Pricing Down-and-Out Power Options with Exponentially Curved Barrier

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Abstract. Power barrier options are options where the payoff depends on an underlying asset raised to a constant number. The barrier determines whether the option is knocked in or knocked out of existence when the underlying asset hits the prescribed barrier level, or not. This paper derives the analytical solution of the power options with an exponentially curved barrier by utilizing the reflection principle and the change of measure. Numerical results show that prices of power options with exponentially curved barrier are cheaper than those of power barrier options and power options.

1. Introduction. Barrier options are path-dependent exotic derivatives – famous for their cheaper price comparing to the standard vanilla options. To our best knowledge, [13] was the first to introduce an analytical solution for a knock-out barrier option. Several studies have highlighted the different aspects of barrier options. In [7], the authors use the binomial model to price barrier options, while [1] uses the probabilistic approach to price barrier options. On the other hand, [12] introduces a simple method that estimates barrier option prices with time-dependent parameters. In addition, [17] uses the reflection theorem within the Black-Scholes framework to produce closed-form solution for barrier options, and shows that barrier options can be used to hedge other financial instrument. In [8], barrier options were extended in order to price Parisian options, and [18, 19] price the variation of Parisian options by further extending the former work. Furthermore, [11] derives a closed-form solution for up-and-in Parisian call options using the “moving window” method.

Most of the studies on barrier options involves a standard straight-line barrier level, including [9] that derives the prices of power barrier options analytically. In
the authors analytically derive a pricing formula of a double barrier options with exponential barriers.

Hence, in this study, we derive a closed-form solution for knock-out power call options with exponentially curved barrier (ECB). We consider the case of ECB to allow an exponential time-dependent barrier level. This paper is organized as follows. In Section 2, we provide the derivation of the closed-form pricing formula for a down-and-out power call option with ECB, and in Section 3, we document the numerical results. Section 4 concludes this paper.

2. The Pricing Formula. The stock price dynamic under $\mathbb{Q}$ measure is

$$dS_t = rdt + \sigma S_t dW_t^\mathbb{Q},$$

(1)

where $r$ is the risk-free interest rate and $\sigma$ is the volatility. The payoff for a down-and-out power call (DOPC) option is defined by

$$\Pi(T) = \max(S^\beta_T - K, 0) \mathbb{1}_{\{S^\beta_T > f_T \land t \in (0, T)\}},$$

(2)

in which $S$ is the price of the underlying asset and $\beta$ is a positive constant with strike price $K$ at expiry date $T$. The latter part of the equation is known as the indicator function in which $f_T$ represents the function of a general curved boundary. Then, we apply the power barrier transformations proposed in [9] and Ito’s Lemma to the indicator function to obtain:

$$\int_0^T \left[(r - \frac{1}{2}\sigma^2)\beta - \frac{f_T'}{f_T}\right]dt + \sigma \beta W_t^\mathbb{Q} > \ln f_0 - \ln S^\beta.$$  

(3)

Hence, by applying the Girsanov theorem, and defining $\hat{m}_T = \min_{0 \leq t \leq T} \hat{W}_t$, Equation (2) can be written as follows:

$$\Pi(T) = \left(\frac{S^\beta_T}{f_0} e^{\sigma \beta \hat{W}_t} - K\right) \mathbb{1}_{\{\hat{W}_T > k \land \hat{m}_t > b\}},$$

(4)

where $k = \frac{1}{\sigma \beta} \ln \left[\frac{Kf_0}{S^\beta_T}\right]$ and $b = \frac{1}{\sigma \beta} \ln \left[\frac{f_T}{S^\beta_T}\right]$. Alternatively, we have:

$$V = e^{-rT} \mathbb{E}^\mathbb{Q}\left[\left(\frac{S^\beta_T}{f_0} e^{\sigma \beta \hat{W}_t} - K\right) \mathbb{1}_{\{\hat{W}_T > k \land \hat{m}_t > b\}}\right].$$

(5)

On that account, to compute the expectation in Equation (5), we find the joint distribution of $(\hat{W}_T, \hat{m}_t)$ under $\mathbb{Q}$ measure by using the reflection principle, given as follows:

$$\mathbb{Q}(\hat{W}_T > w, \hat{m}_T < m) = \mathbb{E}^\mathbb{Q}\left[\frac{1}{Z} \mathbb{1}_{\{\hat{W}_T > w, \hat{m}_T < m\}}\right].$$

(6)

Suppose that the ECB is given by

$$f(\tau) = Be^{\delta \tau},$$

where $\delta$ is the curvature of the barrier level. Therefore, the closed-form pricing formula for DOPC with ECB is shown in the following lemma.

**Lemma 2.1.** For a down-and-out power call option with an exponentially curved barrier, the closed-form pricing formula is as such:
\[ DOPC^{ECB} = S^\beta e^{(\beta-1)(r+\frac{1}{2}\sigma^2)\tau} N(U_{K1}) - Ke^{-r\tau} N(U_{K2}) \]

\[ - S^\beta \left( \frac{B}{S^3} \right)^{2(1+\lambda)} e^{(\beta-1)(r+\frac{1}{2}\sigma^2)\tau} N(U_{B1}) \]

\[ + Ke^{-r\tau} \left( \frac{B}{S^3} \right)^{2\lambda} N(U_{B2}), \]

where

\[ U_{K1} = \frac{\ln \left( \frac{S^3}{Ke^{-r\tau}} \right) + \tau (\hat{\sigma}^2 + \hat{\mu} - \delta)}{\delta \sqrt{\tau}}, \]

\[ U_{K2} = U_{K1} - \hat{\sigma} \sqrt{\tau}, \]

\[ U_{B1} = \frac{\ln \left( \frac{S^3}{Ke^{-r\tau}} \right) + \tau (\hat{\sigma}^2 + \hat{\mu} - \delta)}{\hat{\sigma} \sqrt{\tau}}, \]

\[ U_{B2} = U_{B1} - \hat{\sigma} \sqrt{\tau}, \]

\[ \hat{\sigma} = \sigma \beta, \]

\[ \hat{\mu} = (r - \frac{1}{2}\sigma^2) \beta, \]

\[ \hat{\lambda} = \frac{1}{\sigma^2} (\hat{\mu} - \delta). \]

Proof. We price our DOPC with ECB in which the value is given by the discounted expected value of its payoff under the risk-neutral measure.

\[ DOPC^{ECB} = e^{-r\tau} E^\mathbb{Q} (S^\beta e^{\delta \tau + \sigma \beta \hat{W}_t} - K) \mathbb{1}_{(\hat{W}_T > k \ \& \ \hat{m}_T > b)} \]

\[ = \int_{\max(k,b)}^{\infty} \int_{b}^{w} \left( S^\beta e^{\delta \tau + \sigma \beta \hat{W}_t} - K \right) e^{-\theta w - r\tau - \frac{1}{2} \theta^2 \tau} \]

\[ \times \frac{2(2m - w)}{\tau \sqrt{2\pi \tau}} e^{-\frac{(2m - w)^2}{2\tau}} \ dmdw \]

\[ = \int_{\max(k,b)}^{\infty} \left( S^\beta e^{\delta \tau + \sigma \beta \hat{W}_t} - K \right) e^{-\theta w - r\tau - \frac{1}{2} \theta^2 \tau} \]

\[ \times \left[ \frac{1}{\sqrt{2\pi \tau}} e^{-\frac{w^2}{2\tau}} - \frac{1}{\sqrt{2\pi \tau}} e^{-\frac{(2b - w)^2}{2\tau}} \right] dw \]

\[ = \int_{\max(k,b)}^{\infty} S^\beta e^{(\sigma \beta - \theta)w - (r - \delta) \tau - \frac{1}{2} \theta^2 \tau} \frac{1}{\sqrt{2\pi \tau}} e^{-\frac{w^2}{2\tau}} \ dw \]

\[ - \int_{\max(k,b)}^{\infty} Ke^{-\theta w - r\tau - \frac{1}{2} \theta^2 \tau} \frac{1}{\sqrt{2\pi \tau}} e^{-\frac{w^2}{2\tau}} \ dw \]

\[ - \int_{\max(k,b)}^{\infty} S^\beta e^{(\sigma \beta - \theta)w - (r - \delta) \tau - \frac{1}{2} \theta^2 \tau} \frac{1}{\sqrt{2\pi \tau}} e^{-\frac{(2b - w)^2}{2\tau}} \ dw \]

\[ + \int_{\max(k,b)}^{\infty} Ke^{-\theta w - r\tau - \frac{1}{2} \theta^2 \tau} \frac{1}{\sqrt{2\pi \tau}} e^{-\frac{(2b - w)^2}{2\tau}} \ dw. \]

Then, by completing the square for each integrals, we obtain the closed-form pricing formula (7). Hence, the proof is complete. \[ \Box \]
3. **Numerical Results.** In this section, we document the numerical results for the prices of DOPC with ECB. We compute the prices using Equation (7) with different curvature $\delta$, strike price $K$, and barrier level $B$. Additionally, we use the following hypothetical parameters: $S = 10, r = 0.02, \beta = 2, \sigma = 0.2$ and $\tau = 1$.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$\delta$</th>
<th>$B$</th>
<th>$\text{DOPC}^{\text{ECB}}$</th>
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*Table 1. Prices of DOPC with ECB with different curvature, $\delta$*

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*Table 2. Prices of DOPC with ECB with different strike price, $K$*

<table>
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*Table 3. Prices of DOPC with ECB with different barrier level, $B$*

<table>
<thead>
<tr>
<th>$K$</th>
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*Table 4. Price Comparisons: Power Call, Down-and-Out Power Barrier, and Down-and-Out Power Option with ECB*

Tables 1 to 3 document the prices of DOPC with ECB with different curvatures, strike prices and barrier levels. We illustrate the numerical results as shown in
Figures 1 to 3. It shows that an increasing curvature of the barrier level, the strike price and the barrier level lowers the price of the DOCP with ECB. This is due to the reason that when these three parameters increase in value, the probability of the price of the underlying asset being greater than the strike price gets smaller. Hence, this results in a decrease in the price of DOPC with ECB. On the other hand, we compare the prices of DOPC with ECB, PC and DOPC. Table 4 shows that DOPC with ECB is cheaper than both PC and DOPC while Figure 4 illustrates the price comparison between these three options. By applying exponentially curved barrier, the price of DOPC can become even lower than that of a power call option.
4. Conclusion and future work. Power options are sensitive to even the smallest change in the underlying asset value; hence, the leverage feature they display. By incorporating a barrier in the payoff of power options, the price of the options can be reduced. This was shown in the work of [9] using the standard type of barrier—a straight-line barrier. This study provides the analytical pricing formula for power options with exponentially curved barrier. Numerical results show that the price of the power options with an exponentially curved barrier is cheaper than the power options with a straight-line barrier. Therefore, making the former favorable to investors.

Future research may be directed to different approaches to price power options with exponentially curved barrier, such as lattice model and binomial method. Moreover, the power barrier transformations introduced in [9] may be incorporated to price other variations of barrier options, such as double barrier options and Parisian options.

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REFERENCES

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