Contents

1

Introduction

Since the beginnings of Calculus, differential equations have provided an effective mathematical model for a wide variety of physical phenomena. Consider a system whose state can be described by a finite number of real-valued parameters, say $x = (x_1, \cdots, x_n)$. If the rate of change $\dot{x} = dx/dt$ is entirely determined by the state x itself, then the evolution of the system can be modelled by the ordinary differential equation

$$\dot{x} = g(x). \tag{1.1}$$

If the state of the system is known at some initial time t_0, the future behavior for $t > t_0$ can then be determined by solving a Cauchy problem, consisting of (1.1) together with the initial condition

$$x(t_0) = x_0. \tag{1.2}$$

We are here taking a spectator's point of view: the mathematical model allows us to understand a portion of the physical world and predict its future evolution, but we have no means of altering its behavior in any way. Celestial mechanics provides a typical example of this situation. We can accurately calculate the orbits of moons and planets and exactly predict time and locations of eclipses, but we cannot change them in the slightest amount.

Control theory provides a different paradigm. We now assume the presence of an external agent, i.e. a "controller", who can actively influence the evolution of the system. This new situation is modelled by a **control system**, namely

$$\dot{x} = f(x, u), \qquad u(\cdot) \in \mathcal{U} \tag{1.3}$$

where $\mathcal{U}$ is a family of admissible control functions. In this case, the rate of change $\dot{x}(t)$ depends not only on the state x itself, but also on some external parameters, say $u = (u_1, \cdots, u_m)$, which can also vary in time. The control function $u(\cdot)$, subject to some constraints, will be chosen by a controller in order to modify the evolution of the system and achieve certain preassigned goals — steer the system from one state to another, maximize the terminal value of one of the parameters, minimize a certain cost functional, etc...

In a standard setting, we are given a set of control values $\mathbf{U} \subset \mathbb{R}^m$. The family of admissible control functions is defined as

$$\mathcal{U} \doteq \Big\{ u : \mathbb{R} \mapsto \mathbb{R}^m \,; \quad u \ \text{ measurable}, \quad u(t) \in \mathbf{U} \quad \text{for a.e. } t \Big\}. \tag{1.4}$$

The system (1.1) can then be written as a **differential inclusion**, namely

$$\dot{x} \in F(x) \tag{1.5}$$

where the set of possible velocities is given by

$$F(x) \doteq \Big\{ y \,; \quad y = f(x,u) \quad \text{for some } u \in \mathbf{U} \Big\}. \tag{1.6}$$

Clearly, every admissible trajectory of the control system (1.3) is also a solution of (1.5). Under some regularity assumptions on f, it turns out that the converse is also true: given any absolutely continuous trajectory $t \mapsto x(t)$ of (1.5), one can select a measurable control function $t \mapsto u(t) \in \mathbf{U}$ such that

$$\dot{x}(t) = f(x(t), u(t))$$

at almost every time t. Differential inclusions often provide a convenient approach for the analysis of control systems.

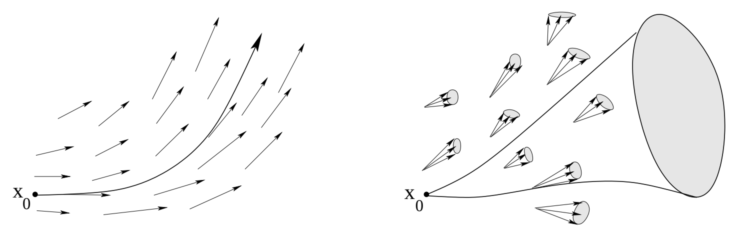

Fig. 1.1. A differential equation vs. a differential inclusion.

Figure 1.1 illustrates the basic difference between an O.D.E and a differential inclusion. In the first case, we have a deterministic model: to each initial state x_0 there corresponds one single trajectory $t \mapsto x(t)$. On the other hand, the evolution described by (1.5) is non-deterministic. Given an initial state x_0, several different trajectories $t \mapsto x(t)$ are possible.

Remark 1.1 Differential inclusions are sometimes used as non-deterministic models, when the future behavior of a system cannot be predicted due to lack of information. It should be clear, however, that is not the point of view of control theory. Here the non-determinacy reflects the possible different strategies of a rational controller, who will make his choices in order to achieve a specific goal.

The control law can be assigned in two basically different ways. In "open loop" form, as a function of time: $t \mapsto u(t)$, and in "closed loop" or *feedback*, as a function of the state: $x \mapsto u(x)$. Implementing an open loop control $u = u(t)$ is in a sense easier, since the only information needed is provided by a clock, measuring time. On the other hand, to implement a closed loop control $u = u(x)$ one constantly needs to measure the state x of the system.

Designing a feedback control, however, yields some distinct advantages. In particular, feedback controls can be more robust in the presence of random perturbations. For example, assume that we seek a control $u(\cdot)$ which steers the system from an initial state P to the origin. If the behavior of the system is exactly described by (1.1), this can be achieved, say, by the open loop control $t \mapsto u(t)$. In many practical situations, however, the evolution is influenced by additional disturbances which cannot be predicted in advance. The actual behavior of the system will thus be governed by

$$\dot{x} = f(x, u) + \eta(t), \tag{1.7}$$

where $t \mapsto \eta(t)$ is a perturbation term. In this case, if the open loop control $u = u(t)$ steers the system (1.1) to the origin, this same control function may not accomplish the same task in connection with (1.7), when a perturbation is present. In Figure 1.2 (left) the solid line depicts the trajectory of the system (1.1), while the dotted line illustrates a perturbed trajectory $\tilde{x}(\cdot)$. We assumed here that the disturbance $\eta(\cdot)$ is active during a small time interval $[t_1, t_2]$. Its presence puts the system "off course", so that the origin is never reached.

Alternatively, one can solve the problem of steering the system to the origin by means of a closed loop control. In this case, we would seek a control function $u = u(x)$ such that all trajectories of the O.D.E.

$$\dot{x} = g(x) \doteq f(x, u(x)) \tag{1.8}$$

approach the origin as $t \to \infty$. This approach is less sensitive to the presence of external disturbances. As illustrated in Figure 1.2 (right), in the presence of an external disturbance $\eta(\cdot)$, the trajectory of the system does change, but our eventual goal – steering the system to the origin – would still be attained.

Various examples of control system are described below.

Example 1.1 (boat on a river). Consider a river with straight course. Using a set of planar coordinates, assume that it occupies the horizontal strip

$$\mathcal{S} \doteq \{(x_1, x_2) : \quad -\infty < x_1 < \infty, \quad -1 \le x_2 \le 1\}.$$

Moreover, assume that speed of the water is given by the velocity vector $\mathbf{v}(x_1, x_2) = (1 - x_2^2,\, 0)$.

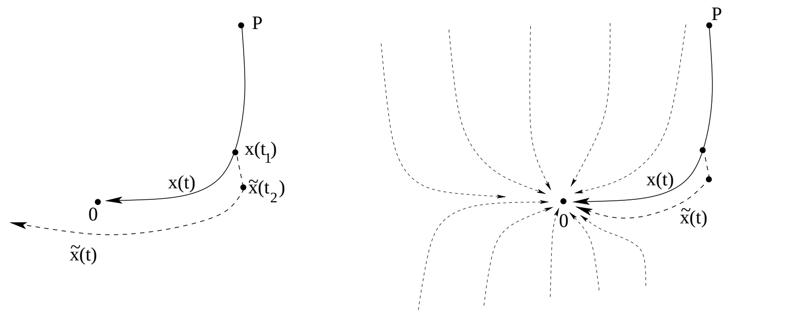

Fig. 1.2. The effect of a perturbation on an open loop and on a feedback control.

If a boat on the river is merely dragged along by the current, its position will be determined by the differential equation

$$(\dot{x}_1, \dot{x}_2) = (1 - x_2^2,\, 0).$$

On the other hand, if the boat is powered by an engine, then its motion can be modelled by the control system

$$(\dot{x}_1, \dot{x}_2) = \mathbf{v} + \mathbf{u} = (1 - x_2^2 + u_1\,,\;\, u_2), \tag{1.9}$$

where the vector $\mathbf{u} = (u_1, u_2)$ describes the velocity of the boat relative to the water. The set $\mathcal{U}$ of admissible controls consists of all measurable functions $\mathbf{u} : \mathbb{R} \mapsto \mathbb{R}^2$ taking values inside the closed disc

$$\mathbf{U} \doteq \left\{ (\omega_1, \omega_2) : \quad \sqrt{\omega_1^2 + \omega_2^2} \leq M \right\}. \tag{1.10}$$

The constant M accounts for the maximum speed (in any direction) that can be produced by the engine.

Given an initial condition $(x_1, x_2)(0) = (\bar{x}_1, \bar{x}_2)$, solving (1.9) one finds

$$\begin{aligned} x_1(t) &= \bar{x}_1 + t + \int_0^t u_1(s)\, ds - \int_0^t \left(\bar{x}_2 + \int_0^s u_2(r)\, dr \right)^2 ds, \\ x_2(t) &= \bar{x}_2 + \int_0^t u_2(s)\, ds \qquad\qquad (-1 \leq x_2 \leq 1). \end{aligned}$$

In particular, the constant control $\mathbf{u} = (u_1, u_2) \equiv (-2/3, 1)$ takes the boat from a point $(\bar{x}_1, -1)$ on one side of the river to the point $(\bar{x}_1, 1)$ on the opposite side, in two units of time. It is not difficult to show that if $M > 0$ the boat can be steered from any point P on the river to any other point Q.

Observe that for the system (1.9)-(1.10) the admissible trajectories coincide with the solutions to the differential inclusion

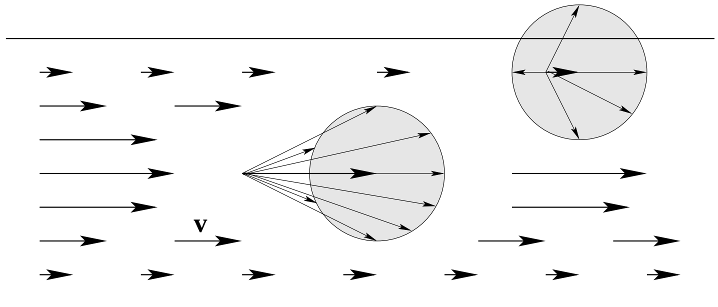

Fig. 1.3. Velocities of the water and of the boat.

$$(\dot{x}_1, \dot{x}_2) \in F(x_1, x_2) \doteq \left\{ (y_1, y_2) : \quad \sqrt{(y_1 - 1 + x_2^2)^2 + y_2^2} \leq M \right\}.$$

Example 1.2 (fishery management). Consider a fish population living in a lake. A simple model describing how its size $x(t)$ varies in time is provided by the O.D.E.

$$\dot{x} = x(\alpha - x). \tag{1.11}$$

Here the constant α describes the maximum sustainable amount of fish which can be present in the lake.

Next, assume that some fish is harvested from the lake, at rate $u = u(t)$. For example, one may think of u as the number of fishermen active at time t. In this case, the evolution of the fish population is described by

$$\dot{x} = x(\alpha - x) - x\,u\,. \tag{1.12}$$

This provides another example of a control system. In a realistic situation, one may select the harvesting rate $u = u(t)$ in order to maximize the total amount of fish caught during a given time interval. Notice that if we adopt a constant harvesting rate $u(t) \equiv \bar{u} < \alpha$, the fish population will approach the asymptotic limit $\bar{x} = \alpha - \bar{u}$. As $t \to \infty$, the choice $u(t) \equiv \alpha/2$ maximizes the average amount of fish caught in unit time. Indeed

$$\bar{x}\,\bar{u} \;=\; (\alpha - \bar{u})\bar{u} \;=\; \max_{\omega \geq 0}\; (\alpha - \omega)\omega\,.$$

In several situations, the optimal harvesting of natural resources leads to control problems of similar type.

Example 1.3 (cart on a rail). Consider a cart which can move without friction along a straight rail (Figure 1.4). For simplicity, assume that it has unit mass. Let $y(0) = \bar{y}$ be its initial position and $v(0) = \bar{v}$ be its initial velocity. If no forces are present, its future position is simply given by

$$y(t) = \bar{y} + \bar{v}t\,.$$

Next, assume that a controller is able to push the cart, with an external force $u = u(t)$. The evolution of the system is then determined by the second order equation

$$\ddot{y}(t) = u(t)\,. \tag{1.13}$$

Calling $x_1(t) = y(t)$ and $x_2(t) = v(t)$ respectively the position and the velocity of the cart at time t, we can rewrite (1.13) as a first order control system:

$$(\dot{x}_1, \dot{x}_2) = (x_2, u). \tag{1.14}$$

Given the initial condition $x_1(0) = \bar{y}$, $x_2(0) = \bar{v}$, solving (1.14) one finds

$$\begin{aligned} x_1(t) &= \bar{y} + \bar{v}t + \int_0^t (t-s)u(s)\,ds, \\ x_2(t) &= \bar{v} + \int_0^t u(s)\,ds. \end{aligned}$$

Assuming that the force satisfies the constraint

$$\big|u(t)\big| \leq 1\,,$$

the control system (1.14) is equivalent to the differential inclusion

$$(\dot{x}_1, \dot{x}_2) \in F(x_1, x_2) = \{(x_2, \omega)\,; \quad -1 \leq \omega \leq 1\}\,.$$

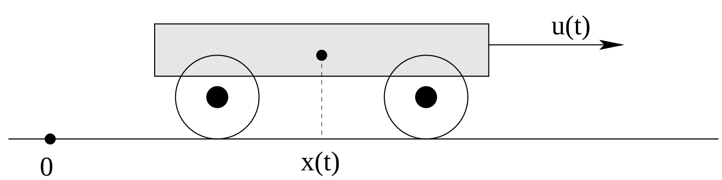

Fig. 1.4. A cart moving along a straight, frictionless rail.

We now consider the problem of steering the system to the origin. More precisely, we want the cart to be at the origin with zero speed. For example, if the initial condition is $(\bar{y}, \bar{v}) = (2, 2)$, this goal is achieved by the open-loop control

$$\tilde{u}(t) = \begin{cases} -1 & \text{if } \ 0 \leq t < 4, \\ 1 & \text{if } \ 4 \leq t < 6, \\ 0 & \text{if } \ t \geq 6. \end{cases}$$

A direct computation shows that $(x_1(t), x_2(t)) = (0, 0)$ for $t \geq 6$. Notice, however, that the above control would not accomplish the same task in connection with any other initial data $(\bar{y}, \bar{v})$ different from $(2, 2)$. This is a consequence of the backward uniqueness of solutions to the differential equation (1.14).

A related problem is that of asymptotic stabilization. In this case, we seek a feedback control function $u = u(x_1, x_2)$ such that, for every initial data $(\bar{y}, \bar{v})$, the corresponding solution of the Cauchy problem

$$(\dot{x}_1, \dot{x}_2) = (x_2\,,\ u(x_1, x_2))\,, \qquad (x_1, x_2)(0) = (\bar{y}, \bar{v})$$

approaches the origin as $t \to \infty$, i.e.

$$\lim_{t\to\infty} (x_1, x_2)(t) = (0, 0).$$

There are several feedback controls which accomplish this task. For example, one can take $u(x_1, x_2) = -x_1 - x_2$.

Because of backward uniqueness, it is clear that there cannot be any Lipschitz continuous feedback $u = u(x_1, x_2)$ which steers every initial condition exactly to the origin within finite time. This goal, however, can be accomplished by the discontinuous feedback law

$$u(x_1, x_2) = \begin{cases} -1 & \text{if} \quad x_2 > 0,\, x_1 \geq -x_2^2/2 \ \text{ or if } \ x_2 \leq 0,\, x_1 > x_2^2/2, \\ 1 & \text{if} \quad x_2 < 0,\, x_1 \leq x_2^2/2 \ \text{ or if } \ x_2 \geq 0,\, x_1 < -x_2^2/2, \\ 0 & \text{if} \quad x_1 = x_2 = 0. \end{cases} \tag{1.15}$$

The multifunction

$$F(x_1, x_2) = \Big\{(x_2, \omega)\,; \quad \omega \in [-1, 1]\Big\}\,,$$

and the trajectories of the corresponding equation

$$(\dot{x}_1,\, \dot{x}_2) = (x_2\,,\ u(x_1, x_2))$$

are shown in figure 1.5.

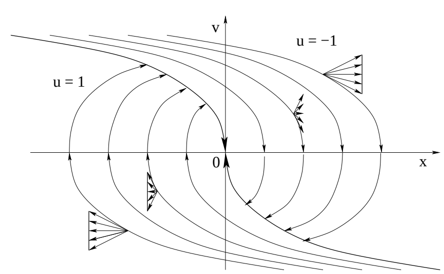

Fig. 1.5. A discontinuous feedback steering every initial point to the origin.

Example 1.4 (car steering). We consider here a mathematical model describing the motion of a car in a large parking lot. At a given time, the position of a car is determined by three scalar parameters: the coordinates (x, y) of its barycenter $B \in \mathbb{R}^2$ and the angle θ giving its orientation, as in Figure 1.6. The driver controls the motion of the car by acting on the gas pedal and on the steering wheel. The control function thus has two scalar components: speed $u(t)$ of the car and the turning angle $\alpha(t)$. The motion is thus described by the control system

$$\begin{cases} \dot{x}_1 = u \cos\theta , \\ \dot{x}_2 = u \sin\theta , \\ \dot{\theta} \ = \alpha\, u . \end{cases} \tag{1.16}$$

It is reasonable here to impose bounds on speed of the car and on the steering angle, say

$$u(t) \in [-m, M] , \qquad \qquad \alpha(t) \in [-\bar{\alpha}, \bar{\alpha}] .$$

A frequently encountered problem is the following: given the initial position, steer the car into a parking spot. The typical maneuver needed for parallel parking is illustrated in Figure 1.6.

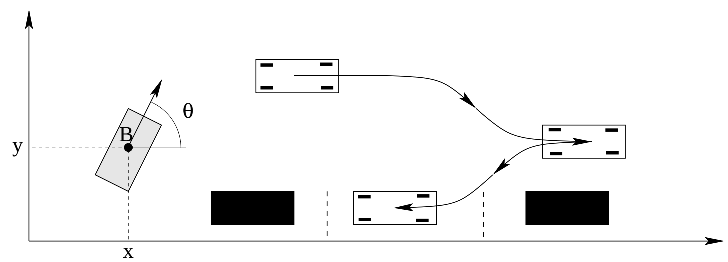

Fig. 1.6. Car parking.

In connection with a control system of the general form (1.3), a wide range of mathematical questions can be formulated.

A first set of problems is concerned with the dynamics of the system. Given an initial state $\bar{x}$, one would like to determine which other states $x \in \mathbb{R}^n$ can be reached using the various admissible controls $u \in \mathcal{U}$. More precisely, given a control function $u = u(t)$, call $t \mapsto x(t, u)$ the solution to the Cauchy problem

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = \bar{x},$$

and define the *reachable set* at time t as

$$R(t) = \{x(t, u); \quad u \in \mathcal{U}\}.$$

For general nonlinear systems, explicit formulas describing $R(t)$ are not available. However, one can analyze several topological and geometric properties of this reachable set. The closure, boundedness, convexity, and the dimension of the set $R(t)$ provide useful information on the control system.

In addition, it is interesting to study whether $R(t)$ is a neighborhood of the initial point $\bar{x}$ for all $t > 0$. In the positive case, the system is said to be *small time locally controllable* at $\bar{x}$. Another important case is when the union of all reachable sets $R(t)$ as $t \to \infty$ includes the entire space $\mathbb{R}^n$. We then say that the system is *globally controllable*.

The dependence of the reachable set $R(t)$ on the time t and on the set of controls $\mathcal{U}$ is also a subject of investigation. For example, if $\mathcal{U}$ is defined by (1.4), one may ask whether the same points in $R(t)$ can be reached by using controls which are piecewise constant, or take values within the set of extreme points of $\mathbf{U}$. Being able to perform the same tasks by means of a smaller set of control functions, easier to implement, is quite relevant in practical applications.

Different kind of problems arise in connection with controls in feedback form. Here one basic goal is to construct a feedback control $u = u(x)$ such that the resulting dynamics determined by the differential equation (1.8) has certain desired properties. For example, one could seek a control which steers every initial state asymptotically toward the origin, or stabilizes the system in a neighborhood of a periodic orbit, etc...

The regularity of the feedback control is often a major issue of investigation. Ideally, one would like the function $x \mapsto u(x)$ to be smooth, or at least continuous. However, for some nonlinear systems it turns out that certain tasks cannot be accomplished by any continuous feedback law. This raises the question of what kind of discontinuities can be allowed in a feedback control, and how to interpret the solution to the resulting O.D.E. (1.8) when the right hand side is a discontinuous function of the state x.

A further key issue related to feedback control is robustness. In general, the differential equation (1.3) provides only an approximate description of reality. External disturbances may affect the evolution of the system. Since these cannot be predicted in advance, it is important to design a control such that the system's behavior will not be much affected by these small perturbations. Continuous feedback laws are usually robust, but the problem can become quite delicate when discontinuous feedbacks are implemented.

A second, very important area of control theory is concerned with **optimal control**. In many applications, among all strategies which accomplish a certain task, one seeks an optimal one, based on a given performance criterion. In mathematical terms, a performance criterion can be defined by an integral functional of the form

$$J = \int_0^T L(t, x, u)\, dt. \tag{1.17}$$

The value of J will have to be optimized among all admissible trajectories of (1.3), with a number of initial or terminal constraints.

For example, among all controls which steer the system from the initial point $\bar{x}$ to some point on a target set Ω at time T, we may seek the one that minimizes the cost functional (1.17). This problem is formulated as

$$\min_{u \in \mathcal{U}} \int_0^T L(t, x, u)\, dt \tag{1.18}$$

subject to

$$\dot{x} = f(t, x, u), \qquad x(0) = \bar{x}, \qquad x(T) \in \Omega\,. \tag{1.19}$$

Observe that if (1.3)-(1.4) takes the simple form

$$\dot{x} = u, \qquad u(t) \in \mathbf{U} = \mathbb{R}^n, \tag{1.20}$$

and if $\Omega = \{\bar{y}\}$ consists of just one point, then we do not have any constraint on the derivative $\dot{x}$. Our problem of optimal control thus reduces to the standard problem in the Calculus of Variations:

$$\min_{x(\cdot)} \int_0^T L(t, x, \dot{x})\, dt\,, \qquad x(0) = \bar{x}, \qquad x(T) = \bar{y}\,. \tag{1.21}$$

Roughly speaking, the main difference between the problem (1.18)-(1.19) and (1.21) is that in (1.21) the derivative $\dot{x}$ is unrestricted, while in (1.18)-(1.19) it is constrained within the closed set $F(x)$ introduced at (1.6).

The basic mathematical theory of optimal control has been concerned with three main issues:

(i) *Existence of optimal controls.* Under a suitable convexity assumption, optimal solutions can be constructed following the direct method in the Calculus of Variations, i.e., as limits of minimizing sequences, relying on compactness and lower semi-continuity properties. When the convexity condition is not satisfied, the problem usually does not admit any optimal solution. In some special cases, however, the existence of optimal control can still be proved, using a variety of more specialized techniques.

(ii) *Necessary conditions for the optimality of a control.* The ultimate goal of any set of necessary conditions is to isolate a hopefully unique candidate for the minimum. The major result in this direction is the celebrated Pontryagin Maximum Principle , which extends to control systems the Euler-Lagrange and the Weierstrass necessary conditions for a strong local minimum in the Calculus of Variations. These first order conditions have been supplemented by several high order conditions, which provide additional information in a number of special cases.

(iii) *Sufficient conditions for optimality.* For some special classes of optimal control problems, one finds a unique control $u^*(\cdot)$ which satisfies the Pontryagin's necessary conditions. In this case, u^* provides the unique solution to the optimization problem.

For general nonlinear systems, however, conditions which guarantee the optimality of a control $u^*(\cdot)$ can only be obtained by a global analysis.

Toward this goal, a standard technique is to embed (1.18)-(1.19) in a family of problems, obtained by varying the initial conditions. The value function V, defined as

$$V(\tau, y) = \min_{u \in \mathcal{U}} \int_{\tau}^{T} L(t, x, u)\, dt$$

subject to

$$\dot{x} = f(t, x, u), \qquad x(\tau) = y, \quad x(T) \in \Omega,$$

can then be characterized as the solution to a first order Hamilton-Jacobi partial differential equation and computed by dynamic programming methods. In turn, from the knowledge of the function V and its gradient $\nabla_x V$, one can determine the optimal control u in feedback form. The strong nonlinearity of the Hamilton-Jacobi equation and the possible lack of regularity of the value function V account for the main difficulties toward a rigorous mathematical analysis. In this direction, a major step forward has been provided by the theory of viscosity solutions.

In addition to the fundamental theory, valid for control systems of the general form (1.3), a wealth of results are available for some special systems which can be analyzed in much greater detail. In particular, consider the linear system with constant coefficients

$$\dot{x} = Ax + Bu, \tag{1.22}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and the matrices A, B have dimension $n \times n$ and $n \times m$, respectively. For a given control $t \mapsto u(t)$, the corresponding solution of (1.22) admits the explicit integral representation

$$x(t, u) = e^{tA} x(0) + \int_0^t e^{(t-s)A} B u(s)\, ds.$$

This allows an in-depth study of all the relevant properties of the system.

Another important class consists of semi-linear systems, having the form

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x) u_i\,, \tag{1.23}$$

where $f_0, f_1, \cdots, f_m$ are smooth vector fields on $\mathbb{R}^n$. In general, there exists no explicit representation for the trajectories of (1.23) in terms of integrals of the control. Nevertheless, a rich mathematical theory has been developed for these systems, applying techniques and ideas from differential geometry and the theory of Lie algebras.