ON A PARABOLIC-ELLIPTIC CHEMOTAXIS-GROWTH SYSTEM WITH NONLINEAR DIFFUSION

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Abstract. This paper considers the following parabolic-elliptic chemotaxis-growth system with nonlinear diffusion
\[
\begin{align*}
    u_t &= \nabla \cdot (D(u) \nabla u) - \nabla \cdot (\chi u^q \nabla v) + \mu u(1 - u^\alpha), & x \in \Omega, t > 0, \\
    0 &= \Delta v - v + u^\gamma, & x \in \Omega, t > 0
\end{align*}
\]
under homogeneous Neumann boundary conditions for some constants \(q \geq 1, \alpha > 0, \text{ and } \gamma \geq 1\), where \(D(u) \geq c D u^{m-1} (m \geq 1)\) for all \(u > 0\) and \(D(u) > 0\) for all \(u \geq 0\), and \(\Omega \subset \mathbb{R}^N (N \geq 1)\) is a bounded domain with smooth boundary. It is shown that when \(m > q + \gamma - \frac{2}{N}\), or \(\alpha > q + \gamma - 1\), or \(\alpha = q + \gamma - 1\) and \(\mu > \mu^*\), where
\[
\mu^* = \begin{cases} 
    \frac{(\alpha+1-m)N-2}{(\alpha+1-m)N+2(\alpha-\gamma)}, & \text{if } m \leq q + \gamma - \frac{2}{N}, \\
    0, & \text{if } m > q + \gamma - \frac{2}{N},
\end{cases}
\]
then the above system possesses a global bounded classical solution for any sufficiently smooth initial data. The results improve the results by Wang et al. (J. Differential Equations 256 (2014)) and generalize the results of Zheng (J. Differential Equations 259 (2015)) and Galakhov et al. (J. Differential Equations 261 (2016)).

1. Introduction. The Keller-Segel system modelling chemotaxis was initially introduced by Keller and Segel [10] in 1970, and it has been well studied in the past four decades (see survey papers [1, 5, 6, 4], for instance). In view of various biological phenomena and the environment for the cells, many variants of the Keller-Segel model have been developed and investigated (see [2, 7, 9, 14, 15, 18, 17, 16, 22, 27, 11, 12, 26, 13, 19, 20, 21, 23] and the references therein, for instance). And other scholars study topological dynamics and control (see [24, 25, 3], for instance).

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In this paper, we consider the following parabolic-elliptic chemotaxis-growth system with nonlinear diffusion and nonlinear signal production

\[
\begin{cases}
  u_t = \nabla \cdot (D(u) \nabla u) - \nabla \cdot (\chi u^\beta \nabla v) + \mu u(1 - u^\alpha), & x \in \Omega, \ t > 0, \\
  0 = \Delta v - v + u^\gamma, & x \in \Omega, \ t > 0, \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
\tag{1}
\]

where \(\Omega \subset \mathbb{R}^N (N \geq 1)\) is a bounded domain with smooth boundary, \(\frac{\partial}{\partial \nu}\) denotes the outward normal derivative on \(\partial \Omega\), \(\chi, \mu, q\) and \(\alpha\) are given positive parameters. \(u\) and \(v\) represent the density of cells and the concentration of chemical substance, respectively. \(D(u)\) is a diffusive coefficient and satisfies

\[
D(u) \in C^2([0, \infty))
\tag{2}
\]

as well as

\[
D(u) \geq c_D u^{m-1} \quad \text{for all } u > 0 \quad \text{and} \quad D(u) > 0 \quad \text{for all } u \geq 0
\tag{3}
\]

with some constants \(c_D > 0\) and \(m \geq 1\). The initial data \(u_0\) satisfies \(u_0 \geq 0\) and \(u_0 \in W^{1,\infty}(\Omega)\).

To motivate our study, we recall some works on system (1). For the case \(D(u) \equiv 1\) and \(q = \alpha = \gamma = 1\), Tello and Winkler [16] proved that when \(\mu > \frac{(N-2)^+ \chi}{N}\), system (1) possesses a global and bounded classical solution for any sufficiently regularized initial data. For the case \(D(u) \equiv 1, q \geq 1\) and \(\gamma \geq 1\), Galakhov, Salieva and Tello [2] recently showed that if either \(\alpha > q + \gamma - 1\) or \(\alpha = q + \gamma - 1\) and \(\mu > \frac{N\alpha - 2}{(m-1)+N\alpha} \chi\), system (1) admits a global and bounded classical solution for any given initial data \(u_0 \in W^{1,\infty}(\Omega)\). And more recently, Hu and Tao [8] proved that the same conclusion still holds for the critical case \(\alpha = q + \gamma - 1\) and \(\mu = \frac{N\alpha - 2}{(m-1)+N\alpha} \chi\). For the case \(D(u)\) fulfilling (2)–(3) and \(q = \gamma = 1\), Wang, Mu and Zheng [18] proved that if either \(\alpha \geq 1\) and \(b > b^*\), where \(b^* = \begin{cases} \frac{(2m)N-2}{(2-m)N} \chi & \text{if } m \leq 2 - \frac{2}{N}, \\
0 & \text{if } m > 2 - \frac{2}{N}, \end{cases}\) or \(\alpha \in (0, 1)\) and \(m > 2 - \frac{2}{N}\), system (1) possesses a global and bounded classical solution for any given \(u_0 \in W^{1,\infty}(\Omega)\). For the case \(D(u)\) fulfilling (2)–(3) and \(\gamma = 1\), Zheng [27] showed that if either \(q + 1 < \min\{\alpha + 1, m + \frac{2}{N}\}\) or \(q = \alpha\) and \(\mu > \frac{(\alpha+1-m)N-2}{(\alpha+1-m)N+2(\alpha-1)} \chi\), system (1) admits a global and bounded classical solution for any sufficiently smooth initial data. Inspired by the above recent works [16, 2, 18, 27], the purpose of this paper is to explore the interaction between nonlinear diffusion, nonlinear cross-diffusion, generalized logistic source and superlinear signal production on the solution of system (1).

We now state the main results of this paper.

**Theorem 1.1.** Let \(\Omega \subset \mathbb{R}^N (N \geq 1)\) be a bounded domain with smooth boundary. Suppose that \(u_0 \in W^{1,\infty}(\Omega)\) is a non-negative function and that \(D(u)\) satisfies (2) and (3). Let \(\chi, \mu, q, \alpha, \beta\) and \(\gamma\) be given positive parameters satisfying \(q \geq 1\) and \(\gamma \geq 1\). If one of the following cases holds:

(i) \(m > q + \gamma - \frac{2}{N}\),

(ii) \(\alpha > q + \gamma - 1\),

then...
Proof. Integrating the first equation in (1), we obtain

Lemma 2.2.

Lemma 2.3.

Suppose

(iii) \( \alpha = q + \gamma - 1 \) and \( \mu > \mu^* \), where

\[
\mu^* = \left\{ \begin{array}{ll}
\frac{(\alpha+1-m)N-2}{(\alpha+1-m)N+2(\alpha-\gamma)} & \text{if } m \leq q - \frac{\gamma}{N}, \\
0 & \text{if } m > q - \frac{\gamma}{N},
\end{array} \right.
\]

then system (1) possesses a global and bounded classical solution.

Remark 1. Our results improve the results given by Wang, Mu and Zheng [18] and generalize the results obtained by Zheng [27] and by Galakhov, Salieva and Tello [2].


In this section, we first begin with the local existence of solutions to system (1), which can be proved by directly adapting the procedure in [18, Lemma 2.1].

Lemma 2.1. Let \( \Omega \subset \mathbb{R}^N \ (N \geq 1) \) be a bounded domain with smooth boundary. Suppose that \( u_0 \in W^{1,\infty}(\Omega) \) is a non-negative function and that \( D(u) \) satisfies (2) and (3). Let \( \chi, \mu, q, \alpha \) and \( \gamma \) be given positive parameters satisfying \( q \geq 1 \) and \( \gamma \geq 1 \). Then there exist \( T_{\max} \in (0, \infty) \) and a pair \( (u,v) \) of nonnegative functions from \( C^0(\bar{\Omega} \times [0,T_{\max}]) \cap C^{2,1}(\Omega \times [0,T_{\max}]) \) solving (1) classically in \( \Omega \times (0,T_{\max}) \). Moreover,

\[
\text{if } T_{\max} < \infty, \text{ then } \lim_{t \to T_{\max}} \sup_{\Omega} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.
\]

The following lemma is a basic property associated with \( u \).

Lemma 2.2. The solution of (1) fulfills

\[
\int_{\Omega} u(\cdot,t)dx \leq M \quad \text{for all } t \in (0,T_{\max}),
\]

where \( M \) is some positive constant.

Proof. Integrating the first equation in (1), we obtain

\[
\frac{d}{dt} \int_{\Omega} u dx = \mu \int_{\Omega} u dx - \mu \int_{\Omega} u^{1+\alpha} dx \quad \text{for all } t \in (0,T_{\max}).
\]

By Young’s inequality, we have \( (\mu+1) \int_{\Omega} u dx \leq \mu \int_{\Omega} u^{1+\alpha} dx + C \), where \( C \) is some positive constant. Thus, we have

\[
\frac{d}{dt} \int_{\Omega} u dx + \int_{\Omega} u dx \leq C.
\]

This yields (5) with \( M := \max\{C, \int_{\Omega} u_0 dx\} \) by Gronwall’s inequality.

Lemma 2.3. Suppose \( m > q + \gamma - \frac{2}{N} \). Then for any \( p > 1 \), there exists some constant \( C(p) > 0 \) such that the solution of (1) satisfies

\[
\|u(\cdot,t)\|_{L^p(\Omega)} \leq C(p) \quad \text{for all } t \in (0,T_{\max}).
\]

Proof. Multiplying the first equation in (1) by \( u^{p-1} \) and integrating by parts over \( \Omega \), we obtain

\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p dx = -\int_{\Omega} D(u) \nabla u \cdot \nabla u^{p-1} dx + \chi(p-1) \int_{\Omega} u^{p+q-2} \nabla u \cdot \nabla u^{p-1} dx
\]

\[
+ \mu \int_{\Omega} u^p dx - \mu \int_{\Omega} u^{p+\alpha} dx
\]

\[
= -(p-1) \int_{\Omega} D(u) u^{p-2} |\nabla u|^2 dx - \frac{(p-1)\chi}{(p+q-1)} \int_{\Omega} u^{p+q-1} \Delta u dx
\]

\[
+ \mu \int_{\Omega} u^p dx - \mu \int_{\Omega} u^{p+\alpha} dx
\]
= -(p - 1) \int_{\Omega} D(u)u^{p-2} |\nabla u|^2 dx - \frac{(p - 1)\chi}{(p + q - 1)} \int_{\Omega} u^{p+q+\gamma-1} dx \\
- \frac{(p - 1)\chi}{(p + q - 1)} \int_{\Omega} u^{p+q-1} v dx + \mu \int_{\Omega} u^p dx - \mu \int_{\Omega} u^{p+\alpha} dx \\
\leq -(p - 1) \int_{\Omega} D(u)u^{p-2} |\nabla u|^2 dx + \frac{(p - 1)\chi}{(p + q - 1)} \int_{\Omega} u^{p+q+\gamma-1} dx \\
+ \mu \int_{\Omega} u^p dx - \mu \int_{\Omega} u^{p+\alpha} dx \quad \text{for all } t \in (0, T_{\text{max}})

due to the nonnegativity of \( v \). By Young’s inequality, we can find some constant \( C_1 > 0 \) such that \( \mu \int_{\Omega} u^p dx \leq \mu \int_{\Omega} u^{p+\alpha} dx + C_1 \) for all \( t \in (0, T_{\text{max}}) \). Inserting it into (7) and using (3), we have

\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p dx + \frac{4c_D(p - 1)}{(m + p - 1)^2} \int_{\Omega} |\nabla u|^{\frac{m+p-1}{2}} dx \\
\leq \frac{(p - 1)\chi}{(p + q - 1)} \int_{\Omega} u^{p+q+\gamma-1} dx + C_1
\]

for all \( t \in (0, T_{\text{max}}) \). In view of the Gagliardo-Nirenberg inequality and (5), we can find some constants \( C_2 = C_2(p) > 0 \) and \( C_3 = C_3(p) > 0 \) such that

\[
\int_{\Omega} u^{p+q+\gamma-1} dx = \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2(p+q+\gamma-1)}{m+p-1}}(\Omega)}^{\frac{2(p+q+\gamma-1)}{m+p-1}} + C_2 \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2(p+q+\gamma-1)}{m+p-1}}(\Omega)}^{\frac{2(p+q+\gamma-1)}{m+p-1}}
\]

\[
\leq C_2 \left\| \nabla u^{\frac{m+p-1}{2}} \right\|_{L^2(\Omega)}^{2(p+q+\gamma-1)} + C_3 \quad \text{for all } t \in (0, T_{\text{max}}),
\]

where

\[
a = \frac{N(m+p-1)}{2} - \frac{N(m+p-1)}{2} + \frac{N(m+p-1)}{2} \in (0, 1)
\]
due to \( m > q + \gamma - \frac{2}{N} \) and \( p > 1, q \geq 1, \gamma \geq 1 \). Since \( m > q + \gamma - \frac{2}{N} \), we have

\[
a \leq \frac{2(p+q+\gamma-1)}{m+p-1} \cdot a = 2 \cdot \frac{p+q+\gamma-2}{m+p+\frac{2}{N}-2} < 2.
\]

Hence, by (9)–(10) and applying Young’s inequality, we can find \( C_4 := C_4(p) > 0 \) such that

\[
\left(1 + \frac{p(p - 1)\chi}{(p + q - 1)}\right) \int_{\Omega} u^{p+q+\gamma-1} dx \leq 4c_Dp(p - 1) \int_{\Omega} |\nabla u^{\frac{m+p-1}{2}}|^2 dx + C_4
\]

for all \( t \in (0, T_{\text{max}}) \). Combining (11) and (8) yields that

\[
\frac{d}{dt} \int_{\Omega} u^p dx + \int_{\Omega} u^p dx \leq C_4 + pC_1 \quad \text{for all } t \in (0, T_{\text{max}}).
\]

By Gronwall’s inequality this shows (6) with \( C(p) := \max\{C_4 + pC_1, \int_{\Omega} u_0^p\} \). 

**Lemma 2.4.** Assume \( \alpha > q+\gamma-1 \). Then for any \( p > 1 \), there exists some constant \( C(p) > 0 \) such that the solution of (1) satisfies

\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq C(p) \quad \text{for all } t \in (0, T_{\text{max}}).
\]
Proof. From (7), we have
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p dx \leq -(p-1) \int_{\Omega} D(u) u^{p-2} |\nabla u|^2 dx + \frac{(p-1)\chi}{(p+q-1)} \int_{\Omega} u^{p+q+\gamma-1} dx \\
+ \mu \int_{\Omega} u^{p} dx - \mu \int_{\Omega} u^{p+\alpha} dx
\]
for all \( t \in (0, T_{\max}) \). Since \( u \geq 0 \) and \( D(u) > 0 \) for all \( u \geq 0 \), we obtain
\[
\frac{d}{dt} \int_{\Omega} u^p dx \leq \frac{p(p-1)\chi}{(p+q-1)} \int_{\Omega} u^{p+q+\gamma-1} dx + \mu p \int_{\Omega} u^p dx - \mu \int_{\Omega} u^{p+\alpha} dx \quad (14)
\]
for all \( t \in (0, T_{\max}) \). Since \( \alpha > q + \gamma - 1 \), we have \( p + \alpha > p + q + \gamma - 1 \). By Young’s inequality, we can find some positive constants \( C_1, C_2 \) and \( C_3 \) such that
\[
\frac{p(p-1)\chi}{(p+q-1)} \int_{\Omega} u^{p+q+\gamma-1} dx \leq \frac{\mu p}{4} \int_{\Omega} u^{p+\alpha} dx + C_1 \quad (15)
\]
and
\[
\mu p \int_{\Omega} u^p dx \leq \frac{\mu p}{4} \int_{\Omega} u^{p+\alpha} dx + C_2 \quad (16)
\]
as well as
\[
\int_{\Omega} u^p dx \leq \frac{\mu p}{2} \int_{\Omega} u^{p+\alpha} dx + C_3 \quad (17)
\]
for all \( t \in (0, T_{\max}) \). Substituting (15)–(16) into (14) and then combining with (17) yield that
\[
\frac{d}{dt} \int_{\Omega} u^p dx + \int_{\Omega} u^p dx \leq C_1 + C_2 + C_3 \quad \text{for all} \ t \in (0, T_{\max}). \quad (18)
\]
Upon an ODE comparison principle we have \( \int_{\Omega} u^p dx \leq \max \{ \int_{\Omega} u_0^p dx, C_1 + C_2 + C_3 \} \) for all \( t \in (0, T_{\max}) \), which yields (13). \( \square \)

Lemma 2.5. Suppose \( \alpha = q + \gamma - 1 \) and \( \mu > \mu^* \), where \( \mu^* \) defined in Theorem 1.1. Then for any \( p > 1 \), there exists some constant \( C(p) > 0 \) such that the solution of (1) satisfies
\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq C(p) \quad \text{for all} \ t \in (0, T_{\max}). \quad (19)
\]
Proof. We first prove the following claim.

Claim. When \( \alpha = q + \gamma - 1 \), for any \( p \in (1, \frac{(q-1)\mu+\chi}{(\chi-\mu)^+}) \) there exists some constant \( C_1 := C_1(p) > 0 \) such that
\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq C_1 \quad \text{for all} \ t \in (0, T_{\max}). \quad (20)
\]
Multiplying the first equation in (1) by \( u^{p-1} \) and integrating over \( \Omega \) as in Lemma 2.3, we obtain
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p dx + \frac{4cD(p-1)}{(m+p-1)^2} \int_{\Omega} |\nabla u|^{m+p-1} dx \\
\leq - \left( \mu - \frac{(p-1)\chi}{(p+q-1)} \right) \int_{\Omega} u^{p+\alpha} dx + \mu \int_{\Omega} u^{p} dx \quad (21)
\]
for all \( t \in (0, T_{\max}) \). For any \( p \in (1, \frac{(q-1)\mu+\chi}{(\chi-\mu)^+}) \), we have \( \mu - \frac{(p-1)\chi}{(p+q-1)} > 0 \). Thanks to Young’s inequality, we can find \( C_2 > 0 \) such that
\[
\mu \int_{\Omega} u^{p} dx \leq \frac{1}{2} \left( \mu - \frac{(p-1)\chi}{(p+q-1)} \right) \int_{\Omega} u^{p+\alpha} dx + C_2.
\]
Therefore, we have
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p dx + \frac{4c_D(p-1)}{(m+p-1)^2} \int_{\Omega} \left| \nabla u^{\frac{m+p-1}{2}} \right|^2 dx \\
\leq - \frac{1}{2} \left( \mu - \frac{(p-1)\chi}{(p+q-1)} \right) \int_{\Omega} u^{p+\alpha} dx + C_2. \tag{22}
\]

Similarly, applying Young’s inequality again, we can find \( C_3 > 0 \) such that
\[
\int_{\Omega} u^p dx \leq \frac{1}{2} \left( \mu - \frac{(p-1)\chi}{(p+q-1)} \right) \int_{\Omega} u^{p+\alpha} dx + C_3 \tag{23}
\]
for all \( t \in (0, T_{\text{max}}) \). Combining (22) with (23) shows that
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p dx + \frac{4c_D(p-1)}{(m+p-1)^2} \int_{\Omega} \left| \nabla u^{\frac{m+p-1}{2}} \right|^2 dx + \int_{\Omega} u^p dx \leq C_2 + C_3 \tag{24}
\]
for all \( t \in (0, T_{\text{max}}) \). An ODE comparison principle implies (20) holds for any \( p \in (1, \frac{(q-1)\mu+\chi}{(\chi-\mu)_+}) \).

When \( \mu \geq \chi \) and \( \alpha = q + \gamma - 1 \), we have obtained the desired result (19) for any \( p > 1 \). However, when \( \mu < \chi \), it is not true. We are now in the position to treat the case \( \mu < \chi \). By arranging (21) and then using the inequality \( u^p \leq u^{p+\alpha} + 1 \), we have
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p dx + \frac{4c_D(p-1)}{(m+p-1)^2} \int_{\Omega} \left| \nabla u^{\frac{m+p-1}{2}} \right|^2 dx \leq \frac{(p-1)\chi}{(p+q-1)} \int_{\Omega} u^{p+\alpha} dx + |\Omega| \tag{25}
\]
for all \( t \in (0, T_{\text{max}}) \).

We divide the proof into two cases: \( m > q + \gamma - \frac{2}{\chi} \) and \( m \leq q + \gamma - \frac{2}{\chi} \).

**Case 1.** \( m > q + \gamma - \frac{2}{\chi} \). From Lemma 2.3 we can obtain that (19) holds for any \( p > 1 \).

**Case 2.** \( m \leq q + \gamma - \frac{2}{\chi} \). Since \( m \leq q + \gamma - \frac{2}{\chi} = \alpha + 1 - \frac{2}{\chi} \) and \( \mu > \frac{(\alpha+1-m)N}{(\alpha+1-m)N + 2(\alpha-\gamma)} \chi \), we have \( \frac{(\alpha+1-m)N}{2} \geq 1 \) and \( \frac{(\alpha+1-m)N}{2} < \frac{(q-1)\mu+\chi}{(\chi-\mu)_+} \). Thus, according to the claim we can pick \( p' \in (\frac{(\alpha+1-m)N}{2}, \frac{(q-1)\mu+\chi}{(\chi-\mu)_+}) \) such that
\[
\|u(\cdot, t)\|_{L^{p'}(\Omega)} \leq C_4(p') \quad \text{for all } t \in (0, T_{\text{max}}) \tag{26}
\]
with some constant \( C_4(p') > 0 \). Thus, we choose \( p > p' \) and use the Gagliardo-Nirenberg inequality and (26) to derive
\[
\int_{\Omega} u^{p+\alpha} dx \\
= \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{2(p+\alpha)}(\Omega)}^{2(p+\alpha)} \\
\leq C_5 \left( \left\| \nabla u^{\frac{m+p-1}{2}} \right\|_{L^{2(p+\alpha)}(\Omega)}^{2(p+\alpha)\theta} \right)^{\frac{\theta}{p+\alpha}} \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{2(p+\alpha)(1-\theta)}(\Omega)}^{2(p+\alpha)(1-\theta)} + \left\| u^{\frac{m+p-1}{2}} \right\|_{L^{2(p+\alpha)}(\Omega)}^{2(p+\alpha)} \right)^{\frac{1-\theta}{p+\alpha}} \\
\leq C_6 \left( \left\| \nabla u^{\frac{m+p-1}{2}} \right\|_{L^{2(p+\alpha)}(\Omega)}^{2(p+\alpha)\theta} \right)^{\frac{1-\theta}{p+\alpha}} + 1 \quad \text{for all } t \in (0, T_{\text{max}}) \tag{27}
\]
with some constants \( C_5 > 0 \) and \( C_6 > 0 \), where
\[
\theta = \frac{(m+p-1)N}{2p'} - \frac{(m+p-1)N}{2(p+\alpha)} \in (0, 1)
\]
due to $p > \frac{(\alpha+1-m)N}{2} > \frac{(\alpha+1-m)N-2\alpha}{2}$. Then we have
\[
\frac{2(p+\alpha)}{m+p-1} \cdot \theta = 2 \cdot \frac{(p+\alpha)N - N \frac{2}{2}}{1 - \frac{(m+1-m)N}{2} + \frac{(p+\alpha)N}{2} - \frac{N}{2}} < 2 \quad (28)
\]
due to $p' > \frac{(\alpha+1-m)N}{2}$. Thus, by Young’s inequality there exists a constant $C_T > 0$ such that
\[
\left(1 + \frac{p(p-1)\chi}{(p+q-1)}\right) \int_{\Omega} u^{p+\alpha} dx \leq \frac{4c_{DP}(p-1)}{(m+p-1)^2} \int_{\Omega} \left|\nabla u^{\frac{m+p-1}{2}}\right|^2 dx + C_T
\]
for all $t \in (0, \max T)$, which combines with (25) yields
\[
\frac{d}{dt} \int_{\Omega} u^p dx + \int_{\Omega} u^p dx \leq C_T + |\Omega|p \quad \text{for all } t \in (0, \max T).
\]
An ODE comparison principle implies (19) holds. \hfill \(\square\)

**Proof of Theorem 1.1.** Using the estimates in Lemma 2.3–Lemma 2.5 with suitably large $p$ and invoking Lemma A.1 in [15], we can find some constant $C_1 > 0$ such that $\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1$ for all $t \in (0, \max T)$. Thus, by standard elliptic regularity theory for the second equation in system (1) we can derive that $\|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C_2$ for all $t \in (0, \max T)$ with some $C_2 > 0$. In view of the extensibility criterion (4), we infer that $\max T = \infty$. Thus, we complete the proof of Theorem 1.1. \hfill \(\square\)

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