) \right) \left( \int_{\Omega} \left| \nabla \varphi \right|^2 \right).
\]
\[ \int_{\Omega} u_{\varepsilon}^2 |\nabla u_{\varepsilon}||\nabla \varphi| \leq \frac{1}{\beta+1} |\nabla u_{\varepsilon}^{\beta+1}(t)||\nabla \varphi||L^2(\Omega) \leq c_8 \left( |\nabla u_{\varepsilon}^{\beta+1}(t)||L^2(\Omega) + 1 \right) \|\varphi\|_{H^\beta(\Omega)}, \]
\[ \leq c_8 \left( |\nabla v_{\varepsilon}^{\beta+1}(t)||L^2(\Omega) + 1 \right) \|\varphi\|_{H^\beta(\Omega)}, \]
\[ \int_{\Omega} |\nabla v_{\varepsilon}||\nabla \varphi| \leq |\nabla v_{\varepsilon}(t)||L^2(\Omega)||\nabla \varphi||L^2(\Omega) \leq \left( |\nabla v_{\varepsilon}(t)||L^2(\Omega) + 1 \right) \|\varphi\|_{H^\beta(\Omega)}, \]
for some constants \(c_7, c_8 > 0\), where we used \(H^k(\Omega) \hookrightarrow L^\infty(\Omega)\). Hence we obtain
\[ \left\| \frac{d}{dt} \int_0^u D^\frac{1}{2}(\sigma + \varepsilon)^{\frac{r-2}{2}} \,d\sigma \right\|_{(H^\nu(\Omega))^*} \]
\[ \leq c_8 \left( |\nabla u_{\varepsilon}^{\frac{r}{2}}||L^2(\Omega) + |\nabla u_{\varepsilon}^{\frac{r}{2}}||L^2(\Omega) + |\nabla u_{\varepsilon}^{\frac{r-1}{2}}||L^2(\Omega) + |\nabla u_{\varepsilon}^{\frac{r-1}{2}}||L^2(\Omega) + |\nabla u_{\varepsilon}^{\frac{r-1}{2}}||L^2(\Omega) + \|\nabla v_{\varepsilon}||L^2(\Omega) + 4 \right) \]
for some constant \(c_9 > 0\). Integrating this estimate upon \((0,T)\), we use (47) and (48) to obtain (43) for \(r \geq \max\{2(m+1), -2q+3m+1, 2(-q+m+2), 4(-q+m+1)\}. \]

We now discuss convergences for \(\{u_{\varepsilon}\}_{\varepsilon > 0}\) and \(\{v_{\varepsilon}\}_{\varepsilon > 0}\).

**Lemma 4.2.** Let \((u_{\varepsilon}, v_{\varepsilon})\) be a solution of (15) on \([0, \infty)\). Then there exist a subsequence of \(\{\varepsilon\}\), which is denoted by itself, and nonnegative functions \(u \in L^\infty(0, \infty; L^\infty(\Omega)), \quad v \in L^\infty(0, \infty; W^{1,\infty}(\Omega))\) such that
\[ \int_0^u D(\sigma) \,d\sigma \in L^2(0,T; H^1(\Omega)) \quad (\forall T > 0) \]
and the following convergences hold as \(\varepsilon \to 0\):
\[ u_{\varepsilon} \to u \quad \text{weakly* in } L^\infty(0, \infty; L^\infty(\Omega)), \quad \nabla \int_0^u D(\sigma + \varepsilon) \,d\sigma \to \nabla \int_0^u D(\sigma) \,d\sigma \quad \text{weakly in } L^2(0,T; (L^2(\Omega))^N) \quad (\forall T > 0), \]
\[ v_{\varepsilon} \to v \quad \text{weakly* in } L^\infty(0, \infty; L^\infty(\Omega)), \quad \nabla v_{\varepsilon} \to \nabla v \quad \text{weakly in } L^\infty(0, \infty; (L^\infty(\Omega))^N). \]

Moreover, let \(r \geq \max\{2(m+1), -2q+3m+1, 2(-q+m+2), 4(-q+m+1)\}. Then one can find a subsequence of \(\{\varepsilon\}\), which is denoted again by itself, such that for all \(T > 0\),
\[ \nabla \int_0^u D^\frac{1}{2}(\sigma + \varepsilon)^{\frac{r-2}{2}} \,d\sigma \to \nabla \int_0^u D^\frac{1}{2}(\sigma)^{\frac{r-2}{2}} \,d\sigma \quad \text{weakly in } L^2(0,T; (L^2(\Omega))^N), \]
\[ u_{\varepsilon} \to u \quad \text{a.e. on } \Omega \times (0,T) \]
as \(\varepsilon \to 0\).

**Proof.** Thanks to (39) we see that \(\{u_{\varepsilon}\}_{\varepsilon > 0}\) is bounded in \(L^\infty(0, \infty; L^\infty(\Omega))\) and arrive at (49). On the other hand, in view of (40) we deduce that \(\{v_{\varepsilon}\}_{\varepsilon > 0}\) and \(\{\nabla v_{\varepsilon}\}_{\varepsilon > 0}\) are bounded in \(L^\infty(0, \infty; L^\infty(\Omega))\) and in \(L^\infty(0, \infty; (L^\infty(\Omega))^N)\), respectively, and so (51) and (52) hold.

We next prove (53) and (54). Let \(r\) and \(T\) be taken as in the lemma. From Lemma 4.1 and the Lions–Aubin compactness theorem (see Simon [23]) we can observe that \(\left\{ \int_0^u D^\frac{1}{2}(\sigma + \varepsilon)^{\frac{r-2}{2}} \,d\sigma \right\}_{\varepsilon > 0}\) admits a strongly converging subsequence.
in $L^2(0, T; L^2(\Omega))$, i.e., there exist a subsequence of $\{\varepsilon\}$ (we denote it again by $\{\varepsilon\}$) and a function $\eta \in L^2(0, T; L^2(\Omega))$ such that
\[
\int_0^{u_\varepsilon} D^2(\sigma + \varepsilon) \sigma^{-2} d\sigma \to \eta \quad \text{strongly in } L^2(0, T; L^2(\Omega)) \quad \text{as } \varepsilon \to 0. \tag{55}
\]
Moreover, boundedness of $\{u_\varepsilon(t)\}_{\varepsilon > 0}$ in $L^p(\Omega)$ for $p \in [1, \infty]$ and the uniform continuity of $D^2$ on $[0, K_\infty + 1]$ imply that for all $\lambda > 0$ there exists $\varepsilon_0 \in (0, 1)$ such that
\[
|D^2(\sigma + \varepsilon) - D^2(\sigma)| \leq \lambda \quad (\varepsilon \in (0, \varepsilon_0), \sigma \in [0, K_\infty])
\]
and hence we obtain
\[
\int_0^{u_\varepsilon} D^2(\sigma + \varepsilon) \sigma^{-2} d\sigma - \int_0^{u_\varepsilon} D^2(\sigma) \sigma^{-2} d\sigma \to 0 \quad \text{strongly in } L^2(0, T; L^2(\Omega)),
\]
and a.e. $\Omega \times (0, T)$ as $\varepsilon \to 0$. Connecting the above convergence to (55) gives us that
\[
\int_0^{u_\varepsilon} D^2(\sigma) \sigma^{-2} d\sigma \to \eta \quad \text{strongly in } L^2(0, T; L^2(\Omega)) \quad \text{and a.e. } \Omega \times (0, T)
\]
as $\varepsilon \to 0$. Since $h(\gamma) := \int_0^\gamma D^2(\sigma) \sigma^{-2} d\sigma$ is a strictly increasing and continuous function, we see that
\[
u_\varepsilon \to h^{-1}(\eta) = u \quad \text{a.e. on } \Omega \times (0, T),
\]
where we used (49) to obtain the final equality. This implies (54). From (42) it follows that $\int_0^u D^2(\sigma) \sigma^{-2} d\sigma \in L^2(0, T; H^1(\Omega))$ and (53) holds.

We finally prove (50) by using Lebesgue’s dominated convergence theorem. Because of the facts that $\max_{\lambda \in [0, K_\infty + 1]} D(r) \leq c_1$ for some $c_1 > 0$ and that for any $\lambda > 0$ there exists $\varepsilon_1 \in (0, 1)$ such that $|D(\sigma + \varepsilon) - D(\sigma)| < \lambda$ ($\sigma \in [0, K_\infty], \varepsilon \in (0, \varepsilon_1)$), we see that
\[
\left| \int_0^{u_\varepsilon} D(\sigma + \varepsilon) d\sigma - \int_0^{u_\varepsilon} D(\sigma) d\sigma \right|^2 \\
\leq \left| \int_0^{u_\varepsilon} D(\sigma + \varepsilon) d\sigma - \int_0^{u_\varepsilon} D(\sigma + \varepsilon) d\sigma + \int_0^{u_\varepsilon} D(\sigma + \varepsilon) d\sigma - \int_0^{u_\varepsilon} D(\sigma) d\sigma \right|^2 \\
\leq c_1 \left( \left| \int_0^{u_\varepsilon} - \int_0^{\varepsilon_1} \right| d\sigma \right| + \lambda \int_0^{u_\varepsilon} d\sigma \right|^2 \\
\to 0 \quad \text{a.e. on } \Omega \times (0, \infty)
\]
as $\lambda \to 0$ (which means $\varepsilon \to 0$), where we used the point wise convergence (54). Applying Lebesgue’s dominated convergence theorem provided by this convergence
together with
\[
\left| \int_0^{u_\epsilon} D(\sigma + \epsilon) \, d\sigma - \int_0^u D(\sigma) \, d\sigma \right|^2 \leq c_1^2 |u_\epsilon - u|^2 \\
\leq 4c_1^2 K_\infty^2 \in L^2(0, T; L^2(\omega))
\]
for any compact set \( \omega \subset \Omega \) will give
\[
\int_0^{u_\epsilon} D(\sigma + \epsilon) \, d\sigma \to \int_0^u D(\sigma) \, d\sigma \quad \text{in} \quad L^2(0, T; L^2(\omega)).
\]
On the other hand, the boundedness (41) asserts that
\[
\int_0^{u_\epsilon} D(\sigma + \epsilon) \, d\sigma \to \tilde{\eta} \quad \text{weakly in} \quad L^2(0, T; H^1(\Omega))
\]
with some \( \tilde{\eta} \in L^2(0, T; H^1(\Omega)) \). These two convergences will give \( \tilde{\eta} = \int_0^u D(\sigma) \, d\sigma \in L^2(0, T; H^1(\Omega)) \), and (50) holds.

We are now in a position to close the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Let us take \( \varphi \in C_c^\infty(\Omega \times [0, \infty)) \) as a test function. Then we fix \( T > 0 \) such that \( \text{supp} \varphi \subset \Omega \times [0, T) \). Multiplying the first equation in (15) by \( \varphi \) and integrating it over \( \Omega \times (0, T) \), we have
\[
\int_0^T \int_\Omega \left( D(u_\epsilon + \epsilon) \nabla u_\epsilon \cdot \nabla \varphi - \frac{u_\epsilon}{u_\epsilon + \epsilon} S(u_\epsilon + \epsilon) \nabla v_\epsilon \cdot \nabla \varphi - u_\epsilon \varphi_t \right) \, dx \, dt = \int_\Omega u_0(x) \varphi(x, 0) \, dx.
\]
(56)
Now we see that
\[
\int_0^T \int_\Omega D(u_\epsilon + \epsilon) \nabla u_\epsilon \cdot \nabla \varphi = \int_0^T \int_\Omega \nabla \left( \int_0^{u_\epsilon} D(\sigma + \epsilon) \, d\sigma \right) \cdot \nabla \varphi \\
\to \int_0^T \int_\Omega \nabla \left( \int_0^u D(\sigma) \, d\sigma \right) \cdot \nabla \varphi
\]
as \( \epsilon \to 0 \) due to (50). Also we deduce from (39), (52) and (54) that
\[
\int_0^T \int_\Omega \frac{u_\epsilon}{u_\epsilon + \epsilon} S(u_\epsilon + \epsilon) \nabla v_\epsilon \cdot \nabla \varphi \to \int_0^T \int_\Omega S(u) \nabla v \cdot \nabla \varphi
\]
as \( \epsilon \to 0 \). Therefore we can pass to the limit in (56) as \( \epsilon \to 0 \) to obtain
\[
\int_0^T \int_\Omega \left( \nabla \left( \int_0^u D(\sigma) \, d\sigma \right) \cdot \nabla \varphi - S(u) \nabla v \cdot \nabla \varphi - u \varphi_t \right) \, dx \, dt = \int_\Omega u_0(x) \varphi(x, 0) \, dx.
\]
By a similar argument we have
\[
\int_0^T \int_\Omega (\nabla v \cdot \nabla \varphi + v \varphi - u \varphi - v \varphi_t) \, dx \, dt = \int_\Omega v_0(x) \varphi(x, 0) \, dx.
\]
Collecting the facts obtained up to here, we can conclude that \((u, v)\) is the desired global weak solution of (1) and (8) follows from (39).

**Acknowledgments.** The authors express their appreciation to the referees for their comments on an earlier version of the manuscript.
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Received May 2017; revised October 2017.

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