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Auxiliary Materials

1.1 Functional spaces and embedding theorems

We shall use the following notation. We shall denote by $\mathbb{R}, \mathbb{C}, \mathbb{Z}$ and $\mathbb{N}$ the sets of real, complex, integer and natural numbers respectively; $\mathbb{Z}_+ = \{x \in \mathbb{Z} | x \geq 0\}$ is the set of nonnegative integers. $\mathbb{R}^n$ is the standard real vector space of dimension n. We denote by D_i the operator of partial differentiation with respect to x_i,

$$D_i u = \frac{\partial u}{\partial x_i} \quad (i = 1, \dots n). \tag{1.1}$$

As usual, we use multi-index notation to denote higher order partial derivatives,

$$D^\gamma = D_1^{\gamma_1} \cdots D_n^{\gamma_n} \, , \; |\gamma| = \gamma_1 + \cdots + \gamma_n \tag{1.2}$$

is a partial derivatives of order $|\gamma|$, for a given $\gamma = (\gamma_1, \cdots, \gamma_n)$, $\gamma_i \in \mathbb{Z}_+$.

Let $u : \Omega \subset \mathbb{R}^n$ be a real function defined on a bounded domain Ω. The space of continuous functions over the bounded domain $\bar{\Omega}$ is denoted by $C(\bar{\Omega})$; the norm in $C(\bar{\Omega})$ is defined in a standard way:

$$\|u\|_{C(\bar{\Omega})} = \sup\{|u(x)| \mid \; x \in \bar{\Omega}\}. \tag{1.3}$$

The space $C^m(\Omega)$ consists of all real functions on Ω which have continuous partial derivatives up to order m. By definition, u belongs to $C^m(\bar{\Omega})$ iff (abbreviation for if and only if) $u \in C^m(\Omega)$ and u and all its partial derivatives up to order m can be extended continuously to $\bar{\Omega}$.

Let $0 < \gamma < 1$ and $k \in \mathbb{Z}_+$. By definition $C^{k,\gamma}(\bar{\Omega})$ denotes the Hölder space of functions $u : \Omega \to \mathbb{R}$ such that, $D^\alpha u : \Omega \to \mathbb{R}$ exists and is uniformly continuous when $|\alpha| = k$ and such that

$$|u|_{k,\gamma} \equiv \sup \left\{ \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\gamma} \mid x, y \in \Omega, \; x \neq y, \; |\alpha| \leq k \right\} \tag{1.4}$$

is finite. For $u \in C^{k,\gamma}(\bar{\Omega})$, we set

$$\|u\|_{k,\gamma} = |u|_{k,\gamma} + \sum_{|\alpha|\le k} \max\{|D^\alpha u(x)| \mid x \in \bar{\Omega}\}. \tag{1.5}$$

We also have

$$C^{k,\gamma}(\partial\Omega) = \{\varphi : \partial\Omega \to \mathbb{R} \mid \text{ there exists } u \in C^{k,\gamma}(\bar{\Omega}) \text{ with } u|_{\partial\Omega} = \varphi\},$$

and for $\varphi \in C^{k,\gamma}(\partial\Omega)$ we set

$$\|\varphi\|_{k,\gamma} = \inf\{\|u\|_{k,\gamma} : \; u|_{\partial\Omega} = \varphi; u \in C^{k,\gamma}(\bar{\Omega})\} \tag{1.6}$$

In cases when it is clear from the context where the function under consideration is defined, we shall sometimes simply write $u \in C^k$ instead of, for example, $u \in C^k(\mathbb{R}^n)$. In several examples we shall use the spaces of functions that are 2π-periodic in every variable x_i $(i = 1, \cdots, n)$. We shall consider such functions as being defined on the n-dimensional torus $T^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$.

We denote by $L_p(\Omega)$, $1 \le p < \infty$, the space of measurable functions with the finite norm

$$\|u\|_{0,p} = \|u\|_{L_p} = \left(\int_\Omega |u(x)|^p dx\right)^{1/p}. \tag{1.7}$$

We denote by $L_\infty(\Omega)$ the space of almost everywhere bounded functions,

$$\|u\|_{0,\infty} = \|u\|_{L_\infty} = \operatorname{vrai\,sup}\{|u(x)| \mid x \in \Omega\} \tag{1.8}$$

(for continuous functions this norm coincides with the norm of $C(\bar{\Omega})$).

The norm in the Sobolev space $W_p^l(\Omega)$, $l \in \mathbb{Z}_+$, $1 \le p < \infty$, is defined by the formula

$$\|u\|_{l,p} = \left(\sum_{|\alpha|\le l} \|D^\alpha u\|_{L_p}^p\right)^{1/p}. \tag{1.9}$$

In the case $p = 2$ this Sobolev space is a Hilbert space and is denoted $H^l(\Omega)$, $H^l(\Omega) = W_2^l(\Omega)$. The scalar product in $H^l(\Omega)$ is defined by the formula

$$(u, v)_l = \sum_{|\alpha|\le l} \int_\Omega D^\alpha u(x) \cdot D^\alpha v(x) dx. \tag{1.10}$$

The space $W_p^l(\Omega)$ is the completion of $C^l(\Omega)$ with respect to the norm (1.9).

The norms $C^{k,\gamma}(T^n)$ and $W_p^l(T^n)$ are defined by (1.4) and (1.9) with $\Omega = (]0, 2\pi[)$. The scalar product and the norm in $H^l(T^n)$, which are equivalent to those defined by (1.10), are defined in terms of Fourier coefficients,

$$(u,v)_l = \sum \tilde{u}(\xi) \cdot \overline{\tilde{v}(\xi)} \cdot (1+|\xi|^2)^l; \ \|u\|_l^2 = \langle u,u\rangle_l, \tag{1.11}$$

where the summation is over $\xi \in \mathbb{Z}^n$; the bar denotes complex conjugation; $\tilde{u}(\xi)$ and $\tilde{v}(\xi)$ are the Fourier coefficients,

$$\tilde{u}(\xi) = (2\pi)^{-n} \int u(x) e^{-ix\cdot\xi}\, dx. \tag{1.12}$$

Here the integral is over $[0,2\pi]^n$;

$$x\cdot\xi = x_1\xi_1 + \cdots + x_n\xi_n. \tag{1.13}$$

The formula (1.11) defines the norm in $H^l(T^n)$ for $l \in \mathbb{R}$ as well as $l \in \mathbb{Z}_+$. We denote by $C^\infty(\bar{\Omega})$ the space $\bigcap\limits_{k\geq 0} C^k(\bar{\Omega})$; by $C_0^\infty(\Omega)$ the set of functions from $C^\infty(\bar{\Omega})$ which vanish on a neighbourhood of the boundary $\partial\Omega$. We shall use also spaces of functions which vanish on $\partial\Omega$. In this case we shall denote the corresponding space as follows:

$$C^{k,\gamma}(\bar{\Omega}) \cap \{u|_{\partial\Omega} = 0\}, \quad W_p^l(\Omega) \cap \{u|_{\partial\Omega} = 0\}. \tag{1.14}$$

We denote the completion of $C_0^\infty(\Omega)$ with respect to the norm of $H^l(\Omega)$ by $H_0^l(\Omega)$ and with respect to the norm of $W_p^1(\Omega)$ by $W_{p,0}^1(\Omega)$. It is well-known that

$$H_0^1(\Omega) = H_1(\Omega) \cap \{u|_{\partial\Omega} = 0\}; \quad W_{p,0}^1(\Omega) = W_p^1 \cap \{u|_{\partial\Omega} = 0\}. \tag{1.15}$$

The Sobolev spaces $H^\rho(\Omega)$ with non integer $\rho \geq 0$, $\rho = k+\beta$, $k \in \mathbb{Z}$, $0 \leq \beta < 1$ are endowed with the norm

$$\|u\|_\rho^2 = \|u\|_k^2 + \int_{|y|\leq\delta} \|u(x+y) - u(x)\|_k^2 \cdot |y|^{-n-2\beta} dy \tag{1.16}$$

($u(x)$ is extended over a δ-neighbourhood of the boundary, see [104]).

By $S(\mathbb{R}^n)$ we denote the class of rapidly decreasing (at ∞) functions $u(x) \in C^\infty(\mathbb{R}^n)$, with

$$(1+|x|)^k\, |D^\alpha u(x)| \leq C_{k,\alpha}$$

for each $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$ and $k \in \mathbb{Z}_+$, where $C_{k,\alpha}$ are constants.

Recall that an operator $j : X \to Y$ between Banach spaces with $X \subseteq Y$ is an embedding iff $j(x) = x$ for all $x \in X$. The operator j is continuous iff $\|x\|_Y \leq \text{constant}\, \|x\|_X$ for all $x \in X$. Further, j is compact iff j is continuous, and every bounded set in X is relatively compact in Y. If the embedding $X \hookrightarrow Y$ is compact, then each bounded sequence $\{x_n\}$ in X has a subsequence $\{x_{n'}\}$ which is convergent in Y.

We shall widely use Sobolev's embedding theorems formulated below.

Theorem 1.1 *Let Ω be a bounded domain in $\mathbb{R}^n$, with smooth boundary $\partial\Omega$ and $0 \le k \le m-1$. (See [104].)*
Then

$$W_p^m(\Omega) \hookrightarrow W_q^k(\Omega), \text{ if } \frac{1}{q} \ge \frac{1}{p} - \frac{m-k}{n} > 0 \tag{1.17}$$

$$W_p^m(\Omega) \hookrightarrow W_q^k(\Omega), \text{ if } q < \infty, \frac{1}{p} = \frac{m-k}{n} \tag{1.18}$$

$$W_p^m(\Omega) \hookrightarrow C^{k,\delta}(\bar{\Omega}), \text{ if } \frac{n}{p} < m - (k+\delta),\ 0 < \delta < 1. \tag{1.19}$$

The first embedding is compact if $\frac{1}{q} > \frac{1}{p} - \frac{m-k}{n}$. The last two embeddings are compact.

Theorem 1.2 *Let $0 \le \beta < \alpha \le 1$ or $\alpha, \beta \in \mathbb{Z}$ with $0 \le \beta < \alpha$ (see [142]). Then the embedding*

$$C^\alpha(\bar{\Omega}) \hookrightarrow C^\beta(\bar{\Omega}) \quad \textit{is compact} \tag{1.20}$$

and for $k + \beta < m + \alpha$, with $0 \le \alpha, \beta \le 1,\ m \ge k \ge 0$ the embeddings

$$C^{m,\alpha}(\bar{\Omega}) \hookrightarrow C^{k,\beta}(\bar{\Omega}) \ \textit{ are compact.} \tag{1.21}$$

1.2 Linear Elliptic Boundary value problems

Notation. Let Ω be a bounded domain in $\mathbb{R}^n$. For $\alpha = (\alpha_1, \cdots, \alpha_n)$ an n-tuple of nonnegative integers, recall that $D^\alpha = \prod_{i=1}^n \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}$, $|\alpha| = \sum_{i=1}^n \alpha_i$ and let $\xi^\alpha = \prod_{i=1}^n (\xi_i)^{\alpha_i}$ if $\xi \in \mathbb{C}^n$.

Every linear differential operator L of order $2m$ ($m \in \mathbb{N}$) has the form

$$Lu = \sum_{|\alpha| \le 2m} a_\alpha(x) \cdot D^\alpha u. \tag{1.22}$$

All coefficients $a_\alpha(x)$ are assumed to be real.

The partial differential operator defined by (1.22) is called elliptic of order $2m$ if its principal symbol,

$$p_0(x,\xi) = \sum_{|\alpha|=2m} a_\alpha(x) \cdot \xi^\alpha,$$

has the property that $p_0(x,\xi) \ne 0$ for all $x \in \Omega$, $\xi \in \mathbb{R}^n \setminus \{0\}$.

The differential operator L defined by (1.22) is called uniformly elliptic in Ω, if there is some $c > 0$, such that

$$(-1)^m \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \geq C|\xi|^{2m} \text{ for every } x \in \Omega, \xi \in \mathbb{R}^n \setminus \{0\} \qquad (1.23)$$

Throughout we assume that $\partial\Omega$ is a smooth $(n-1)$-manifold.

Suppose now that L is elliptic and of order $2m$.

Let $\{m_i,\ 1 \leq i \leq m\}$ be distinct integers with $0 \leq m_i \leq 2m-1$, and suppose that for $1 \leq i \leq m$ we prescribe a differential operator B_i of order m_i on $\partial\Omega$, by

$$B_i u(x) = \sum_{|\alpha|\leq m_i} b_{\alpha,i}(x) D^\alpha u(x), \quad i = 1, \cdots, m. \qquad (1.24)$$

The family of boundary operators $B = \{B_1, \cdots, B_m\}$ is said to satisfy the Shapiro-Lopatinski covering condition with respect to L provided that the following algebraic condition is satisfied. For each $x \in \partial\Omega$, $\mathbf{N} \in \mathbb{R}^n \setminus \{0\}$ normal to $\partial\Omega$ at x and $\xi \in \mathbb{R}^n \setminus \{0\}$ with $\langle \xi, \mathbf{N} \rangle = 0$, consider the $(m+1)$ polynomials of a single complex variable

$$\begin{aligned} &\tau \longmapsto p_0(x, \xi + \tau\mathbf{N}), \\ &\tau \longmapsto \sum_{|\alpha|=m_i} b_{\alpha,i}(x) \cdot (\xi + \tau\mathbf{N})^\alpha \equiv p_{0,i}(x, \xi, \tau), \quad 1 \leq i \leq m. \end{aligned} \qquad (1.25)$$

Let $\tau_1^+, \cdots, \tau_m^+$ be the m complex zeros of $p_0(x, \xi + \tau\mathbf{N})$ which have positive imaginary part. Then $\{p_{0,i}(\tau)\}_{i=1}^m$ are assumed to be linearly independent modulo $\prod_{i=1}^m (\tau - \tau_i^+) = M^+(x, \xi, \mathbf{N}, \tau)$, i.e., after division by $M^+(x, \xi, N, \tau)$ all the various remainders are linearly independent.

In other words, let

$$p'_{0,i}(x, \xi, \mathbf{N}, \tau) = \sum_{k=0}^{m-1} b_{i,k}(x, \xi, \mathbf{N}) \cdot \tau^k, \quad i = 1, \cdots, m$$

be the remainders after division by $M^+(x, \xi, \mathbf{N}, \tau)$. Then the condition of the Shapiro-Lopatinski implies that

$$D(x, \xi, N) = \det \|b_{ik}(x, \xi, \mathbf{N})\| \neq 0 \qquad (1.26)$$

for all $x \in \partial\Omega$, and for all $\mathbf{N} \in \mathbb{R}^n \setminus \{0\}$ normal to $\partial\Omega$ at x and $\xi \in \mathbb{R}^n \setminus \{0\}$ with$\langle \xi, \mathbf{N} \rangle = 0$.

Definition 1.1 *We say that* $(L, B_1, \cdots, B_m)$ *defines an elliptic boundary value problem of order* $(2m, m_1, \cdots, m_m)$ *if* L *given by (1.22), is uniformly elliptic and of order* $2m$*, each* B_i *given by (1.24) has order* $m_i, 0 \leq m_i \leq 2m-1$*,* *the m_i's are distinct, $\partial\Omega$ is non characteristic to B_i at each point and $\{B_i\}_{i=1}^m$ satisfy the Shapiro-Lopatinski condition with respect to L (see [95]).*

We have the following lemma (see [9, 54]).

Lemma 1.1 *Let $(L, B_1, \cdots, B_m)$ define an elliptic boundary value problem of order $(2m, m_1, \cdots, m_m)$. Then*

$$(L \circ \triangle^l, B_1 \circ \tilde{\triangle}^l, \cdots, B_m \circ \tilde{\triangle}^l, L \circ \frac{\partial u}{\partial \mathbf{N}})$$

defines an elliptic boundary value problem of order $(2k+2l, m_1+2l, \cdots, m_m+2l, 2m+1)$ where $\tilde{\triangle}$ is the Laplace-Beltrami operator, $l \in N$.

Proof: The principal symbol of $L \circ \triangle^l$ is $|\xi|^{2l} \cdot p_0(x,\xi)$, so it is clear that $L \circ \triangle^l$ is uniformly elliptic.

Let $x \in \partial\Omega$ and $\xi, \mathbf{N} \in \mathbb{R}^n \setminus \{0\}$, with $\langle \xi, \mathbf{N} \rangle = 0$ and $\mathbf{N}$ normal to $\partial\Omega$ at x. It is obvious that the principal symbol operators $B_i \circ \tilde{\triangle}^l$ and $L \circ \frac{\partial}{\partial \mathbf{N}}$ at $\xi + \tau\mathbf{N}$ are $\psi_l(\xi) \cdot p_{0i}(x, \xi + \tau\mathbf{N})$ and $\tau p_0(x, \xi + \tau\mathbf{N})$ respectively, where $\psi_l(\xi) \neq 0$.

If $\tau_1^+, \cdots, \tau_m^+$ are the m roots of $p_0(x, \xi + \tau\mathbf{N}) = 0$ having positive imaginary part, then the $m+1$ roots of $|\xi + \tau\mathbf{N}|^2 \cdot p_0(x, \xi + \tau\mathbf{N}) = 0$ with positive imaginary part are given by $\tau_1^+, \cdots, \tau_m^+$, $i \cdot \frac{|\mathbf{N}|}{|\xi|}$.

We must show that if $\lambda_1, \cdots, \lambda_{m+1} \in \mathbb{C}$ and $h(\tau)$ is a polynomial with

$$\begin{aligned} \psi_l(\xi) \textstyle\sum_{i=1}^m \lambda_i \cdot p_{0i}(x, \xi + \tau\mathbf{N}) + \lambda_{m+1} \tau p_0(x, \xi + \tau\mathbf{N}) = \\ h(\tau) \cdot \left(\tau - \frac{i|\mathbf{N}|}{|\xi|}\right) \cdot \textstyle\prod_{i=1}^m (\tau - \tau_i^+) \end{aligned} \tag{1.27}$$

then $\lambda_i = 0$, $1 \leq i \leq m+1$ and $h(\tau) \equiv 0$. Due to the assumption that $(B_1, \cdots, B_m)$ satisfy the covering condition it is not difficult to see that $\lambda_1 = \cdots = \lambda_m = 0$. But then the right-hand side of (1.27) has more roots with positive imaginary part than does the left-hand side, so that $\lambda_{m+1} = 0$ and $h(\tau) \equiv 0$. □

With appropriate smoothness conditions on the coefficients (see Lemma 2.2 below), elliptic boundary value problems induce linear Fredholm operators in Sobolev spaces. Here the spaces $W_p^{2m+k-m_i-1/p}(\partial\Omega)$ with the fractional differentiation order $2m + k - m_i - \frac{1}{p}$ play a decisive role. Before giving a precise definition we wish to point out *a priori* the most important property of these spaces, i.e. the surjective boundary operator

$$T : C^\infty(\bar{\Omega}) \to C^\infty(\partial\Omega)$$

which assigns to each function $u \in C^\infty(\bar{\Omega})$ its classical boundary value Tu on $\partial\Omega$, can be extended uniquely to a continuous linear surjective operator

$$T : W_p^{2m+k}(\Omega) \to W_p^{2m+k-m_i-1/p}(\partial\Omega).$$

Here $k \geq 0$ and $m \geq 1$ are integers, and $1 < p < \infty$ (we are mainly interested in the case $p = 2$, $W_2^{2m}(\Omega) = H^{2m}(\Omega)$). Then Tu is described naturally as the generalized boundary value of $u \in W_p^{2m+k}(\Omega)$. These functions u have generalized derivatives $D^\alpha u$ up to order $2m+k$ on Ω. The functions $D^\alpha u$ with $|\alpha| \leq m_i$ have generalized boundary values which all lie in $W_p^{2m+k-m_i-1/p}(\partial\Omega)$, since $m_i < 2m$. Consequently, $B_i u \in W_p^{2m+k-m_i-1/p}(\partial\Omega)$ also. The differential operators L and the boundary operator B_i are thus to be understood in the space of generalized derivatives on Ω and as generalized boundary values respectively.

Definition of the space $W_p^{m-1/p}(\partial\Omega)$.

Let Ω be an open subset of $\mathbb{R}^n$ with sufficiently smooth boundary and $\{U_i\}_{i=1}^l$ be an open covering of $\bar{\Omega}$ with diffeomorphisms $\varphi_i : U_i \to \mathbb{R}^n, \varphi_i \in C^m(U_i)$, such that $\varphi_i(U_i) = V_1 = \{y \in \mathbb{R}^n \mid |y| < 1\}$ if $U_i \subset \Omega$, and

$$\varphi_i(U_i \cap \bar{\Omega}) = V_1^+ = \{y \in \mathbb{R}^n \mid |y| < 1, y_n \geq 0\},$$

$$\varphi_i(U_i \cap \partial\Omega) = \tilde{V}_1 = \{y \in \mathbb{R}^n \mid |y| < 1, y_n = 0\} \text{ if } U_i \cap \partial\Omega \neq \emptyset.$$

Let $\chi_i(x)$ be a partition of unity subordinated to $\{U_i\}_{i=1}^l$ and let $\lambda_i(y) := \chi_i(\varphi_i^{-1}(y))$.

For each $u(x) \in C^m(\partial\Omega)$, $0 < \delta < 1$, $p > 1$ we define the norm:

$$\|u\|'_{m-\delta,p,\partial\Omega} = \Bigg\{ \sum_{i \in I'} \Bigg[\sum_{|\alpha| \leq m-1}' \int_{\tilde{V}_1} |D_y^\alpha(\lambda_i(y) \cdot u_i(y))|^p \, dy' + \sum_{|\alpha| \; = m-1}' \int_{\tilde{V}_1} \int_{\tilde{V}_1} |D_y^\alpha(\lambda_i(y) \cdot u_i(y)) - D_z^\alpha(\lambda_i(z) \cdot u_i(z)) \,|^p \cdot \frac{dy'dz'}{|y'-z'|^{n+p-1-\delta p}} \Bigg] \Bigg\}^{\frac{1}{p}}, \tag{1.28}$$

$u_i(y) = u(\varphi_i^{-1}(y))$, $y' = (y_1, \cdots, y_{n-1})$, $I' \subset \{1, \cdots, l\}$ such that $U_i \cap \partial\Omega \neq \emptyset$ and $\sum'$ implies that the sum is taken over those α for which $\alpha_n = 0$, $\alpha = (\alpha_1, \cdots, \alpha_n)$.

By definition, the norm in $W_p^{m-\frac{1}{p}}(\partial\Omega)$, $p > 1$ is defined as the norm $\|\cdot\|'_{m-\frac{1}{p},p,\partial\Omega}$. For more details see [104].

Let us return to the discussion of elliptic boundary value problems. We first recall some results regarding linear Fredholm operators. Let X and Y be real Banach spaces. By $L(X,Y)$ we denote the Banach space of bounded linear operators from X to Y. An operator T in $L(X,Y)$ is called Fredholm if the kernel (nullspace) $\ker T := \{x \in X : Tx = 0\}$ has finite dimension and the image (range) of T, $R(T) := \{Tx, x \in X\}$ is of finite codimension in Y, that is $\operatorname{codim} R(T) = \dim Y/R(T) < \infty$. For a Fredholm operator $T : X \to Y$, the numerical Fredholm index of T, $\operatorname{ind}(T)$ is defined by

$$\mathrm{ind}(T) = \dim\ker T - \mathrm{codim}(R(T)).$$

Lemma 1.2 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with $\partial\Omega$ smooth. Suppose that $s > n/2$, $a_\alpha \in H^s(\Omega)$ if $|\alpha| \le 2m$, while $b_{\alpha,i} \in H^{s+2m-m_i}(\partial\Omega)$ and $i = 1, \cdots m$. Then the following three assertions are equivalent:*

(i) *The operator $A = (L, B_1, \cdots, B_m)$*

$$A : H^{s+2m}(\Omega) \longrightarrow H^s(\Omega) \times \prod_{i=1}^{m} H^{s+2m-m_i}(\partial\Omega) \tag{1.29}$$

is an elliptic boundary value problem of order $(2m, m_1, \cdots, m_m)$

(ii) *The operator $A = (L, B_1, \cdots B_m)$ is Fredholm*

(iii) *There is some $c > 0$, such that if $u \in H^{s+2m}(\Omega)$, then*

$$\|u\|_{2m+s} \le c\left[\|Lu\|_s + \sum_{i=1}^{m} \|B_i(x,D)u\|_{2m+s-m_i-\frac{1}{2}} + \|u\|_s\right]. \tag{1.30}$$

Proof. If each $a_\alpha \in C^s(\Omega)$ and each $b_{\alpha,i} \in C^{2k+s-m_i}(\Omega)$, then a priori estimate (1.30) is contained in [5]. It is not difficult to see that (1.30) also holds under the present smoothness conditions. Thus, in fact a priori estimate (1.30) and equivalence (i) and (iii) follows from [5]. Equivalence (i) and (iii) to (ii) can be proved analogously to [6]. □

Remark 1.1: Of course, the Fredholm index of $(L, B_1, \cdots, B_m)$ need not be equal to 0. If L is uniformly elliptic and $B_i u(x) = \left(\frac{\partial}{\partial \mathbf{N}}\right)^{i-1} u(x)$ for $1 \le i \le m$, then the index

$$A = (L, B_1, \cdots, B_m) : H^{2m+s}(\Omega) \to H^s(\Omega) \times \prod_{i=1}^{m} H^{2m+s-m_i-\frac{1}{2}}(\partial\Omega)$$

is 0. (see [9, 93]).

Remark 1.2: (C^γ-theory). The a priori estimates(1.30) remain valid if we choose the following B- spaces for $0 < \gamma < 1$:

$$X = C^{2m+s,\gamma}(\bar{\Omega}),\ Y = C^{s,\gamma}(\bar{\Omega}),\ Z = C(\bar{\Omega}), Y_j = C^{2m+s-m_i,\gamma}$$

i.e.

$$\|u\|_X \le \mathrm{constant}(\|Lu\|_Y + \sum_{j=1}^{m} \|B_j u\|_{Y_j} + \|u\|_Z). \tag{1.31}$$

Remark 1.3: The important fact is that the index of corresponding operators is the same in both theories.

Remark 1.4: As shown in [5, 6] the terms $\|u\|_s$ and $\|u\|_Z$ in (1.30), (1.31) disappear if $\dim\ker A = \{0\}$, where $Au = (Lu, B_1 u, \cdots, B_m u)$.

1.3 Superposition operators

The investigation of nonlinear equations in the following chapters relies on properties of mappings of the form $u \mapsto f(u)$ in the spaces $C^\alpha(\bar{\Omega})$ and $L_p(\Omega)$, $H^l(\Omega)$.

Definition 1.2 *Let $\Omega \subset \mathbb{R}^n$ be a domain. We say that a function*

$$\Omega \times \mathbb{R}^m \ni (x,u) \longmapsto f(x,u) \in \mathbb{R}$$

satisfies the Carathéodory conditions if

$$u \longmapsto f(x,u) \text{ is continuous for almost every } x \in \Omega$$

and

$$x \longmapsto f(x,u) \text{ is measurable for every } u \in \Omega.$$

Given any f satisfying the Carathéodory conditions and a function $u : \Omega \to \mathbb{R}^m$, we can define another function by composition

$$Fu(x) := f(x, u(x)). \tag{1.32}$$

The composed operator F is called a Nemytskii operator. In this section we state some important results on the composition of $C^\alpha(\bar{\Omega}), L^p(\Omega), H^l(\Omega)$ with nonlinear functions (some of them without proof [84, 142]).

Proposition 1.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and*

$$\Omega \times \mathbb{R}^m \ni (x,u) \longmapsto f(x,u) \in \mathbb{R}$$

satisfy the Carathéodory conditions. In addition, let

$$|f(x,u)| \leq f_0(x) + c(1+|u|)^r \tag{1.33}$$

where $f_0 \in L_{p_0}(\Omega)$, $p_0 \geq 1$, and $rp_0 \leq p_1$. Then the Nemytskii operator F defined by (1.32) is bounded from $L_{p_1}(\Omega)$ into $L_{p_0}(\Omega)$, and satisfies

$$\|F(u)\|_{0,p_0} \leq C_1 \cdot (1 + \|u\|_{p_1}^r) \tag{1.34}$$

Proof: By (1.33) and (1.7)

$$\begin{aligned} \|F(u)\|_{0,p_0} &\leq \|f_0(x)\|_{0,p_0} + C\|1\|_{0,p_0} + C\||u|^r\|_{0,p_0} \\ &\leq C' + C\left(\int_\Omega |u|^{rp_0} dx\right)^{\frac{1}{p_0}} = C' + \|u\|_{0,p_0 r}^r. \end{aligned} \tag{1.35}$$

Since Ω is bounded, then by Hölder's inequality

$$\|v\|_{0,q} \leq C(\Omega)\|v\|_{0,p} \quad \text{when } 1 \leq q \leq p,\ v \in L_p(\Omega) \tag{1.36}$$

where $C(\Omega) = (\text{mes})(\Omega)^{\frac{1}{q}-\frac{1}{p}}$. Inequalities (1.35) and (1.36) with $q = rp_0$ and $p = p_1$ imply (1.34). □

It is well-known that the notions of continuity and boundedness of a nonlinear operator are independent of one another ([84]). It turns out that the following is valid.

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let*

$$\Omega \times \mathbb{R}^m \ni (x, u) \longmapsto f(x, u) \in \mathbb{R}$$

satisfy the Carathéodory conditions. In addition, let $p \in (1, \infty)$ and $g \in L_q(\Omega)$ (where $\frac{1}{p} + \frac{1}{q} = 1$) *be given, and let f satisfy*

$$|f(x, u)| \leq C|u|^{p-1} + g(x).$$

Then the Nemytskii operator F defined by (1.32) is a bounded and continuous map from $L_p(\Omega)$ to $L_q(\Omega)$.

For a more detailed treatment, the reader can consult [84, 142].

Theorem 1.4 *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary and let*

$$\Omega \times \mathbb{R} \ni (x, u) \mapsto f(x, u) \in \mathbb{R}$$

satisfy the Carathéodory conditions. Then for $s > n/2$, f induces

1) a continuous mapping from $H^s(\Omega)$ into $H^s(\Omega)$ if $f \in C^s$,

2) a continuously differentiable mapping from $H^s(\Omega)$ into $H^s(\Omega)$ if $f \in C^{s+1}$.

Proof: First we consider the simplest case, that is $f = f(u)$ is independent of x. By the Sobolev embedding theorem, we have $H^s(\Omega) \subset C(\bar{\Omega})$. Hence we have $f(u) \in C(\bar{\Omega})$ for every $u \in H^s(\Omega)$. Moreover, if u is in $C^{(s)}(\bar{\Omega})$, we can obtain the derivatives of $f(u)$ by the chain rule, and in the general case, we can use approximation by smooth functions. Note that all derivatives of $f(u)$ have the form of a product involving a derivative of f and derivatives of u. The first factor is in $C(\bar{\Omega})$, while any l-th derivative of u lies in $H^{s-l}(\Omega)$, which imbeds into $L_{2n/(n-2(s-l))}(\Omega)$ if $s - l < \frac{n}{2}$.

We can use this fact and Hölder's inequality to show that all derivatives of $f(u)$ up to order s are in $L_2(\Omega)$; moreover, it is clear from this argument that f is actually continuous from $H^s(\Omega)$ into $H^s(\Omega)$. A proof of the differentiability in this special case is that $f = f(u)$ is based on the relation

$$f(u) - f(v) = \int_0^1 f_u'(v + \theta(u - v))(u - v)d\theta$$

and the same arguments as before.

Let us now consider the general case, that is $f = f(x, u)$. Let $|\alpha| \le s$. We must show that

$$u \longmapsto D^\alpha F(u) \tag{1.37}$$

defines a continuous map of $H^s(\Omega)$ into $L_2(\Omega)$. It is not difficult to see that (1.37) is a finite sum of operators of the form

$$u(x) \longmapsto g(x, u(x)) \cdot D^\gamma u(x) \tag{1.38}$$

where $|\gamma| = \gamma_1 + \cdots + \gamma_n \le s$, while g is a partial derivative of f order at most s. It is obvious that D^γ is continuous from $H^s(\Omega)$ into $L_2(\Omega)$ for $|\gamma| \le s$. On the other hand, the continuous embedding of $H^s(\Omega)$ in $C(\bar{\Omega})$ implies that

$$u(x) \longmapsto g(x, u(x))$$

is continuous from $H^s(\Omega)$ into $C(\bar{\Omega})$. Thus

$$u(x) \longmapsto g(x, u(x)) \cdot D^\gamma u(x)$$

defines a continuous map of $H^s(\Omega)$ into $L_2(\Omega)$ and hence so does $u \longmapsto D^\alpha F(u)$. □

As before, let $p \in \mathbb{N}$ and $\tilde{p}$ denote the number of multi-indices with $|\alpha| \le p$ and let Ω be a bounded domain in $\mathbb{R}^n$.

Corollary 1.1 *An analogous result is valid for a continuity of the operator*

$$Fu(x) = f(x, u(x), \cdots, D^p u(x)) : H^{s+p}(\Omega) \to H^s(\Omega)$$

where $p, s \in \mathbb{N}$ with $s > \frac{n}{2}$ and $f : \Omega \times \mathbb{R}^{\tilde{p}} \to \mathbb{R}$ is C^s.

Corollary 1.2 *Let* $p, s \in \mathbb{N}$ with $s > \frac{n}{2}$ *and*

$$f : \Omega \times \mathbb{R}^{\tilde{p}} \to \mathbb{R} \textit{ be } C^{s+1}.$$

Then the operator $F : H^{s+p}(\Omega) \to H^s(\Omega)$ *defined by*

$$Fu(x) = f(x, u(x), \cdots, D^p u(x))$$

is Fréchet differentiable from $H^{s+p}(\Omega)$ *into* $H^s(\Omega)$.

We will show some continuity and C^1-differentiability results for a nonlinear differential operator of the form $Au(x) = f(x, u(x), ..D^{2p}u(x))$ in Hölder spaces. They are based on the following theorems 1.5 and 1.6.

Theorem 1.5 *Let the function* $f(x, y) = f(x, y_1, \cdots, y_{\tilde{p}})$ *be defined on* $\bar{\Omega} \times \mathbb{R}^{\tilde{p}}$ *and satisfy the following conditions:*

1) $f(x,0)=0$

2) *For any* $R>0$, $\sup_{|y|\le R}|\frac{\partial^2 f}{\partial y_i\partial y_j}|\le C(R)$, $\sup_{|y|\le R}\|f\|_{C^{1,\alpha}(\bar{\Omega})}\le C(R)$, *where* $C(R)$ *is constant depending on* R.

Let $u_1(x),\cdots,u_{\tilde{p}}(x)\in C^\alpha(\bar{\Omega}),\ 0<\alpha<1,\ \|u_i\|_{C^\alpha(\bar{\Omega})}\le R,\ i=1,\ldots,\tilde{p}$. *Then*

$$\|f(x,u_1(x),\cdots,u_{\tilde{p}}(x))\|_{C^\alpha(\bar{\Omega})}\le C_1(R)\cdot\sum_{i=1}^{\tilde{p}}\|u_i\|_{C^\alpha(\bar{\Omega})} \tag{1.39}$$

Proof: Obviously,

$$\begin{aligned}f(x,y,\ldots,y_{\tilde{p}})=\int_0^1\frac{d}{dt}f(x,ty_1,\ldots,ty_{\tilde{p}})dt&=\sum_{j=1}^{\tilde{p}}y_j\int_0^1\frac{\partial f(x,ty_1,\ldots,ty_{\tilde{p}})}{\partial y_j}dt\\&=\sum_{j=1}^{\tilde{p}}\ \varphi_j(x,y_1,\ldots,y_{\tilde{p}})\cdot y_j\end{aligned}$$

where

$$\varphi_j(x,y_1,\ldots,y_{\tilde{p}})=\int_0^1\frac{\partial f(x,ty_1,\ldots,ty_{\tilde{p}})}{\partial y_j}dt$$

Hence

$$f(x,u_1(x),\cdots,u_{\tilde{p}}(x))=\sum_{j=1}^{\tilde{p}}\varphi_j(x,u_1(x),\cdots,u_{\tilde{p}}(x))\cdot u_j(x).$$

Since $C^\alpha(\bar{\Omega}),0<\alpha<1$ is a Banach algebra, we have

$$\|f(x,u_1(x),\cdots u_{\tilde{p}}(x))\|_{C^\alpha}\le\sum_{j=1}^{\tilde{p}}\|\varphi_j(x,u_1(x),\cdots u_{\tilde{p}}(x))\|_{C^\alpha}\cdot\|u_j\|_{C^\alpha}.$$

Hence we have to prove that

$$\sup_{|y|\le R}\|\varphi_j(x,u_1(x),\cdots,u_{\tilde{p}}(x))\|_{C^\alpha}\le C_1(R).$$

Indeed

$$\begin{aligned}&|\varphi_j(x+\xi,u_1(x+\xi),\cdots,u_{\tilde{p}}(x+\xi))-\varphi_j(x,u_1(x),\cdots,u_{\tilde{p}}(x))|\\&\le|\varphi_j(x+\xi,u_1(x+\xi),\cdots,u_{\tilde{p}}(x+\xi))-\varphi_j(x,u_1(x+\xi),\cdots,u_{\tilde{p}}(x+\xi))|\\&+|\varphi_j(x,u_1(x+\xi),\cdots,u_{\tilde{p}}(x+\xi))-\varphi_j(x,u_1(x),\cdots,u_{\tilde{p}}(x))|.\end{aligned} \tag{1.40}$$

The first term on the right-hand side of (1.40) is bounded by $C(R)\cdot|\xi|^\alpha$. The second term is bounded by

$$\sup_{|y|\le R}|\frac{\partial\varphi_j}{\partial y_k}|\cdot|\varphi_j(x,u_1(x+\xi),\cdots,u_{\tilde{p}}(x+\xi))-\varphi_j(x,u_1(x),\cdots,u_{\tilde{p}}(x))|\le C_R R|\xi|^\alpha. \tag{1.41}$$

The estimates (1.40) and (1.41) yield (1.39). □

Theorem 1.6 *Let the function $f(x,y)=f(x,y_1,\cdots,y_{\tilde{p}})$ be defined on $\bar{\Omega}\times R^{\tilde{p}}$ satisfy the following conditions:*

1) $f(x,0)=0,\ grad_y f(x,0)=0$

2) For any $R>0$, $\sup_{|y|\le R}\|f(x,y)\|_{C^{2,\alpha}(\bar{\Omega})}\le C(R)$ and $\sup_{|y|\le R}|\frac{\partial^3 f}{\partial y_i\partial y_j\partial y_k}|\le C(R)$,

where $C(R)$ is constant depending on R. Let as before, $u_1(x),\cdots,u_{\tilde{p}}(x)\in C^\alpha(\bar{\Omega})$ with $\|u_i\|_{C^\alpha(\bar{\Omega})}\le R, i=1,\cdots,\tilde{p}$.
Then the following estimate holds.

$$\|f(x,u_1(x),\cdots,u_{\tilde{p}}(x))\|_{C^\alpha(\bar{\Omega})}\le C_2(R)\cdot\sum_{i=1}^{\tilde{p}}\|u_i\|^2_{C^\alpha}. \tag{1.42}$$

Proof: Obviously we have

$$f(x,y_1,\cdots,u_{\tilde{p}})=\sum_{i,j=1}^{\tilde{p}}g_{ij}(x,y_1,\cdots,y_{\tilde{p}})\cdot y_i\cdot y_j$$

so we can write

$$f(x,u_1(x),\cdots,u_{\tilde{p}}(x))=\sum_{i,j=1}^{\tilde{p}}g_{ij}(x,u_1(x),\ldots,u_{\tilde{p}}(x))\cdot u_i(x)\cdot u_j(x)$$

and we have

$$\|f(x,u_1(x),\cdots,u_{\tilde{p}}(x)\|_{C^\alpha(\bar{\Omega})}\le\sum_{i,j=1}^{\tilde{p}}\|g_{ij}(x,u_1(x),\cdots,u_{\tilde{p}}(x)\|_{C^\alpha}\|u_i\|_{C^\alpha}\|u_j\|_{C^\alpha} \tag{1.43}$$

Due to Theorem 1.5 we obtain

$$\|g_{ij}(x,u_1(x),\cdots,u_{\tilde{p}}(x)\|_{C^\alpha(\bar{\Omega})}\le C_0(R) \tag{1.44}$$

Hence the estimates (1.43) and (1.44) yield (1.42)

$$\|f(x,u_1(x),\cdots,u_{\tilde{p}}(x)\|_{C^\alpha(\bar{\Omega})}\le C_2(R)\cdot\sum_{i=1}^{\tilde{p}}\|u_i\|^2_{C^\alpha}$$

□

We apply Theorems 1.5 and 1.6 to the operator

$$Au(x) = f(x, u(x), \cdots, D^{2p}u(x))$$

where the function $f(x, y_1, \cdots, y_{\tilde{p}})$ satisfy conditions of Theorems 1.5 and 1.6, respectively. Hence we have

$$\|Au\|_{C^{2p,\alpha}} \leq C(R) \cdot \|u\|_{C^\alpha}$$

Moreover as it follows from Theorem 1.6 $A \in C^1, A'(0) = 0$ and

$$\|A'(u+h) - A'(u)\|_{L(C^{2p,\alpha}, C^\alpha)} \leq C \cdot \|h\|_{C^{2p,\alpha}(\bar{\Omega})}.$$

Remark 1.5: As shown in the proofs of Theorems 1.5 and 1.6, continuity and differentiability of the operator $Au(x) = f(x, u(x), \cdots, D^{2p}u(x))$ between $C^{2p,\alpha}(\bar{\Omega})$ and $C^\alpha(\bar{\Omega})$ remains valid under slightly weaker conditions on a given function $f(x, y_1, \cdots, y_{\tilde{p}})$. We leave these as exercises for the reader.

In the investigation of nonlinear boundary value problems related to pseudodifferential operators and in particular nonlinear Riemann-Hilbert problems we need properties of the Nemytskii operators in the spaces $H^s(S^1)$ or $C^{p,\alpha}(S^1)$, where S^1 is the unit circle. We recall some of the properties which will be used often in the sequel. The norm in $C^\alpha(M)$ is given by

$$\|f\|_{C^\alpha(M)} = \|f\|_C + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}, M = S^1.$$

As before, by $C^{k,\alpha}(M)$ we denote the space of Hölder continuous functions, which have derivatives up to order k, with $D^k f \in C^\alpha(M)$. Let F be a superposition operator defined by

$$Fu(x) = f(x, u(x)), \quad x \in M.$$

The following theorems are not hard to prove (although not obvious).

Theorem 1.7 *Let $k \in \mathbb{R}+$. Then the superposition operator $F : E_1 \to E_2$ defined by $Fu(x) = f(x, u(x))$ acts as a bounded operator in each of the following cases (see also [133])*

1) $f \in C(S^1 \times \mathbb{R}, \mathbb{R}), \quad E_1 = C(S^1), \quad E_2 = C(S^1)$
2) $f \in C^1(S^1 \times \mathbb{R}, \mathbb{R}), E_1 = C^\alpha(S^1), E_2 = C^\alpha(S^1), \ 0 < \alpha < 1$

Theorem 1.8 *Let $k \in \mathbb{R}+, 0 < \alpha < 1$. Then the superposition operator $F : E_1 \to E_2$ defined by $Fu(x) = f(x, u(x))$ is m times continuously differentiable in each of the following cases*

1) $D^{0,j} f \in C^k(S^1 \times \mathbb{R}, \mathbb{R}), \quad E_1 = C^k(S^1), \quad E_2 = C^k(S^1)$
2) $D^{0,j} f \in C^{k+1}(S^1 \times \mathbb{R}, \mathbb{R}), E_1 = C^{k,\alpha}(S^1), E_2 = C^{k,\alpha}(S^1),$

The $j-th$ derivative of F is given by

$$D^{0,j}F(x,u(x))h_1(x)\dots h_j(x) = D^jF(f)(h_1,\dots h_j)(x).$$

Analogous results are valid in Sobolev spaces:

Theorem 1.9 *Let $X = Y = H^s(S^1)(s \geq 1)$ be the Sobolev space of real functions $x(\tau)$ on the circumference of a circle, where $0 \leq \tau < 2\pi; f(\tau,x)$ is a smooth real function, $x \in \mathbb{R}, 0 \leq \tau < 2\pi$. Then the operator $F : H^s(S^1) \to H^s(S^1)$ defined by $Fx(\tau) = f(\tau, x(\tau))$ is continuous.*

Proof: It is not difficult to see, that

$$\left(\frac{d}{d\tau}\right)^k f(\tau,x(\tau)) = \sum_{\substack{p+q\leq k\\ r_1+\dots+r_q=k-p\\ r_j\geq 0}} C_{p,q,r_1\dots r_q} \frac{\partial^{p+q} f(\tau,x(\tau))}{\partial\tau^p\cdot\partial x^q} x^{(r_1)}(\tau)\cdots x^{(r_q)}(\tau)$$

where $C_{p,q,r_1\dots r_q}$ are some constants. If $x(\tau) \in H^s$, then it follows that the derivatives $\left\{\frac{d^l x(\tau)}{d\tau^l} | 0 \leq l \leq s-1\right\}$ are continuous. Therefore in $\frac{d^s}{d\tau^s} f(\tau,x(\tau))$ all terms without ones are continuous. The last term is equal to $\frac{d^s x(\tau)}{d\tau^s} \times Q(\tau)$ where $Q(\tau)$ is a continuous function, hence also square integrable. As a consequence of these arguments we obtain continuity. □

Remark 1.6: An analogous result holds for vector functions, and also in the multidimensional case, for functions on arbitrary smooth compact manifold with boundary.

1.4 Linear Riemann-Hilbert Problems

Let G_q be a given bounded, q-connected domain, with boundary $\partial G_q = \bigcup_{j=1}^q \Gamma j$ consisting of q separated Jordan curves $\Gamma_j, j = 1,\dots q$, where Γ_1 encloses all the others. For the positive direction on the contour $\Gamma = \partial G_q$ we take that which leaves the domain G_q on the left, i.e. the contour Γ_1 is to be traversed anticlockwise and the contours $\Gamma_2,\dots \Gamma_q$ clockwise. Let $\zeta = t_j(\tau_j)$ for $0 \leq \tau_j < 2\pi, j = 1,\dots q$ be a parametric representations of Γ_j. A linear Riemann-Hilbert problem (abbreviated with RHP) on G_q consists of finding a function $w(z) = u(z) + iv(z)$ holomorphic in G_q, continuous on the closure $\bar{G}_q$, and satisfying the boundary conditions

$$a(\zeta)\cdot u(\zeta) + b(\zeta)\cdot v(\zeta) = c(\zeta) \text{ for } \zeta \in \partial G_q \tag{1.45}$$

where $a(\zeta), b(\zeta), c(\zeta)$ are given real continuous functions on ∂G_q. The function $a(\zeta)+ib(\zeta)$ is called the symbol of the linear RHP. Throughout this section we assume that $a(\zeta) + ib(\zeta) \neq 0$ on the boundary ∂G_q. Moreover, without loss of generality, we assume that $a^2(\zeta)+b^2(\zeta)|_{\Gamma j}=1$, otherwise, we can divide both sides of the boundary conditions (1.45) by $\sqrt{[a^2(\zeta)+b^2(\zeta)]|_{\Gamma j}}$ and reduce to the case just mentioned. As $a(\zeta)+ib(\zeta)|_{\Gamma j}\neq 0$ we can choose a branch of the argument function, such that $\arg[a(\zeta)+ib(\zeta)]|_{\Gamma j}$ is continuous. We define the winding number of $[a(\zeta)+ib(\zeta)]_{\Gamma j}$ about zero by

$$\mathrm{wind}[a(\zeta)+ib(\zeta)]_{\Gamma j}=\frac{1}{2\pi}\left(\varphi_j(2\pi)-\varphi_j(0)\right)$$

where $\varphi_j(\tau)=\arg\left\{a(t_j(\tau))+ib(t_j(\tau))\right\}, j=1,\dots q$ Then the integer

$$\chi=\chi_1+\ldots+\chi_q,\ \ \text{where}\ \ \chi_j=\mathrm{wind}[a(\zeta)+ib(\zeta)]|_{\Gamma j}$$

(with all contours Γ_j traversed in the above established positive direction) is called the index of the linear RHP.

In this section we restrict ourselves to problems which have index zero, for the following reasons:

a) A nonlinear Riemann-Hilbert problem which is considered in Chapter 5, generates a nonlinear map, whose diagonal representations is connected with linear RHP whose index is equal to zero.
b) Some problems of Hydrodynamics, namely the plane potential flow of an inviscid incompressible fluid past a cylinder with porous surfaces under various filtration laws, are reduced to the nonlinear RHP just mentioned. (see [10]).

   Mathematically reduction of non-vanishing index to the case when the index is zero, is more or less standard both for linear and nonlinear RHP.

First we start with linear RHP for a simply-connected case. There is no loss of generality in assuming that $G_1=\{z|\ |z|<1\}$. In this case linear Riemann-Hilbert problems can be solved explicitly. As in the linear elliptic nonlinear boundary value problems we will consider C^α and H^s theory for linear Riemann-Hilbert problems. Let $G_1=\{z|\ |z|<1\}, \Gamma=\partial G_1$ and τ be angular coordinate on Γ where $a(e^{i\tau}), b(e^{i\tau}), c(e^{i\tau})$ are given real functions. We shall sometimes write $a(\tau), b(\tau), c(\tau), w(\tau)=u(\tau)+iv(\tau)$, in place of $a(e^{i\tau}), b(e^{i\tau}), c(e^{i\tau}), w(e^{i\tau})=u(e^{i\tau})+iv(e^{i\tau})$ respectively , $0\leq\tau<2\pi$; this should not lead to misunderstanding.

Lemma 1.3 *Let $a, b, c, \in H^s(\Gamma), s\geq 1$ and assume that*

$$a(\tau)+ib(\tau)\neq 0 \ \textit{and}\ \int d\arg(a(\tau\pm ib(\tau))=0.$$

Then boundary values of solutions of the linear RHP are given by the formulas

$$\begin{aligned} v_1(\tau) &:= \arg(b(\tau)-ia(\tau)),\ u_1(\tau):=-H_0v_1(\tau)\\ u_2(\tau)+iv_2(\tau) &= e^{u_1(\tau)+iv_1(\tau)} \end{aligned} \tag{1.46}$$

$(u_2(\tau) + iv_2(\tau)$ is a solution of the homogeneous linear RHP)

$$v_3(\tau) := \frac{c(\tau)}{(u_2(\tau) + iv_2(\tau)) \cdot (b(\tau) + ia(\tau))}$$

$$u_3(\tau) = H_0 v_3(\tau) + \lambda, u(\tau) + iv(\tau) = (u_3(\tau) + iv_3(\tau))(u_2(\tau) + iv_2(\tau)), \tag{1.47}$$

where

$$\lambda \in \mathbb{R} \text{ is arbitrary, } H_0 u(\tau) := \frac{1}{2\pi} \text{p.v.} \int_0^{2\pi} u(\sigma) \cot \frac{\tau - \sigma}{2} ds \tag{1.48}$$

$u(\tau) + iv(\tau)$ is a solution of the nonhomogeneous linear RHP. Note that in (1.48) *the integral is taken in the sense of principal value.*

For the proof see [60, 64],

Remark 1.7: Recall the following facts. The Hilbert operator H_0 defined by (1.48) is a bounded linear operator on the spaces

1) $L_p(\Gamma), 1 < p < \infty$
2) $W_p^k(\Gamma),$ with $k \in \mathbb{R}_+, 1 < p < \infty$
3) $C^{k,\alpha}(\Gamma)$ with $k \in \mathbb{R}_+, 0 < \alpha < 1$

In each case $\ker H_0 = \{1\}, \operatorname{Im} H_0 = \left\{ u | \int_0^{2\pi} u(e^{i\tau}) d\tau = 0 \right\}$. Note that, $H_0 \notin L(C^k(\Gamma))$, with $k \in \mathbb{R}_+$ and $H_0 \notin L(W_p^k(\Gamma))$ with $k \in \mathbb{R}_+, p = 1, p = \infty$.

The following Lemmas 1.4 and 1.5 will be useful in Chapter 5.

Lemma 1.4 *Let $a, b, c, \in C(\Gamma), a(\tau) + ib(\tau) \neq 0, wind(a + ib) = 0$ and $w_\lambda(z) = u_\lambda(z) + iv_\lambda(z)$ be solutions of the linear Riemann-Hilbert problem $a(\tau) + ib(\tau) = c(\tau)$, whose boundary values are given by formulas* (1.46) *and* (1.47).

1) If $a, b, c, \in C^{s,\alpha}(\Gamma)$ with $s \in \mathbb{R}_+, 0 < \alpha < 1$, then $w_\lambda \in C^{s,\alpha}(\Gamma)$
2) If $a, b, c, \in H^s(\Gamma)$ with $s \in \mathbb{R}_+, s \geq 1$ then $w_\lambda \in H^s(\Gamma)$.

Proof: A proof of Lemma 1.4 is a direct consequence of the representations formula (1.47), mapping properties of Hilbert and Nemytskii operators (see Theorems 1.8, 1.9, and Remark 1.7). □

Remark 1.8: The assertion of Lemma 1.4 remains valid if we replace $a, b, c, \in H^s(\Gamma) = W_2^s(\Gamma)$ by $a, b, c, \in W_p^s(\Gamma)$ with $s \geq 1, 1 < p < \infty$.

Remark 1.9: We emphasize that the regularity of the solution does not coincide with the regularity of the coefficients in the case $p = 1, p = \infty$ or $\alpha = 0$.

As in the case of linear elliptic boundary value problems the linear RHP can be rewritten as an operator equation. It admits considerably many important linear RHP from the operator theory point of view. Indeed, since $w(\tau) = u(\tau) + iv(\tau))$ are the boundary values of a holomorphic function in $G_1 = \{z|\ |z| < 1\}$, its real and imaginary parts are given by the Hilbert operator (see [60]).

$$v(\tau) = \frac{1}{2\pi}\text{p.v.}\int_0^{2\pi} u(\sigma)\cot\frac{\tau-\sigma}{2}d\sigma + \lambda = Hu(\tau)$$

$$u(\tau) = -Hv(\tau)$$

where the integral is taken in the sense of principal value. The operator H is defined up to a real constant λ; if λ is fixed, then the operator is denoted by H_λ. We note that

$$\int_0^{2\pi} H_\lambda(\tau)d\tau = 2\pi\lambda.$$

Hence condition (1.45) is equivalent to the following linear singular integral equations $A(\lambda, u) = c(\tau)$

$$A(\lambda, u) := a(\tau)u(\tau) + b(\tau)H_\lambda u(\tau) \tag{1.49}$$

Lemma 1.5 *Let $a, b, c \in H^s(\Gamma), s \geq 1$ with $a(\tau) + ib(\tau) \neq 0$ and $wind[a(\tau) + ib(\tau)] = 0$. Then operator $A(\lambda, u) : \mathbb{R} \times H^s(\Gamma) \to H^s(\Gamma)$, defined by (1.49) restricted to the plane $\lambda = const$, i.e. $\tilde{A} = A(\lambda, u)|_{\lambda=const} : H^s(\Gamma) \to H^s(\Gamma)$ is Fredholm of index zero.*

Proof: From the representation formula of solutions (1.46) and (1.47) it follows that the operator $A : (\lambda, u) \to a(\tau)u(\tau) + b(\tau)H_\lambda u(\tau)$ has one-dimensional kernel and null-dimensional cokernel.. Hence $A : \mathbb{R} \times H^s(\Gamma) \to H^s(\Gamma), A(\lambda, u) = a(\tau)u(\tau) + b(\tau)H_\lambda u(\tau)$ is Fredholm and the index of A is equal to one, $indA = 1$. Then its restriction on the plane $\lambda = const$, i.e.

$$\tilde{A} = A(\lambda, u)|_{\lambda=const} \quad : H^s(\Gamma) \to H^s(\Gamma)$$

has either one dimensional kernel and cokernel or null-dimensional kernel and cokernel which follows from the following Proposition 1.2, i.e. $\tilde{A}$ is Fredholm of index zero.

Proposition 1.2 *Let X, Y be (for the sake of simplicity) Hilbert spaces, $X = X_\circ \bigoplus \mathbb{R}^N$* and *$A : X \to Y$ be a linear Fredholm operator of index m. Then the restriction of A to the subspace $X_\circ$ is also Fredholm and has index*

$$indA|_{X_\circ} = m - N.$$

A proof of Proposition 1.2 is based on a direct computation of the index. We leave a proof of this simple fact to the reader.

Lemma 1.6 *Let* $a, b, c \in H^s(\Gamma), s \geq 1$ *with* $a(\tau)+ib(\tau) \neq 0, wind(a+ib) = 0$. *Then for solutions of linear RHP*

$$a(\tau)u(\tau) + b(\tau)v(\tau) = c(\tau)$$

the following a priori estimates are valid.

$$\|u\|_s \leq \Phi_s \left(\sup|a+ib| + \sup|a+ib|^{-1}\right) \cdot \left(\|a\|_s + \|b\|_s)^3(\|g\|_s + \|u\|_{s-1}\right),$$

where Φ_s *is some positive monotone function.*

Proof: The proof is a direct consequence of the representation formula (1.46), (1.47) of the solution of linear RHP and the mapping properties of H_0 and Nemytskii operators with the generating functions $x \to e^x$ and $x \to e^{-x}$.

Remark 1.10: An analogous a priori estimate is also valid for the C^α-norm. A proof is based on the same arguments.

Serious difficulties appear when considering linear Riemann-Hilbert problems for multiply connected domains, since an analytic function in a multiply-connected domain may not be single-valued. Hence here naturally, there occur solutions which are analytic but not single-valued. If according to the conditions of the problem it is required to deduce single-valued solutions, there arise nontrivial difficulties in separating out such solutions. Throughout this book we are interested only in single-valued analytic functions. Another peculiarity for a linear RHP in a multiply connected domain is the following: in contrast to the simply-connected domain case, which are always solvable, when $\chi = 0$ (χ- index of linear RHP, (Lemma 1.3)), the linear RHP in the multiply connected case is not always solvable, even if $\chi = 0$. For a proof of solvability we have to introduce another additional characteristic, the so-called condition of single-valuedness. Hence two subcases are possible, depending on whether the conditions of single-valuedness are satisfied or not. More precisely:

Definition 1.3 *We say that for linear RHP (1.45),*

$$A(\zeta)u(\zeta) + B(\zeta)v(\zeta) = c(\zeta)$$

the condition of uniqueness is satisfied, if the known function

$$w(\zeta) = \arctan\frac{B(\zeta)}{A(\zeta)}, \zeta \in \Gamma = \partial G_q$$

is the boundary value of the real part of an analytic and single-valued function in G_q.

For multiply-connected domains the following is valid: ([60]). Let $\chi = 0$ and

1) the conditions of uniqueness are not satisfied.
   Then the homogeneous Riemann-Hilbert problem has no solution. The nonhomogeneous problem is solvable if and only if $q - 2$ conditions of orthogonality are satisfied

2) the conditions of uniqueness are satisfied.
   Then the homogeneous linear RHP has one solution. The non-homogeneous problem is solvable if $q-1$ conditions are satisfied.

1.5 Degree Theory in Finite Dimensional Spaces

We start with some definitions. Let $f : M^m \to N^n$ be a smooth map from a manifold of dimension m to a manifold of dimension n. We denote by $T_x M^m$ the tangent space of M at x. Let S be the set of all $x \in M$ such that

$$f'(x) : T_x M^m \to T_{f(x)} N^n$$

has rank less than n (i.e. is not onto). Then S will be called the set of critical (singular) points, $f(S)$ the set of critical (singular) values, and the component $N^n \setminus f(S)$ the set of regular values. One of the important facts is:

Theorem 1.10 *If $f : M^m \to N^n$ is a smooth map between manifolds of dimension $m \geq n$, and if $y \in N^n$ is a regular value, then the set $f^{-1}(y) \subset M^m$ is a smooth manifold of dimension $m - n$.*

For simplicity we assume that M, N are open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. From theorem 1.10 it follows that the regular values play an important role. This leads naturally to the question: are regular values generic?

Theorem 1.11 (Sard's theorem) *If $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ is C^k-mapping on the open set Ω with $k > \max(0, m-n)$, then the set of singular values of f has n-dimensional Lebesgue measure zero. Consequently, the set of regular values of f is dense in $\mathbb{R}^n$.*

A proof can be found in [101, 117]. Now we are in the position to define a concept of degree. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f : \bar{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$ be continuous.

What is degree? Roughly speaking, degree is the "number of solutions" of the equation $f(x) = y$. To be more precise we proceed as follows: Let $y \notin f(\partial\Omega)$, $\delta = \text{dist}(y, f(\partial\Omega))$ and $\tilde{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a smooth approximation of f satisfying the conditions:

1) $\sup\limits_{x\in\Omega} \|\tilde{f}(x) - f(x)\| < \delta/2$
2) If $\tilde{f}^{-1}(y) \neq \emptyset$, then $\tilde{f}^{-1}(y) = \{x_1, \ldots, x_N\}$ such that

$$\det\left(\frac{\partial \tilde{f}}{\partial x}\right)|_{x=x_k} \neq 0, k = 1, \cdots N.$$

Existence of such approximations follows from Sard's theorem. Then by definition: degree for smooth approximation is defined by

$$\deg(\tilde{f}, \Omega, y) = \sum_{k=1}^{N} \operatorname{sgn} \det \left(\frac{\partial \tilde{f}}{\partial x} \right) |_{x=x_k} \tag{1.50}$$

It is not difficult to see that $\deg(\tilde{f}, \Omega, y)$ does not depend on the choice of $\tilde{f}$ that satisfies

$$\sup_{x \in \Omega} \|\tilde{f} - f\| < \delta/2.$$

This justifies:

Definition 1.4 $\deg(f, \Omega, y) := \deg(\tilde{f}, \Omega, y)$.

In the literature, the degree of a continuous map as defined above is called the Brouwer degree of f with respect to Ω and y. The idea was first developed by Brouwer (in 1912, [23]). We denote this degree by $\deg_B(f, \Omega, y)$.

Remark 1.11: Brouwer degree can be regarded as a generalized winding number. Indeed, suppose that Ω is a bounded domain in $\mathbb{C}$, and $\partial\Omega$ is a smooth Jordan curve. Let $f = f_1 + i f_2$, be holomorphic in Ω and continuous in $\bar{\Omega}$. If $z_0 = x_0 + i y_0$ is a zero of the function $f(z) = f_1(x, y) + i f_2(x, y)$ (without loss of generality we take $z_0 = 0$) of multiplicity k, then we have

$$f(z) = z^k \cdot g(z)$$

in a neighbourhood U of $z_0 = 0$, where $g(0) \neq 0$. We identify $\mathbb{C}$ with $\mathbb{R}^2$. By the Cauchy-Riemann equations,

$$J_f(z) := \det \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \left(\frac{\partial f_1}{\partial x} \right)^2 + \left(\frac{\partial f_2}{\partial x} \right)^2 \geq 0.$$

Hence, J_f is nonzero in $\tilde{U} = U \setminus \{0\}$ and $\tilde{U}$ contains exactly k solutions of $f(z) = \delta$, none of them multiple, if $\delta \neq 0$ and is small enough. Similar remarks apply to each of zeros contained in Ω. Thus $\deg_B(f, \Omega, \delta)$ in this case, is simply the number of solutions of $f(z) = \delta$ in Ω. But it is not difficult to see that $\deg_B(f, \Omega, \delta) = \deg_B(f, \Omega, 0)$.

Therefore $\deg_B(f, \Omega, 0) =$ number of zeros of f in Ω, counting multiplicity.

Thus we obtain

$$\deg_B(f, \Omega, 0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z)} dz. \tag{1.51}$$

Here we give (without proof) a list of basic properties of Brouwer degree, which will be useful in the next chapters. They can be found in [23, 94, 105, 120]. Some of them can easily be proved by the reader independently. They are:

1) If $\deg_B(f, \Omega, y) \neq 0$, then $y \in f(\Omega)$.
2) If $F(t, x) = f_t(x)$ is a homotopy and $y \notin f_t(\partial\Omega)$ for $0 \leq 1$, then $\deg_B(f_t, \Omega, y)$ is independent of $t \in [0, 1]$.
3) $\deg_B(f, \Omega, y)$ is constant on components of $\mathbb{R}^n \setminus f(\partial\Omega)$.

   Notation. If A is a connected subset of $\mathbb{R}^n \setminus f(\partial\Omega)$, we write $\deg_B(f, \Omega, A)$ for $\deg_B(f, \Omega, y), (y \in A)$
4) (Boundary value dependence). If $f, g \in C(\bar{\Omega})$ and $f = g$ and $\partial\Omega$, then $\deg_B(f, \Omega, y) = \deg_B(g, \Omega, y)$ (provided that $y \notin f(\partial\Omega)$).
5) (Poincaré (1886), Bohl (1904) theorem). Let $f, g \in C(\bar{\Omega})$ and suppose that for all $x \in \partial\Omega$, the line segment $[f(x), g(x)]$ does not contain y. Then $\deg_B(f, \Omega, y) = \deg_B(g, \Omega, y)$
6) Suppose that $f \in C(\bar{\Omega})$ and $y \notin f(\partial\Omega)$. Then for all $\xi \in \mathbb{R}^n$, $\deg_B(f, \Omega, y) = \deg_B(f - \xi, \Omega, y - \xi)$, where $f - \xi$ denotes the mapping $x \mapsto f(x) - \xi$.
7) Suppose that $f_t, t \in [0, 1]$ is a homotopy in $C(\bar{\Omega})$ and y_t is a continuous path in $\mathbb{R}^n$. If $y_t \notin f_t(\partial\Omega)$, then $\deg_B(f_t, \Omega, y_t)$ is independent of $t \in [0, 1]$.
8) Decomposition of the domain. Let $y \notin f(\partial\Omega)$ and $f \in C(\bar{\Omega})$. If Ω is the disjoint union of open sets $\Omega_i (i = 1, 2 \ldots)$, then
$$\deg_B(f, \Omega, y) = \sum_i \deg_B(f, \Omega_i, y)$$
9) The excision property. If $K \subset \bar{\Omega}$ is closed and $y \notin f(K)$, then
$$\deg_B(f, \Omega, y) = \deg_B(f, \Omega \setminus K, y)$$
10) Generalized homotopy property. Suppose that $\tilde{\Omega}$ is a bounded, open subset of $[0,1] \times \mathbb{R}^n$ and $f : \tilde{\Omega} \to \mathbb{R}^n$ is continuous. Let f_t denote the mapping of $x \to f(t, x)$ and $\tilde{\Omega}_t = \{x | (t, x) \in \tilde{\Omega}\} \subset \mathbb{R}^n$, with $\partial\tilde{\Omega}_t = \{x | (t, x) \in \partial\tilde{\Omega}\}$. If $y \notin f_t(\tilde{\Omega}_t)$ for $0 \leq t \leq 1$, then
$$\deg(f_t, \tilde{\Omega}_t, y) = const.$$
11) Multiplication theorem (Leray). Let $f \in C(\bar{\Omega})$ and D be a bounded, open set containing $f(\bar{\Omega})$. Let $\triangle = D \setminus f(\partial\Omega)$, and suppose that the components of $\triangle$ are $\triangle_j$ $(j = 1, 2 \ldots)$. If $g \in C(\bar{D})$ and $y \notin g(f(\partial\Omega)) \bigcup g(\partial D)$ then
$$\deg_B(g \circ f, \Omega, y) = \sum_j \deg_B(g, \triangle_j, y) \cdot \deg_B(f, \Omega, \triangle_j). \tag{1.52}$$

Remark 1.12: The summation in (1.52) is finite, because $g^{-1}(y)$ is compact. These properties have many useful applications both in analysis and in topology. We will restrict ourselves to the topological applications. They are:

1) Brouwer fixed point theorem

Proposition 1.3 *Let* $B = \{x| \ \|x\| \leq 1\}$ *be a closed unit ball in* $\mathbb{R}^n$*. If* $f \in C(B)$ *and* $f(B) \subset B$*, then* f *has a fixed point in* B*.*

2) "Invariance of a normal"

Proposition 1.4 *Let* Ω *be a bounded, open subset of* $\mathbb{R}^n$ *containing the origin; suppose that* n *is odd. If* $f \in C(\bar{\Omega})$ *and* $0 \notin f(\partial\Omega)$*, then there are* $y \in \partial\Omega, \lambda \neq 0$ *such that* $f(y) = \lambda y$*.*

3) Odd mapping theorem

Proposition 1.5 *Let* Ω *be a bounded, open, symmetric subset of* $\mathbb{R}^n$ *containing the origin 0. If* $f : \bar{\Omega} \to \mathbb{R}^n$ *is continuous,* $0 \notin f(\partial\Omega)$*, and for all* $x \in \partial\Omega$

$$\frac{f(x)}{\|f(x)\|} \neq \frac{f(-x)}{\|f(-x)\|}$$

then $\deg(f, \Omega, 0)$ *is an odd number.*

4) Borsuk-Ulam theorem

Let X and Y be finite-dimensional spaces over $\mathbb{R}(\mathbb{C})$ with

$$\dim Y < \dim X$$

and let S be the unit sphere in X. If $f : S \to Y$ is continuous map, then there is a point $x \in S$ such that $f(-x) = f(x)$.

Corollary 1.3 *If* f *is an odd function then there is an* $x \in S$ *such that* $f(x) = 0$*.*

5) Covering theorem of Ljusternik-Schnirelman-Borsuk

If the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is covered by p closed sets $A_1 \dots A_p$ with $1 \leq p \leq n$, then at least one A_i contains an antipodal pair of points $\{x, -x\}$.

Corollary 1.4 *For* $p = n + 1$ *there is a covering of* S^{n-1} *by closed sets* $A_1 \dots A_{n+1}$ *such that no* A_i *contains an antipodal pair.*

6) Bread-Ham-Cheese theorem

If $B_1 \dots B_n$ are bounded measurable subsets of $\mathbb{R}^n$ with $n \geq 1$, then there is an $(n-1)$-dimensional plane which divides all the B_j in half.

7) The Jordan separation theorem ([94])

Let M and N be compact subsets of $\mathbb{R}^n$ which are homeomorphic. Then either $\mathbb{R}^n \setminus M$ and $\mathbb{R}^n \setminus N$ have the same finite number of components, or both have a countably infinite number.

8) Invariance of domain

Let Ω be an open subset of $\mathbb{R}^n$ (not necessarily bounded). If $f : \Omega \to \mathbb{R}^n$ is one to one and continuous, then $f(\Omega)$ is open.

Remark 1.13: The Brouwer degree can be extended to a continuous mappings $f : M \to N$, when the manifolds M and N are oriented and have the same finite dimension.

Remark 1.14: Using suspension mapping theory ([105, 106]) and framed cobordism theory one can generalize degree to mappings $f : S^n \to S^k (n > k)$ and $f : M \to S^k$ respectively, where S^n, S^k are spheres, M is an arbitrary compact, manifold without boundary. Details of this and related materials may be found in Nirenberg [101, 105, 106].