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Introduction

The aim of this book is to present a new theory, based on conservation law models, for traffic flow on networks. The thematic is suitable for various applications including: urban car traffic [27], data flows on telecommunication networks [41] and supply chains models [8, 53].

The main inspiration is that of understanding traffic behavior in urban context in order to answer to several questions: where to install traffic lights or stop signs; how long the cycle of traffic lights should be; where to construct entrances, exits, and overpasses. The aims of this analysis are principally represented by the maximization of cars flow, and the minimization of traffic congestions, accidents and pollution.

Classically, the network models of transportation systems are assumed to be static, but these models do not allow a correct simulation of heavily congested urban road networks. For this reason, traffic engineers have been studying dynamic traffic assignment or *within-day* models, thus rendering necessary the use of time advancing mathematical models (traffic simulation models). These models, principally created from static network traffic assignments, can be roughly classified in microscopic, mesoscopic and macroscopic (see [10] and the references therein). The main problems of this approach consist in the fact that it does not properly reproduce the backward propagation of shocks and in the difficulty of collecting experimental data to test the models. Various other ideas have been developed by researchers studying traffic from other perspectives, see for instance [9, 38, 63, 68, 89, 91, 106]. In many cases, the attention was focused on a single road or on small portions of an urban network.

In the 1950s James Lighthill and Gerald Whitham, two experts in fluid-dynamics, (and independently P. Richards) thought that the equations describing the flow of water could also describe the flow of car traffic. These equations in fluid dynamics are a set of partial differential equations known as the Euler or Navier-Stokes equations, expressing the conservation of mass, momentum and energy. The basic idea is to look at large scales so to consider cars as small particles and their density as the main quantity to be considered. In any case, it is reasonable to assume the conservation of the number of cars,

thus leading again to a conservation law. As traffic jams display sharp discontinuities, there is a correspondence between traffic jams and shock waves. Therefore, fluid-dynamic models for traffic flow seem the most appropriate to detect some phenomena as shocks formation and propagation on roads, since solutions can develop discontinuities in a finite time even starting from smooth initial data (see [19]).
This nonlinear framework, based on the conservation of cars, is described by the equation:

$$\partial_t \rho + \partial_x f(\rho) = 0, \tag{1.0.1}$$

where $\rho = \rho(t,x)$ is the density of cars, with $\rho \in [0, \rho_{max}]$, $(t,x) \in \mathbb{R}^2$ and ρ_{max} is the maximum density of cars on the road; $f(\rho)$ is the flux, which can be written $f(\rho) = \rho v$ with v the average velocity of cars. In most cases one assumes that v is a function of ρ only, thus also $f = f(\rho)$ and its graph is called the *fundamental diagram*. We make this assumption, moreover, for simplicity, we let f be concave and have a unique maximum $\sigma \in]0, \rho_{max}[$ (the non concave case is discussed along the book).

A simple choice for the velocity is that of a linear decreasing function:

$$v(\rho) = v_{max}\,(\rho_{max} - \rho), \tag{1.0.2}$$

thus the resulting flux is given by:

$$f(\rho) = v_{max}\,\rho\,(\rho_{max} - \rho),$$

see Figure 1.1.

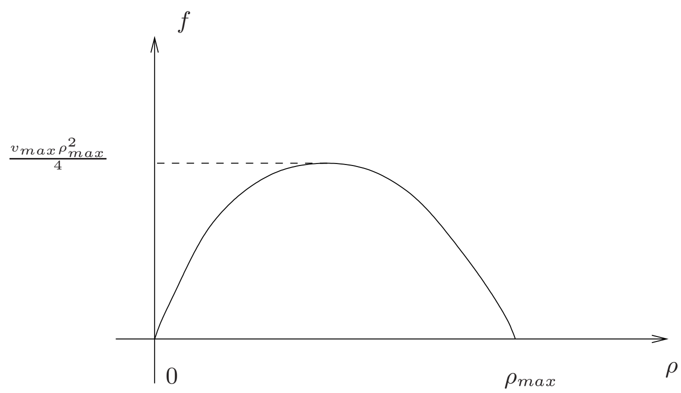

Fig. 1.1. The flux function when the velocity is given by (1.0.2).

To illustrate the behavior of solutions to (1.0.1), we start focusing on the simplest example of junction (or of network) which is a traffic light. Assume that the traffic light is positioned at $x = 0$ and consider the initial density of cars given by:

$$\rho_0(x) = \begin{cases} \rho_{max}, & \text{if } x \le 0, \\ 0, & \text{if } x > 0, \end{cases} \tag{1.0.3}$$

so that the road before the light is full of cars and empty in front of the light. This typical datum corresponds to a situation in which the traffic light is red, so that cars are in a queue at the light. Assume that the green starts at time $t = 0$, then cars start to pass through. This is well detected from the solution to (1.0.1) with initial datum given by (1.0.3). In fact the evolution is given by:

$$\rho(t,x) = \begin{cases} \rho_{max}, & \text{if } x < f'(\rho_{max})t, \\ (f')^{-1}(\frac{x}{t}), & \text{if } f'(\rho_{max})t < x < f'(0)t, \\ 0, & \text{if } x > f'(0)t; \end{cases}$$

see Figure 1.2. For a fixed time $t > 0$, the solution is equal to ρ_{max} to the left of the point $f'(\rho_{max})t$, hence there is a queue beyond this point; the solution vanishes to the right of the point $f'(0)t$, hence no car reached yet this point; finally, in the middle, there is a decreasing density, which is the effect of progressive acceleration of cars at the green light.

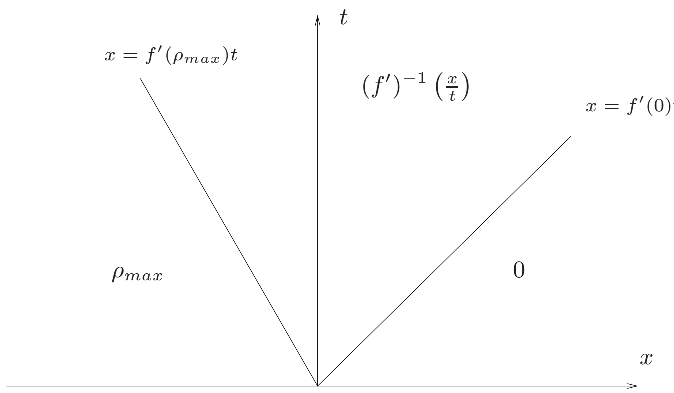

Fig. 1.2. The evolution in the case of a traffic light, when the red light turns green.

The same model describes well the phenomenon of queue formation when a red light starts. Assume now that the initial datum is given by $\rho_0(x) \equiv \sigma$. This typical datum corresponds to a situation in which the traffic light is green, so that cars are flowing through. Assume that the red starts at time $t = 0$, then it is the same as imposing the flux to be zero at $x = 0$. This can be mathematically described by considering the road split in two parts and assign the flux to be zero at $x = 0$ (same as giving boundary data). In this case the solution is given by:

$$\rho(t,x) = \begin{cases} \sigma, & \text{if } x \leq -v_{max}\,\sigma\,t, \\ \rho_{max}, & \text{if } -v_{max}\,\sigma\,t < x \leq 0, \\ 0, & \text{if } 0 < x < v_{max}\,\sigma\,t, \\ \sigma, & \text{if } x > v_{max}\,\sigma\,t; \end{cases}$$

see figure 1.3. Therefore, for a fixed time $t > 0$, a queue is forming at the light and cars are piling up at the point $-v_{max}\,\sigma\,t$; in front of the light there is an empty region up to the point $v_{max}\,\sigma\,t$.

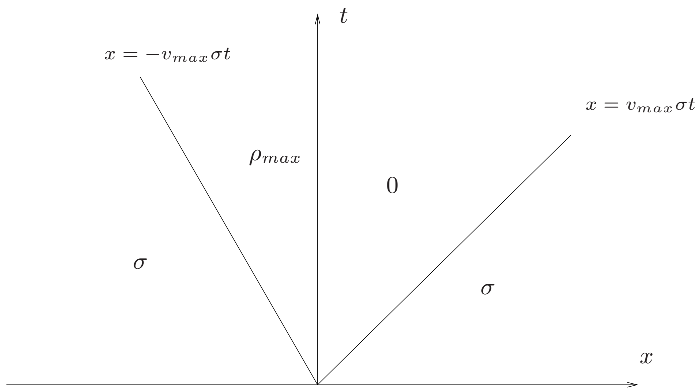

Fig. 1.3. The evolution in the case of a traffic light turning red.

Let us now pass to treat the case of a general network, formed by a finite collection of roads that meet at some junctions. Due to finite speed of waves in solutions to (1.0.1), it is enough to assign the dynamics at each junction separately to obtain an evolution on the whole network. To illustrate the main difficulties, it is enough to focus on a simple junction with one incoming and two outgoing roads. We consider the particular situation in which the incoming road is occupied by cars with maximum density, while the outgoing roads are empty, see Figure 1.4. Indicating by $\rho_{0,i}$ the initial datum on road

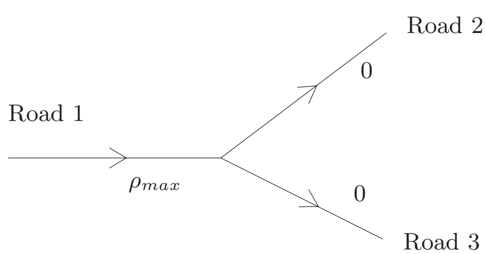

Fig. 1.4. A particular situation in a junction with 1 incoming road and 2 outgoing roads.

i, $i = 1, \ldots, 3$, in formulas we have:

$$\rho_{0,1}(x) = \rho_{max}, \quad \rho_{0,2}(x) = \rho_{0,3}(x) = 0. \tag{1.0.4}$$

There are two extreme behaviors of cars one may expect. Namely: all cars flow towards the first outgoing road or all cars flow towards the second outgoing road. These two behaviors in fact define two different solutions on the network ρ and $\tilde{\rho}$:

$$\rho_1(t,x) = \begin{cases} \rho_{max}, & \text{if } x < f'(\rho_{max})t, \\ (f')^{-1}(\frac{x}{t}), & \text{if } f'(\rho_{max})t < x < 0, \end{cases}$$

$$\rho_2(t,x) = \begin{cases} (f')^{-1}(\frac{x}{t}), & \text{if } 0 \leqslant x < f'(0)t, \\ 0, & \text{if } x > f'(0)t, \end{cases} \qquad \rho_3(t,x) = 0,$$

while $\tilde{\rho}_1 = \rho_1$, $\tilde{\rho}_2 = \rho_3$ and $\tilde{\rho}_3 = \rho_2$.
Notice that both solutions conserve cars quantity through the junction and this can be expressed as:

$$\sum_{\text{incoming roads}} \text{incoming fluxes} = \sum_{\text{outgoing roads}} \text{outgoing fluxes}.$$

Therefore the solely conservation of cars is not sufficient to isolate a unique solution. Moreover, the example shows the necessity of taking into account drivers preferences, in the sense of assigning traffic distribution coefficients, which prescribe the percentage of cars going to each outgoing road. For a general junction with n incoming roads and m outgoing roads, this can be resumed in a general rule:

(A) there exists a traffic distribution matrix

$$A = \begin{pmatrix} \alpha_{n+1,1} & \cdots & \alpha_{n+1,n} \\ \vdots & \vdots & \vdots \\ \alpha_{n+m,1} & \cdots & \alpha_{n+m,n} \end{pmatrix}, \tag{1.0.5}$$

where $0 \leqslant \alpha_{j,i} \leqslant 1$ for every $i \in \{1, \ldots, n\}$ and for every $j \in \{n+1, \ldots, n+m\}$ and

$$\sum_{j=n+1}^{n+m} \alpha_{j,i} = 1 \tag{1.0.6}$$

for every $i \in \{1, \ldots, n\}$. The coefficient $\alpha_{j,i}$ gives the percentage of cars flowing from the i-th incoming road to the j-th outgoing one.

If we denote by f_i and f^j, respectively, the fluxes on the i-th incoming road and on the j-th outgoing one, the rule (A) can be mathematically expressed by the formula

$$\begin{pmatrix} f^1 \\ \vdots \\ f^m \end{pmatrix} = A \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}. \tag{1.0.7}$$

First notice that rule (A) is in agreement with conservation of cars through the junction. In fact from (1.0.6):

$$\sum_j f^j = \sum_j \sum_i \alpha_{j,i}\, f_i = \sum_i \sum_j \alpha_{j,i}\, f_i = \sum_i f_i.$$

One may expect that rule (A) is sufficient to describe in a unique fashion the dynamics at junctions. Unfortunately this is not the case!
Consider again a junction with one incoming and two outgoing roads and the initial data (1.0.4). A solution satisfying rule (A) is simply given by the functions

$$\rho_1(t,x) = \rho_{max}, \quad \rho_2(t,x) = \rho_3(t,x) = 0,$$

i.e. cars do not cross the junction. Notice that this is clearly a solution to (1.0.1) on each road, in fact all derivatives vanish, and it conserves the number of cars through the junction, since all fluxes vanish. Moreover, for every traffic distribution matrix A, this solution always respects rule (A)! Indeed, given any distribution matrix $A = (\alpha_{j,i})$, equation (1.0.7) reads simply $0 = 0$.
The modelling counterpart of such phenomenon is the fact that the attitude of drivers to cross the junction is not detected by rule (A). The solution above captures the situation in which, for some reason, no driver wants to cross the junction. On the contrary, it is reasonable to assume that the will of drivers is that of reaching their final destination as fast as possible. We thus fix another rule

(B) The number of cars passing the junction is the maximum possible (respecting rule (A)).

Notice that the previous rule is equivalent to maximize the fluxes in incoming roads, i.e. to maximize the functional

$$\sum_{i=1}^{n} f_i.$$

We show that, using rules (A) and (B), one can isolate a unique solution on networks, in case $m \geq n$ (and under some generic assumption). Moreover, in case of a single incoming road, it is easy to check that rule (B) is equivalent to maximize the average velocity on the incoming roads.

One can also treat junctions where the number of incoming roads is greater than the number of outgoing ones. But, in this case, if not all cars can go through the junction then there should be a yielding rule between incoming roads. This corresponds to fix *right of way* parameters, which permit to find a unique solution. More precisely, the i–th parameter indicates the percentage, among cars passing through the junction, coming from the i–th incoming road. The details about the mentioned rules are showed in Section 5.2.

The book develops a complete theory for car traffic, including source-destination models and traffic regulation problems. Also it contains some results for telecommunication networks. We illustrate below the content of each chapter.

1.1 Book Chapters

This book is organized as follows. Chapter 2 deals with hyperbolic systems of conservation laws. We introduce the basic definitions and give the basic tool to prove existence and uniqueness of solutions.

In Chapter 3, some fluidodynamic macroscopic models for traffic on a single road are presented. First we describe in detail the first order model of Lighthill, Whitham and Richards (LWR model). Then we pass to the second order models (i.e. systems of two equations) proposed by Payne and Whitham and by Aw and Rascle. Various other more complex models are included, e.g. multi-lane and multi-population.

Chapter 4 is devoted to the study of a general network, composed by a finite number of edges and vertices. The general approach is presented: Riemann problems at junctions and wave-front tracking algorithms. In particular, we give the definition of Riemann solver at a junction.

Chapter 5 is also focused on road networks, where on each road the scalar LWR model determines the evolution of car traffic. At each junction, we construct the solution satisfying rules (A) and (B) (or with the right of way parameters.) Existence of a solution to Cauchy problems on the whole network is granted. The method, based on the wave-front tracking procedure, described in Chapter 4, makes use of total variation estimates on the fluxes, weak convergence and big waves tracing. The solution happens to be not Lipschitz continuous in the L^1-norm with respect to the initial condition.

In Chapter 6, the Aw-Rascle model is put on a road network. At each junction, we consider again rules (A) and (B), but, in this case, they are not sufficient for uniqueness of solutions. Hence additional rules are in order and we propose three alternative ones. Then stability properties of solutions at junctions are studied for each additional rule. Finally, a solution for the Cauchy problem at a single junction is obtained for initial data, which are small perturbations of stable equilibria.

Chapter 7 deals with an extended model of network, containing sources and destinations. The situation on each road is no more described just by the car density, but also by the traffic types, distinguished on the base of sources and destinations. The resulting model is more complicated than that of Chapter 5, and we consider several equations describing the evolution of traffic-type functions. Using a new Riemann solver at junctions, existence of solutions is proved for perturbations of network equilibria.

In Chapter 8, a typical traffic regulation problem is discussed: when constructing a junction, with some traffic flux expected, is it preferable a traffic light or a circle? Using the model of Chapter 5, we assume that drivers arriving at the junction distribute on the outgoing roads according to some expected coefficients. A comparison between the light and the circle is developed showing the behavior of solutions and, in particular, detecting the situation of stuck traffic.

Then we pass to consider the flow of information on a telecommunication network encoded in packets, in Chapter 9. The analogy with fluids comes from considering packets as particles. Our idea is to look at the network at an intermediate time scale so that packets transmission happens at a faster level but the equilibria of the whole network are reached only as asymptotic. This permits to construct a model relying on a macroscopic description. A new Riemann solver is introduced, to better mimic a router policy, getting a more stable situation in which solutions depend in a Lipschitz fashion from initial data.

Chapter 10 deals with some numerical algorithms to simulate the behavior of the urban traffic flow. We focus on the Lighthill-Whitham-Richards model on each road network and, at junctions, the Riemann solver proposed in Chapter 5. Some numerical schemes, based on the Godunov and Kinetic schemes, are proposed and tested on some networks.

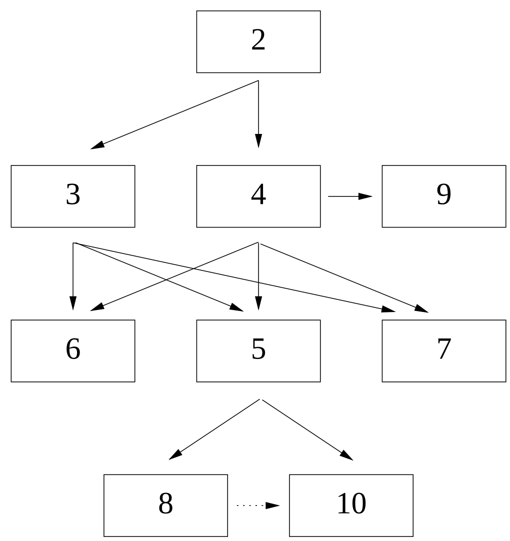

Fig. 1.5. Table of links among book chapters.

A complete course on a fluid-dynamic models for urban traffic can be based on all Chapters excluding Chapter 9. For a short course on urban traffic, the readers can focus on Chapters from 2 to 5 and possibly choose one of Chapters 6, 7, 8 and 10.
A short course on telecommunication networks can be based on Chapters 2, 4, 5 and 9.