Contents

Part II APPLICATIONS OF EQUIVARIANT DEGREE TO DIFFERENTIAL EQUATIONS AND BIFURCATION PROBLEMS

1

Introduction

1.1 Motivation and Subject

It usually comes as a great disappointment in mathematics when a person realizes that not every equation can be solved explicitly. This trust in a "perfect" science is even further damaged when it becomes clear that this is the fate of most of the equations, with perhaps a few exceptions well-known from high school. Applied mathematics, with its sophisticated numerical weapons, seems to be the only way to search for a "black cat in a dark room." However, the computations should be based on intelligent strategies, beginning with the answers to the serious questions about the existence and multiplicity of solutions. *Degree theory* has proved itself as a powerful tool for the detection of single and multiple solutions, which can be confined to small regions, and consequently, more effectively computed. The standard *existence, additivity, homotopy and normalization properties* of the degree simply mean that it is an algebraic 'count' of the solutions, which is not affected by small perturbations or even larger deformations. The Brouwer degree theory is a well-understood device used to study the existence and multiplicity of solutions as well as bifurcations occuring in evolution models. *Why should anyone look for much more sophisticated tools?*

To answer this question, let us consider the system of equations

$$f(\lambda, \vec{x}) = f\left(\lambda, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} \lambda & 1 & 0 & 1 \\ 1 & \lambda & 1 & 0 \\ 0 & 1 & \lambda & 1 \\ 1 & 0 & 1 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} x_2^2 x_4^2 \\ x_1^2 x_3^2 \\ x_2^2 x_4^2 \\ x_1^2 x_3^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.1}$$

It is clear that for every λ, the vector $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$ is a *trivial* (i.e. obvious) solution to the system (1.1), where a possible bifurcation of a branch of nontrivial solutions may occur for $\lambda = 0$. A standard method to investigate this phenomenon is to apply the degree to 'count' solutions near zero for $\lambda_- < 0$ and then comparing it with the 'count' for $\lambda_+ > 0$. If these degrees are different, then we can conclude the existence of a 'leak,' i.e. a branch of nontrivial solutions bifurcating from $(0,0)$ (see Figure 1.1). However, this is not the case here. The local degrees around zero of $f(\lambda_-, \cdot)$ and $f(\lambda_+, \cdot)$, are both equal to -1, so unfortunately, Brouwer degree theory can not help us to detect if a bifurcation takes place.

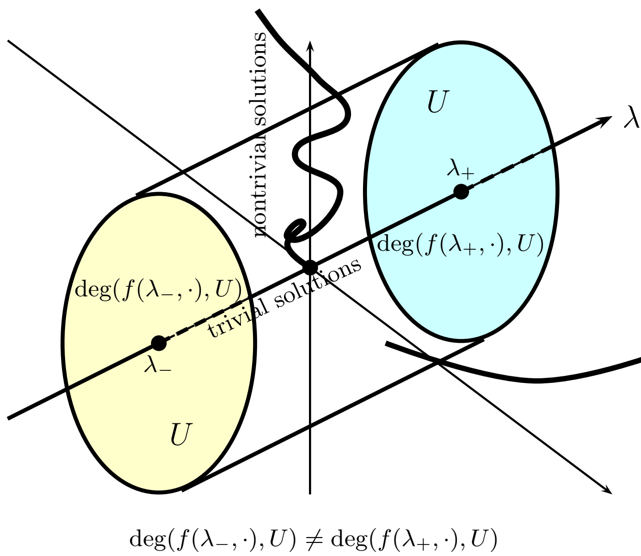

Fig. 1.1. Detecting bifurcation using local degree

For many mathematical models of natural phenomena, very often, their closeness to the real world problems is reflected (on top of their nonlinear character) in the presence of *symmetries* that are related to some physical or geometric regularities. In such a case, multiplicity of solutions is not just a possibility but, in fact, it is the reality. Obtaining knowledge of symmetries of the expected solutions and their *classification* constitute a separate important problem required for a complete analysis of the model including description of its solutions according to their algebraic properties. For instance, phase transitions in crystals correspond exactly to the changes of their symmetries; in a mechanical system, presence of symmetries allows to decrease the number of its degrees of freedom; breaking spherical symmetry of a hydrogen atom by introducing a magnetic field gives rise to the elimination of its degeneracy resulting in splitting energy levels (Zeeman's phenomena), etc.

Our book is devoted to the *applied aspects of the so-called equivariant degree theory*, which allows "counting" solutions, in the same way as the usual degree, but according to their symmetry properties.

Let us point out that the system (1.1) admits clear symmetries. To be more specific, let us consider a square with vertices x_1, x_2, x_3 and x_4 (see Figure 1.2). Any square symmetry σ corresponding to a permutation of the square's vertices, leads to the interchange of the corresponding variables in (1.1). The resulting system differs from the original one only by the order of the equations, and, moreover, this system can be reinstated in its original form simply by applying the symmetry inverse to σ. By the same token, system (1.1) *commutes* with the square symmetries. This phenomenon is commonly called *equivariance*.

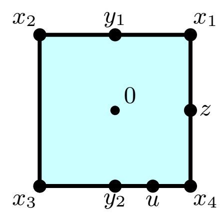

Fig. 1.2. Symmetries of the system (1.1)

We would like to indicate that symmetric properties of the linearizations of (1.1) alter as λ crosses 0, which, as we will see it later, causes the change of the (local) equivariant degree at zero. The information provided by the equivariant degree allows us to predict the existence of at least 4 branches of nontrivial solutions bifurcating from $(\lambda, \vec{x}) = (0, 0)$ with their minimal symmetries.

Let us explain more precisely the symmetries involved in the system (1.1), which form the dihedral group D_4 of order 8. The group D_4 consists of four rotations of the square: 1, r, r^2, r^3:

$$1 : \begin{cases} x_1 \longrightarrow x_1 \\ x_2 \longrightarrow x_2 \\ x_3 \longrightarrow x_3 \\ x_4 \longrightarrow x_4 \end{cases}, \quad r : \begin{cases} x_1 \longrightarrow x_2 \\ x_2 \longrightarrow x_3 \\ x_3 \longrightarrow x_4 \\ x_4 \longrightarrow x_1 \end{cases}, \quad r^2 : \begin{cases} x_1 \longrightarrow x_3 \\ x_2 \longrightarrow x_4 \\ x_3 \longrightarrow x_1 \\ x_4 \longrightarrow x_2 \end{cases}, \quad r^3 : \begin{cases} x_1 \longrightarrow x_4 \\ x_2 \longrightarrow x_1 \\ x_3 \longrightarrow x_2 \\ x_4 \longrightarrow x_3 \end{cases},$$

and four reflections: κ, κr, κr^2, κr^3

$$\kappa : \begin{cases} x_1 \longleftrightarrow x_4 \\ x_2 \longleftrightarrow x_3 \end{cases}, \quad \kappa r : \begin{cases} x_1 \longleftrightarrow x_3 \end{cases}, \quad \kappa r^2 : \begin{cases} x_1 \longleftrightarrow x_2 \\ x_3 \longleftrightarrow x_4 \end{cases}, \quad \kappa r^3 : \begin{cases} x_2 \longleftrightarrow x_4 \end{cases}.$$

The following subgroups of D_4 represent symmetries of other configurations of points inside the square shown on Figure 1.2.

$D_4 = \{1, r, r^2, r^3, \kappa, \kappa r, \kappa r^2, \kappa r^3\}$ — symmetries of the set $\{0\}$
$D_2 = \{1, r^2, \kappa, \kappa r^2\}$ — symmetries of the set $\{y_1, y_2\}$
$\widetilde{D}_2 = \{1, r^2, \kappa r, \kappa r^3\}$ — symmetries of the set $\{x_1, x_3\}$
$D_1 = \{1, \kappa\}$, — symmetries of the set $\{z\}$
$\widetilde{D}_1 = \{1, \kappa r\}$, — symmetries of the set $\{x_1\}$
$\mathbb{Z}_1 = \{1\}$, — symmetries of the set $\{u\}$

There are also the subgroups $\mathbb{Z}_4 = \{1, r, r^2, r^3\}$ and $\mathbb{Z}_2 = \{1, r^2\}$, which are irrelevant for our discussion at this moment. For this particular example, the equivariant degree of the map f (with no free parameter) is a sequence of integers indexed by the conjugacy classes of subgroups representing possible types of symmetries of the solutions (in contrast to the Brouwer degree, it takes into account the hierarchy of subspaces of $\mathbb{R}^4$ fixed by the subgroups). Namely, in the case of the system (1.1) we have:

$$D_4\text{-Deg}(f(\lambda_-, \cdot), U) = (D_4) - (D_2) + (D_1) - (\widetilde{D}_1),$$
$$D_4\text{-Deg}(f(\lambda_+, \cdot), U) = (D_4) - (D_2),$$

thus

$$D_4\text{-Deg}(f(\lambda_-, \cdot), U) - D_4\text{-Deg}(f(\lambda_+, \cdot), U) = (D_1) - (\widetilde{D}_1). \tag{1.2}$$

where U is a neighborhood of 0 in $\mathbb{R}^4$. The hierarchy of the subgroups of D_4 representing symmetries in the space $\mathbb{R}^4$ is illustrated on Figure 1.3.

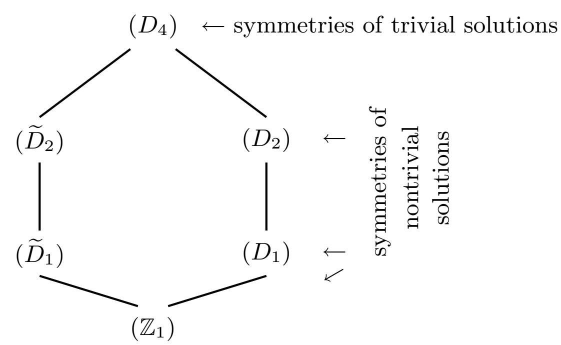

Fig. 1.3. Lattice of “symmetry subgroups” in the D_4-representation $\mathbb{R}^4$

Since, by formula (1.2), the local equivariant degree is changing as λ crosses $\lambda = 0$, there is a branch on nontrivial solutions bifurcating from $(\lambda, \vec{x}) = (0, 0)$. Observe, first, that $D_x f(0,0)|_{\text{span}\{(1,1,1,1)\}}$ is non-singular, thus, by the Implicit Function theorem, there cannot be a bifurcating branch of nontrivial solutions with (D_4)-symmetries. So, from Figure 1.3, the largest possible types of symmetries of nontrivial solutions are $(\widetilde{D}_2)$ and (D_2). On the other hand, in formula (1.2), we have non-zero coefficients for $(\widetilde{D}_1)$ and (D_1), therefore, it follows that the equation admits at least 2 branches of nontrivial solutions with at least $\widetilde{D}_1$-symmetries and at least 2 branches of nontrivial solutions with at least D_1-symmetries.

1.2 Main Goal

The equivariant bifurcation of the type described above, being a starting point for our discussion, was studied in [156] and we refer the reader to this book for more information and examples. Our *actual goal* is to show that the equivariant degree (with free parameters) can be:

(i) applied effectively to many different kinds of mathematical models, even those using sophisticated functional settings; and
(ii) evaluated in a standard way without the use of advanced mathematical theories.

(i) *Application scheme: reduction to basic maps.* To clarify the first point, let us discuss an interesting example of a ring of four identical coupled oscillators, which was used by Turing (cf. [242]) to model various situations in biology, chemistry and electrical engineering. Since in many chemical or biological oscillators the time needed for transport or processing of chemical components or signals may be of considerable length, adding temporal delays to this model seems to be quite reasonable:

$$\begin{cases} \dot{x}_1(t) = -\alpha x_1(t) - \alpha\big[g(x_4(t-1)) - 2g(x_1(t-1)) + g(x_2(t-1))\big] \\ \dot{x}_2(t) = -\alpha x_2(t) - \alpha\big[g(x_1(t-1)) - 2g(x_2(t-1)) + g(x_3(t-1))\big] \\ \dot{x}_3(t) = -\alpha x_3(t) - \alpha\big[g(x_2(t-1)) - 2g(x_3(t-1)) + g(x_4(t-1))\big] \\ \dot{x}_4(t) = -\alpha x_4(t) - \alpha\big[g(x_3(t-1)) - 2g(x_4(t-1)) + g(x_1(t-1))\big] \end{cases} \tag{1.3}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function such that $g(0) = 0$ and $g'(0) = 1$. It is clear that the system (1.3) has D_4-symmetries and $\vec{x}(t) := (x_1, x_2, x_3, x_4) = \vec{0}$ is a trivial solution to (1.3) for every α. By following the standard linearization procedure at $\vec{x} = \vec{0}$ (see, for instance, [258]), one considers the system

$$\begin{cases} \dot{x}_1(t) = -\alpha x_1(t) - \alpha\big[x_4(t-1) - 2x_1(t-1) + x_2(t-1)\big] \\ \dot{x}_2(t) = -\alpha x_2(t) - \alpha\big[x_1(t-1) - 2x_2(t-1) + x_3(t-1)\big] \\ \dot{x}_3(t) = -\alpha x_3(t) - \alpha\big[x_2(t-1) - 2x_3(t-1) + x_4(t-1)\big] \\ \dot{x}_4(t) = -\alpha x_4(t) - \alpha\big[x_3(t-1) - 2x_4(t-1) + x_1(t-1)\big] \end{cases} \tag{1.4}$$

and, by substituting $\vec{x}(t) = e^{\lambda t}\vec{z}$, $\vec{z} \in \mathbb{C}^4$, $\lambda \in \mathbb{C}$, into (1.4), one obtains the following characteristic equation for (1.3)

$$\begin{cases} \lambda z_1 = -\alpha z_1 - \alpha e^{-\lambda}\left[z_4 - 2z_1 + z_2\right] \\ \lambda z_2 = -\alpha z_2 - \alpha e^{-\lambda}\left[z_1 - 2z_2 + z_3\right] \\ \lambda z_3 = -\alpha z_3 - \alpha e^{-\lambda}\left[z_2 - 2z_3 + z_4\right] \\ \lambda z_4 = -\alpha z_4 - \alpha e^{-\lambda}\left[z_3 - 2z_4 + z_1\right] \end{cases} \tag{1.5}$$

where $\vec{z} := (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$. It is well-known that in the case the zero solution changes its stability as α crosses a certain value α_o, in particular, when there is a characteristic value λ crossing the imaginary axis, then a Hopf bifurcation takes place in (1.3), i.e. there is a bifurcation of small amplitude periodic solutions from $(\alpha_o, \vec{0})$. It turns out that symmetries are also helpful to identify those values of α for which there exists a purely imaginary characteristic value. The space $\mathbb{C}^4$ can be decomposed into a direct sum of three subspaces (each of them being an irreducible complex D_4-subrepresentation), namely,

$$\mathbb{C}^4 = V_0 \oplus V_1 \oplus V_2,$$

where $V_0 := \text{span}\{(1,1,1,1)\}$, $V_1 := \text{span}\{(-1,0,1,0),(0,-1,0,1)\}$, and $V_2 := \text{span}\{(1,-1,1,-1)\}$, and the characteristic equation (1.5) can be written using this direct decomposition as the following three equations:

$$\begin{cases} \lambda + \alpha & = 0 \quad \text{for } V_0; \\ \lambda + \alpha - 2\alpha e^{-\lambda} & = 0 \quad \text{for } V_1; \\ \lambda + \alpha - 4\alpha e^{-\lambda} & = 0 \quad \text{for } V_2. \end{cases} \tag{1.6}$$

By substituting into the second equation $\lambda = i\beta$, we find easily that $\beta_o = \frac{\pi}{3}$ and $\alpha_o = -\frac{\pi}{3\sqrt{3}}$ is a solution to (1.6). Moreover, it can be verified that, when α crosses α_o, a pair of characteristic values enter the right half-plane of the complex plane at $\pm i\beta_o$. Therefore, we clearly have an occurrence of the Hopf bifurcation. However, questions related to the multiplicity of the bifurcating periodic solutions and their symmetries still remain.

By applying the standard techniques based on introducing the unknown period as an additional parameter, and reformulating this bifurcation problem in the appropriate functional space W, we obtain an equation with $D_4 \times S^1$-symmetries, where $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ denotes the symmetries related to shiftings of the temporal variable. More precisely, the bifurcation problem can be reduced to studying the equation $\mathfrak{F}_\varsigma(\lambda, x) = 0$, where $\mathfrak{F}_\varsigma : \overline{\Omega} \to \mathbb{R} \oplus W$ is $D_4 \times S^1$-equivariant and $\Omega \subset \mathbb{R}^2 \oplus W$ is a suitable bounded open subset. In a natural way one can associate to $\mathfrak{F}_\varsigma$ the equivariant degree $\deg_{D_4 \times S^1}(\mathfrak{F}_\varsigma, \Omega)$. The complex topological nature of $\deg_{D_4 \times S^1}(\mathfrak{F}_\varsigma, \Omega)$ makes difficult its practical computation. Fortunately, all the essential information relevant to the *symmetric Hopf bifurcation* is contained in the so-called *primary part* of $\deg_{D_4 \times S^1}(\mathfrak{F}_\varsigma, \Omega)$, in which, for the sake of simplicity, we consider only the coefficients related to the first mode of the 'Fourier decomposition' of W. We denote this 'simplified' degree by $D_4 \times S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega)_1$. The nontriviality of the coefficients of the latter degree allows us to classify the bifurcating solutions according to their symmetries.

Let us point out that $D_4 \times S^1$-Deg$(\mathfrak{F}_\varsigma, \Omega)_1$ can be completely evaluated following two main strategies:

(a) using the standard homotopy argument and symmetric spectral information contained in the linearization, one can deform $\mathfrak{F}_\varsigma$ into a map 'built up' of simple canonical maps (the so-called *basic maps*) $\mathfrak{b} : \mathbb{R} \oplus \mathcal{V} \to \mathcal{V}$ given by

$$\mathfrak{b}(t, v) = \big((1 - \|v\|) + it\big)v,$$

and defined on the set $\big\{(t, v) \in \mathbb{R} \oplus \mathcal{V} : |t| < 1,\ \frac{1}{2} < \|v\| < 2\big\}$, where $\mathcal{V}$ is an *irreducible* $D_4 \times S^1$-representation;

(b) a "pre-fabricated list" of $D_4 \times S^1$-degrees $\deg_{\mathcal{V}}$ of basic maps combined with certain multiplicativity properties allows to evaluate the exact value of $D_4 \times S^1$-Deg$(\mathfrak{F}_\varsigma, \Omega)_1$.

In our case, we have

$$D_4 \times S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega)_1 = \begin{cases} -(\mathbb{Z}_4^t) - (D_2^d) - (\widetilde{D}_2^d) + (\mathbb{Z}_2^-) + (D_1^z) \\ +(\widetilde{D}_1^z) + (D_1) + (\widetilde{D}_1) - 2(\mathbb{Z}_1), \end{cases} \tag{1.7}$$

where

$$\mathbb{Z}_4^t = \{(1,1), (r, i), (r^2, -1), (r^3, -i)\} \tag{1.8}$$

$$D_2^d = \{(1,1), (r^2, -1), (\kappa, 1), (\kappa r^2, -1)\} \tag{1.9}$$

$$\widetilde{D}_2^d = \{(1,1), (r^2, -1), (\kappa r, 1), (\kappa r^3, -1)\} \tag{1.10}$$

$$D_1^z = \{(1,1), (\kappa, -1)\} \tag{1.11}$$

$$\widetilde{D}_1^z = \{(1,1), (\kappa r, -1)\} \tag{1.12}$$

$$\mathbb{Z}_2^- = \{(1,1), (r^2, -1)\}, \tag{1.13}$$

are the subgroups of $D_4 \times S^1$. The existence property of the equivariant degree combined with formula (1.7) imply that there are at least 6 branches of non-constant periodic solutions bifurcating from zero. More precisely, 2 of these branches admit the symmetries at least $\mathbb{Z}_4^t$, i.e. the shifting by $p/4$ of the temporal variable t causes at least the rotation of the p-periodic solution, 2 branches of non-constant periodic solutions have at least the symmetries D_2^d, and 2 other branches of non-constant periodic solutions have at least symmetries $\widetilde{D}_2^d$, i.e. the shifting by $p/2$ causes at least the reflection of the p-periodic solution.

(ii) *Computational scheme.* Let us stress further that, in order to compute the primary equivariant degree of $\mathfrak{F}_\varsigma$ and make the above conclusions, we need only the information about the characteristic values of (1.5) and their symmetries, i.e. the so-called *isotypical decomposition* of the corresponding 'eigenspaces.'

The primary degree, which provides a complete topological classification of the symmetric Hopf bifurcation, can be computed directly from the tables of the primary degrees of basic maps combined with the above mentioned *multiplicativity property*. This functorial property of the primary equivariant degree, being a key feature of the main paradigm of this book – **equivariant topology, via algebra and computer routines, in service of applied mathematics**, deserves a special attention. Its idea can be traced back to the Brouwer degree taking its values in the *ring* $\mathbb{Z}$. Similarly, in the equivariant case without free parameter, the range $A(\Gamma)$ of the Γ-equivariant degree also admits a structure of a ring (the so-called *Burnside ring*). In both cases, the addition in the ring reflects unions of zeros, while the multiplication corresponds to a Cartesian product of zeros. It is reasonable to expect a similar algebraic parallelism in the case of the equivariant degree with at least one free parameter.

To illustrate the basic idea, return to system (1.3). The considered system admitted D_4-symmetries and the invariant $D_4 \times S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega)_1$ associated with the D_4-symmetric Hopf bifurcation, had values ranging over the $\mathbb{Z}$-module $A_1(D_4 \times S^1)$ generated by the so-called *twisted subgroups* of $D_4 \times S^1$ (cf. the list (1.8)–(1.13)). Actually, the final formula (1.7) was obtained using the $A(D_4)$-module structure on $A_1(D_4 \times S^1)$. It turns out that the use of such a structure is a common ingredient in the equivariant degree treatment of the Hopf bifurcation problems admitting an arbitrary compact Lie group Γ of symmetries. Fortunately, for many reasonable compact Lie groups Γ, the multiplication tables for the $A(\Gamma)$-module $A_1^t(\Gamma \times S^1)$ can be established *a priori* by standard routines (the symbol 't' indicates here the connection with twisted subgroups). Of course, in such a case we need to further restrict (when Γ is infinite) the values of the equivariant degree to the module $A_1^t(\Gamma \times S^1)$ (we will use the symbol $\Gamma \times S^1\text{-Deg}^t$ for this restricted degree and will call it *twisted degree*).

A general scheme for the computation of $\Gamma \times S^1\text{-Deg}^t(\mathfrak{F}_\varsigma, \Omega)$ is shown in Figure 1.4.

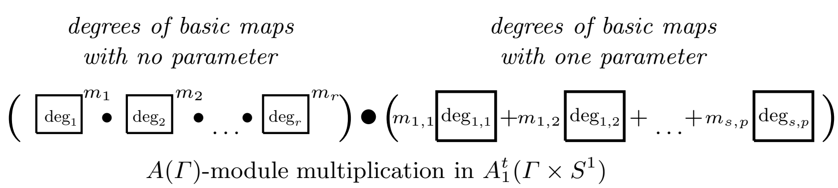

Fig. 1.4. Degrees of basic maps and the computation of the degree

The symbols $\deg_j$ on Figure 1.4 stand for Γ-degrees of $-\text{Id}$ on irreducible subrepresentations, while $\deg_{j,l}$ is the twisted degree of basic maps. Observe that the above formula emanates from a similar product expression well-known in the non-equivariant case. Namely, the first factor reflects the change of orientation caused by multiplicities of the corresponding eigenvalues of the linearization, while the second one is related to the change of stability measured by the corresponding crossing numbers. The numbers m_i and $m_{j,l}$ are the equivariant replacements of these objects.

Based on the above arguments, we claim that the *twisted* equivariant degree can actually be evaluated without direct connection to its theoretical roots lying in equivariant topology, homotopy theory and bordism theory, and, in fact, all the involved tasks can be completely computerized. The only additional background behind the machinery of computer routines, which is needed for applying the equivariant degree to concrete models, are the representation theory and basic properties of classical groups and their subgroups. However, a proper understanding of any applied problem with symmetries requires this knowledge anyway.

Anybody, who got even a little taste in dealing with applied problems admitting non-abelian group symmetries, has certainly also experienced frustration caused by the large numbers of subgroups involved. A common 'allergic' reaction to the equivariant methods results from a feeling that the mosaic-like diversity of symmetry subgroups resembles tensors in the old fashion differential geometry. It is our believe that the expression indicated on Figure 1.4 gives a unified framework for studying symmetric Hopf bifurcation problems allowing to computerize tedious and bland algebraic computations.

1.3 Historical Roots and Topological Aspects of Equivariant Degree

Although our monograph is mainly addressed to applied mathematicians, we also believe that it may be of interest to a more general audience with particular interests in Equivariant Analysis. Keeping this in mind, we feel obliged to include a short discussion of topological, geometric and analytic origins of the equivariant degree.

Equivariant Analysis deals with the impact of symmetries (represented by a certain group G and translated as the equivariance of the corresponding operators) on the existence, multiplicity, stability and topological structure of solutions of non-linear equations, bifurcation phenomena (local and/or global, with one or more parameters), the applicability of different kinds of approximation schemes, etc. In particular, it involves natural symmetries in mathematical models, such as systems of ODEs and FDEs with symmetries, among them the symmetric Hamiltonian systems, PDEs and FPDEs on symmetric domains, phase transition, and symmetry breaking, to mention a few.

Equivariant degree introduced in [127] and rigorously studied in [131] for abelian groups, being one of the main topological tools of the Equivariant Analysis, is an alternative to other topological methods, such as variational methods (minimax theory, Conley index, Morse-Floer complex (cf. [43, 44, 53, 89, 90, 188, 215])), singularity theory (cf. [105, 106, 107]) and reduction to the fixedpoint subspaces (cf. [84, 85, 86]).

To be more specific, given a compact Lie group G, orthogonal G-representations V and W, an open invariant subset $\Omega \subset W$ and a continuous equivariant map $f : (\overline{\Omega}, \partial\Omega) \to (V, V \setminus \{0\})$, one can assign to f the equivariant degree $\deg_G(f, \Omega)$ taking its values in the equivariant homotopy group of maps

$$\partial([0,1] \times B) \to (\mathbb{R} \times V) \setminus \{0\}, \tag{1.14}$$

where B is a large ball in W centered at the origin. Moreover, $\deg_G(f, \Omega)$ satisfies all the natural properties expected from any reasonable "degree theory," in particular, existence, homotopy invariance, excision, suspension, additivity (up to one suspension), etc. Roughly speaking, the equivariant degree "measures" (equivariant) homotopy obstructions for $f_{|\partial\Omega}$ to have an equivariant extension without zeros over $\overline{\Omega}$ (composed of several orbit types). From this point of view the historical roots of the equivariant degree theory can be traced back to several mathematical fields, which we list below.

(i) **Borsuk-Ulam Theorems.** In the early thirties, K. Borsuk [47] established that the Brouwer degree of an odd map of a finite-dimensional sphere into itself is odd, i.e. he observed for the first time that symmetries can lead to restrictions on possible values of the degree. After Borsuk, the following development of the theory was primarily due to P.A. Smith and M.A. Krasnosel'skii. Smith introduced a special homology theory on a category of $\mathbb{Z}_p$-spheres for a prime p which was used to express degrees of equivariant maps via homological characteristics of the corresponding actions (the so-called Smith indices) and via degrees of the restrictions of the maps in question to the relevant sets of fixed points (when defined). This gave rise to the "homological approach" (we refer to [46] and [92] where the recent achievements of this approach are presented).

Krasnosel'skii discovered a deep connection between the degree of equivariant maps and the problem of equivariant extensions of maps (cf. [148]), which initiated the so-called "geometric approach" in this field. We refer to [162] for the recent achievements of the geometric approach for arbitrary compact Lie groups and [131], where the case of linear abelian group actions is studied in detail.

(ii) **Fundamental Domains and Equivariant Extensions.** Using the standard techniques (see, for instance [71]), the problem of equivariant extensions can be reduced to the one of extensions over a *free* G-space. The last problem was attacked using mainly two methods: (a) *extension of partial sections* *of associated bundles* (see [134, 135, 167, 175, 176, 71]) and (b) *extension by means of the so-called fundamental domains* (see [162] for a detailed exposition of this concept for arbitrary compact Lie group actions and [131] for linear abelian group actions). In fact, the notion of a fundamental domain has a tie with different mathematical disciplines, namely: (α) fundamental polygon for isometry groups of Riemannian manifolds, (β) Weierstrass section in invariant theory, (γ) Poincaré section in ODEs, to mention a few. Note that the general equivariant extension problem is a subject of the equivariant retract theory (see [15]). For more information about further contributions in this area, we refer to [14, 16].

(iii) **Equivariant Homotopy Groups of Spheres.** As is clear from (1.14), the equivariant degree takes its values in the appropriate equivariant homotopy group of spheres. This object was studied (especially in the stable setting) by algebraic topologists (see [170]) using rather sophisticated techniques (for instance, Adams and Wirthmuller isomorphisms). On the other hand, similarly to the non-equivariant case, equivariant homotopy groups turn out to be connected with the equivariant bordism theory, which, in general, also amounts to the machinery being rather technical in nature (see [238]).

However, in applications one usually deals with equivariant homotopy groups of maps (1.14) with $W = \mathbb{R}^k \oplus V$, where G acts trivially on $\mathbb{R}^k$ (the corresponding stable equivariant homotopy group is denoted by Π_k^G). The latter group in many cases admits a very transparent description. In 1970, Segal [229] conjectured that for any finite group G , $\Pi_0^G \simeq A(G)$, where $A(G)$ denotes the Burnside ring of G. This result was proved by Kośniowski [147], and independently by Rubinsztein [219] with a gap eliminated by Dancer [65] (see also [243]). T. tom Dieck [71] proved the same result for a general compact Lie group, giving a convenient definition of the Burnside ring in this case. The further development (in particular, computation of Π_1^G) was mostly due to people working in nonlinear analysis rather than to algebraic topologists (see [31, 38, 39, 131, 184])).

Finally, we should mention different versions of the so-called equivariant Hopf theorem (see [71, 241, 165], where the equivariant cohomology groups were used) as important preludes to the equivariant degree (see also [162], where a slightly more general result was obtained by using the non-equivariant obstruction theory in compliance with the fundamental domain concept).

(iv) **Equivariant General Position Theorems.** Many theoretical questions related to the equivariant homotopy classification of maps $(\overline{\Omega}, \partial\Omega) \to (V, V \setminus \{0\})$, where V and $\Omega \subset W$ are as in (1.14), can be reduced to the following ones: (a) *to separate zeros having different orbit types*, (b) *to choose representatives of equivariant homotopy classes admitting reasonable transversality/regularity conditions.*

The first problem, in the case $W = \mathbb{R}^k \oplus V$, can be settled using the concept of normality (see [77] for the case $G = S^1$ and [100] for an arbitrary G). To treat the case of an arbitrary W (for G being abelian), Ize *et al.* used the concept of complementing maps (see [131]; cf. [44]); similar concepts combined with G-invariant one-dimensional foliation techniques were used in [162] to compute the Brouwer degree of equivariant maps between G-manifolds (G being arbitrary).

The second problem is more delicate: the equivariance "conflicts" with regularity (for instance, due to the restriction requirements on the dimensions of orbits of zeros). Therefore, one has to look for special types of transversality requirements which are compatible with such techniques as the induction over orbit types and the suspension operation. In the case $W = \mathbb{R}^k \oplus V$ some useful concepts of equivariant regularity were suggested in [77] and [100], where it was also proved that any equivariant map admits an arbitrarily close approximation by a normal regular map (see also [31, 39, 154, 159] for the related results).

Finally, we should refer to [264, 189], where elegant piece-wise linear equivariant general position results are presented (see also [162] for the related topics).

(v) **Generalized Topological Degree, Primary Degree and Topological Invariants of Equivariant Gradient Maps.** Among the non-equivariant topological constructions that are close in spirit to the equivariant degree, the generalized topological degree introduced by Gęba *et al.* in [102] is of particular importance, where the Alexandroff compactification was used and the additivity property was proved in the stable case only, by using cohomotopy operations.

In the case of an autonomous system (which, of course, involves an S^1-action being present implicitly), Fuller (cf. [96]) defined a special index being a rational number. On the other hand, being motivated by studying bifurcations of periodic solutions to Hamiltonian systems, E.N. Dancer introduced an invariant (see [67]), also being a rational number, defined for S^1-equivariant gradient maps. Probably, the rational invariants defined by Fuller and Dancer may be counted as the first "equivariant predecessors" of the primary S^1-degree.

Using regular normal approximations, Gęba *et al.* (independently of Ize *et al.*) introduced in [77] the S^1-equivariant degree, which takes its values in

$$\mathbb{Z}_2 \oplus \bigoplus_{i \in \mathbb{N}} \mathbb{Z} =: A_s \oplus A_p, \quad A_s := \mathbb{Z}_2,\ A_p := \bigoplus_{i \in \mathbb{N}} \mathbb{Z}. \tag{1.15}$$

This degree, in the setting it was defined, coincides with the S^1-degree presented in [128]. The connection between the Fuller index and S^1-degree was indicated in [77] (see also [131], p. 231). The coefficients of the S^1-degree, which belong to the free $\mathbb{Z}$-module A_p indicated in (1.15), correspond exactly to the primary part of the S^1-degree. Combining normal maps techniques with methods of differential geometry, the primary degree was extended to the case of an arbitrary compact Lie group in [100]. Computations done by Peschke (cf. [211]), which are related to the "primary part" of Π_k^G, allow to recognize the primary degree as a part of the equivariant degree introduced by Ize *et al.* The importance of the *bi-orientability* notion, which plays an essential role in the construction of the primary degree, was also first observed in [211].

Observe, finally, that Dancer's idea to associate topological invariants to S^1-equivariant gradient fields, was further developed in several directions: (i) S. Rybicki defined the S^1-degree for equivariant orthogonal maps (an extension of gradient maps, see [222]), (ii) for abelian compact Lie groups, the G-degree for orthogonal G-maps was introduced and studied by Ize and Vignoli (cf. [131]), (iii) in the case of an arbitrary compact Lie group, the corresponding construction was done by Gęba (cf. [99], see also [226]).

(vi) **Geometric Obstruction Theory and J-Homomorphism in Multiparameter Bifurcations.** The last, but not the least stream of ideas preceding the equivariant degree is related to the multiparameter bifurcation. On the one hand, Ize [124] developed the geometric obstruction machinery (i.e., the obstruction theory without cohomologies), on the other hand, the role of the Whitehead J-homomorphism was discovered independently by Alexander and York [8] and Ize. For the detailed exposition of these and related concepts we refer to [126].

Based on the fact that the equivariant degree has emanated from different mathematical fields rooted in a variety of ideas and methods, the generality of its construction is essential in order to fit the diversity of problems that have inspired its creation. However, the generality causes more problems than it helps to solve. Therefore, to avoid a "nothing about everything" situation, one is forced to make the appropriate choices, namely to restrict (i) the classes of considered groups, (ii) the classes of studied representations, (iii) the number of free parameters, (iv) the parts of the equivariant degree, etc. Although, we are mainly concerned with problems relative to Γ-symmetric Hopf bifurcations, which in a natural way lead to the twisted $\Gamma \times S^1$-degree, the symmetric steady-state bifurcation and its interaction with the Hopf bifurcation is linked to the secondary parts of the equivariant degree. In contrast to the primary degree, the computational techniques for the secondary degree allowing us to use tables prepared in advance, are still needed. Actually, we have established (in the case of one free parameter) the method for computation of the corresponding equivariant homotopy groups (range of the secondary degree), which is, of course, the first necessary step in this direction. Next, we can indicate at least three situations for which several parts of the secondary degree may be evaluated:

(a) the component relevant to the so-called "invariant part" of the equivariant degree (i.e. the one related to the subspace fixed by the whole group) allows the standard non-equivariant methods (see [91]).
(b) some cases of the abelian group action allow direct computations based on the use of fundamental domains (see [131]);
(c) some cases of $SO(3) \times S^1$-representations considered in Section 7.3 of our book allow a reduction to the computations of the primary degree.

Probably, to treat the general case, one should intelligently combine the (non-equivariant) bordism methods with studying the geometry of the corresponding fundamental domains.

1.4 Overview of the Book

Let us discuss in more detail the contents of our book. In Chapter 2, we included some preliminary results, which are used later in this book. In Section 2.1, we introduce the equivariant jargon frequently used throughout the book. In Section 2.2, we give a brief overview of some basic concepts and constructions from representation theory. The geometric approach to the equivariant extension problem, based on the use of a fundamental domain, is discussed in Section 2.3. Section 2.4 is devoted to review of foundations of the smooth equivariant topology. In Section 2.5, we discuss certain integer-valued characteristics (the so-called $n(L, H)$-numbers) of subgroups of a compact Lie group, which play an important role in the computations of the primary equivariant degree. Some aspects of oriented and framed bordism theory are presented in Section 2.6. To keep this chapter of reasonable size, we focused on the ideas and constructions rather than technical details and proofs, which can be easily found in appropriate references.

Chapter 3 is entirely devoted to the construction of the equivariant degree and its properties. As the stable (equivariant) homotopy groups of spheres are the range of the (equivariant) degree, this concept (both in non-equivariant and equivariant context) is discussed in Section 3.1. In this section, we also introduce the stable equivariant homotopy group Π^G, which will play the role of the range for the equivariant degree (notice that the range of the Brouwer degree, representing the usual zero stable homotopy group of sphere, is $\mathbb{Z}$). The equivariant degree is introduced in Section 3.2, where its construction is also compared with the degree in non-equivariant setting. Our definition of the equivariant degree is a slight modification of the one introduced in [127] (see also [131]). Let us point out that similarly to the non-equivariant case (see for example [151]), the formal definition of the equivariant degree plays no significant role in its computations, but instead its properties are used for this purpose. In the same way as the transversality theorem is used for studying the homotopy groups of spheres and related "degrees," the regular normal approximations are applied for the computations of the equivariant homotopy groups and the equivariant degree. This concept, which is discussed in Section 3.3, is used (in Section 3.4) to prove the decomposition of the group Π^G. More precisely, in the case of n-dimensional free parameter space, Π^G is a direct sum of subgroups $\Pi(H)$ indexed by the conjugacy classes (H) of subgroups in G with $\dim W(H) \leq n$ (where $W(H)$ stands for the Weyl group of H). In Section 3.4, we also apply the notion of the equivariant framed bordism to describe the subgroups $\Pi(H)$. Section 3.5 is devoted to the computations of the primary groups $\Pi(H)$, i.e. $\dim W(H) = n$. First, we show, using the equivariant bordism approach, that $\Pi(H) = \mathbb{Z}$ or $\mathbb{Z}_2$, depending on the biorientability property of $W(H)$. Next, to unveil the geometric roots underlying the equivariant bordism argument, we also give an alternative proof based on the use of a fundamental domain. Some additional properties of fundamental domains are also established. In Section 3.6, we are interested in the secondary groups $\Pi(H)$, for the one free parameter case, i.e. $\dim W(H) = 0$. We combine the standard bordism theory with the fundamental domain approach to completely compute the group $\Pi(H)$ in the considered cases.

In Chapter 4, we introduce an axiomatic definition of the primary degree in the case of one free parameter. In Section 4.1, we formulate a list of axioms for the primary degree. In order to make the axiomatic approach workable, in Sections 4.2—4.4, we pay special attention to the primary S^1-degree. The basic maps appear here in the simplest form in the context, which will be carried over to the general case in Chapter 5. Section 4.5 is rather technical in nature. It contains procedures and explicit formulae needed for the computations of the S^1-degree, which are used in later applications. Finally, by bringing together the S^1-degree techniques with the geometry underlying the numbers $n(L, H)$, we establish, in Section 4.6, the so-called *Recurrence Formula*, which plays a key role in practical computations of the primary G-equivariant degree (for arbitrary compact Lie group). To some extent, this formula may be counted as a Borsuk-Ulam type result for equivariant maps with one free parameter.

Chapter 5 is devoted to computations of twisted degrees of *basic maps.* After a short discussion of *twisted subgroups* of $\Gamma \times S^1$ (in Section 5.1), we introduce, in Sections 5.2, the notion of the *G-equivariant twisted degree.* In Sections 5.3 and 5.4, we establish the lattices of conjugacy classes of subgroups for Γ being the quaternionic units, dihedral, tetrahedral, octahedral, icosahedral, $O(2)$, and $SO(3)$ group. In addition, we provide the reader with a complete list of the numbers $n(L, H)$ for all the above-mentioned groups. Based on this information, we apply the Recurrence Formula to compute twisted degrees of basic maps in all the considered cases (see Sections 5.5–5.8).

In Chapter 6, we discuss the algebraic properties of the twisted equivariant degree in the case of one free parameter. In Section 6.1, we analyze the structure of the Burnside ring $A(\Gamma)$, and present several examples. In Section 6.2, we focus our attention on a product group $G = \Gamma \times S^1$, for which we prove that $A_1^t(\Gamma \times S^1)$ (the range of the twisted G-equivariant degree) has a structure of an $A(\Gamma)$-module. In fact, $A_1^t(\Gamma \times S^1)$ is a submodule of the Euler ring $U(G)$ (cf. [71]). We also present (see Sections 6.3 and 6.4) the complete multiplication tables for a series of examples of groups $G = \Gamma \times S^1$ (with a special consideration given to the group $SO(3) \times S^1$), which will be used in the applications. In Section 6.5, we prove the multiplicativity property for the twisted degree.

Chapter 7 is an illustration of problems arising in connection with the computations of the secondary groups $\Pi(H)$ and the corresponding parts of the equivariant degree. By the results obtained in Section 3.6, $\Pi(H) = \Pi_*(H) \oplus \mathbb{Z}_2$, where $\Pi_*(H) = H_1((\mathbb{R} \oplus V)_H/W(H))$ (here "$H_1(\cdot)$" stands for the first homology group). In the case $(\mathbb{R} \oplus V)_H$ is simply connected, $\Pi_*(H) = W(H)/[W(H), W(H)]$. However, the simply-connectedness is a restrictive requirement, which is not often satisfied. In Section 7.1, we discuss some special cases, when simple topological arguments can be applied to compute $\Pi_*(H)$. In Section 7.2, we utilize an elementary homology theory to deal with $SO(3) \times S^1$-representations of a special type to compute the groups $\Pi_*(H)$. In Section 7.3, we give a formula for connecting the twisted degree with the secondary one (in some special settings). This formula is illustrated, in Section 7.4, by several examples.

In Chapter 8, which concludes the theoretical part of our book, we illustrate how the twisted degree can be adapted to study the so-called orthogonal G-equivariant maps ($G = \Gamma \times S^1$ with Γ finite). This class of maps is a natural generalization of gradient fields of G-invariant functionals. Although, this chapter is rather "methodical" in nature, it provides a computational basis to study simple (but still important) variational problems with symmetries.

The rest of the book is devoted to the applications of the equivariant degree theory to dynamical systems with symmetries. The role of Chapter 9 is two-fold: (i) to open the road to infinite-dimensional extensions of the equivariant degree (see Section 9.1), and (ii) to lay down the standard steps of the equivariant degree treatment for differential equations with symmetries. As a starting point for our discussion of the symmetric Hopf bifurcation (see Section 9.2), we have chosen the simplest class of equations – an autonomous 1-st order system of ODEs (without symmetries) leading to an S^1-equivariant problem. A passage to a symmetric system of ODEs is discussed in Section 9.3. We construct the $\Gamma \times S^1$-equivariant map $\mathfrak{F}_\varsigma$ associated with the occurrence of the Hopf bifurcation, and introduce the notion of the so-called *isotypical crossing number*. The strategies established in Chapter 9 lead to a customary framework for the equivariant degree method, which can be applied, without significant modifications, to a large class of (more sophisticated) equations, for instance, FDEs, PDEs, etc.

In Chapter 10, the equivariant degree method is applied to a system of functional differential equations. In Section 10.1, we present a general parametrized system of symmetric delayed functional differential equations. Section 10.2 is devoted to the computation of the equivariant degree $\omega(\alpha_o, \beta_o) = \Gamma \times S^1\text{-Deg}(\mathfrak{F}_\varsigma, \Omega)$ — the equivariant homotopy invariant associated with this bifurcation problem. In Section 10.3, we present, as an example, a system of parametrized symmetric functional differential equations describing a symmetric configuration of identical oscillators. We establish computational formulae for the bifurcation invariant $\omega(\alpha_o, \beta_o)$ in terms of the spectrum of the characteristic operator. In Section 10.4, we present the Hopf bifurcation results, based on the computations of $\omega(\alpha_o, \beta_o)$ for the configurations of identical oscillators with dihedral, tetrahedral, octahedral and icosahedral symmetry groups. In Section 10.5, we apply the equivariant degree method to a Hopf bifurcation problem for a system of symmetric neutral functional differential equations. The equivariant setting and the application of the equivariant degree follows exactly the same lines that we presented in Chapter 9. As an example, in Sections 10.5–10.6, we consider a symmetric configuration of identical lossless transmission lines, which is modelled by a system of neutral equations with a finite group of symmetries. The occurrence of the Hopf bifurcation is investigated and the equivariant degree method is applied for specific configurations of transmission lines with dihedral, tetrahedral, octahedral and icosahedral symmetry groups. In Section 10.8, we analyze the problem of continuation of symmetric branches of non-constant periodic solutions (i.e. a global Hopf bifurcation problem) for symmetric configurations of identical oscillators, with dihedral or tetrahedral symmetries.

In Chapter 11, we consider a system of functional parabolic differential equations. Section 11.1 is devoted to a review of some well-known facts about unbounded Fredholm operators. In Section 11.2, we discuss a general setting, appropriate for symmetric functional parabolic differential equations, which is used for studying equivariant bifurcation problems. In Section 11.3, we discuss the Hopf bifurcation problem for a general functional parabolic system of differential equations with symmetries. A symmetric system of n coupled Hutchinson equations describing an interaction between identical population models, is discussed in Section 11.4. In Section 11.5, we present the Taylor-Couette model, an abstract setting for this type of $O(2)$-symmetric parabolic systems and the results for the computation of the local bifurcation invariant, which are next applied to the Taylor-Couette system. The Hopf bifurcation for the Couette problem and the classification of the symmetries of bifurcating periodic solutions is discussed in Subsection 11.5.4.

In Chapter 12, we extend the approach introduced by Hirano and Rybicki (cf. [118]) to the class of Γ-symmetric van der Pol systems with Γ being a finite group. In contrast to the bifurcation problems considered in the previous chapters, we present here several *existence results* for the studied equations. After a short exposition of the van der Pol model (in Section 12.1), we describe the equivariant fixed-point setting related to this problem (see Section 12.2 where we also briefly discuss the Hirano-Rybicki approach). The computational formulae required for this setting are presented in Section 12.3. Based on the *a priori* estimates obtained in [118], we establish a general existence result for Γ-symmetric van der Pol systems in Section 12.4. We conclude this chapter with the existence and multiplicity results for the van der Pol systems with dihedral, tetrahedral, octahedral and icosahedral symmetry groups.

In Chapter 13, we apply the results from Chapter 8 to illustrate how the equivariant degree method can be used to study variational problems.

The remaining part of the book is devoted to appendices. In Appendix A1, we explain how to set up Maple© routines in order to computerize tedious calculations of the multiplication tables and twisted degrees of basic maps. As an example, in Sections A1.2–A1.4, simple Maple© procedures for the groups A_5 and $A_5 \times S^1$ are presented. These procedures are created based on the computational formulae for basic maps (cf. Chapter 5) and multiplication tables for the $A(\Gamma)$-module $A_1^t(\Gamma \times S^1)$ (cf. Chapter 6), which constitute the core of the application scheme of the equivariant degree method. In Appendix A2, we present some algebraic ideas and methods allowing to obtain "semi-algebraic" results on the existence of bounded solutions to homogeneous polynomial ODEs. In Appendix A3, we (i) view the problem of the existence of (homogeneous) equivariant maps between representation spheres as a factorization procedure through symmetric powers of representations, and (ii) discuss possible generalizations of the classical Atiyah-Tall theorem on the existence of equivariant maps between G-representation spheres with G being a finite p-group.

1.5 Other Related Publication

As *Walter de Gruyter* published recently in the Nonlinear Analysis series the monograph *Equivariant Degree Theory* by J. Ize and A. Vignoli (see [131]), we cannot avoid making comparison of that book with ours. Although, both monographs are devoted to the equivariant degree, in our opinion, one can recognize essential differences in their presentations of the subject, approach and applications.

(i) The authors of [131] deal almost exclusively with abelian group actions. On the other hand, just by observing one's surroundings one can easily notice a multitude of non-abelian symmetries appearing in natural phenomena. We should point out that all our examples and applications are related to nonlinear problems involving non-abelian groups. However, while Ize and Vignoli were able to accomplish completeness of their exposition of the abelian equivariant degree theory, our book, for obvious reasons, is dealing only with several (but important) specific non-abelian group actions. The difficult task of keeping balance between *Scylla* of generality and *Charybdis* of completeness is the usual dilemma of an adventurer entering the realm of mathematical discovery.

(ii) The approach adopted in [131] is essentially based on the usage of a fundamental cell, geometric equivariant obstructions and complementing maps with "efforts in keeping the mathematical background to a minimum." Our method is combining the idea of a fundamental domain with elements of bordism theory and regular normal approximations. In addition, we do not hesitate to use such mathematical tools as elementary homology theory to describe possible values of the equivariant degree (i.e. equivariant homotopy groups). However, our computational formulae are completely free from heavy formalism of the algebraic topology and, we believe, can be understood by applied mathematicians.

(iii) Regarding the applications, a theory can be used by researchers in mathematics in order to advance their own research in another area. Also, it can be applied outside mathematics as a tool to solve problems even without deeper knowledge of all its technical aspects. Although [131] contains many important and elegant examples, their applications are rather theoretical in nature. In our book, we were guided by the idea that this dual nature of any theory motivated by applications requires, besides a formal presentation, an easy to use set of instructions explaining how to make use of its results. Eventually, a computer software implementing obtained algorithms could be created for even more efficient treatment of complex symmetric problems. In conclusion, a distinguished character of our book is explained in its title — *Applied Equivariant Degree*.

All this makes us believe that another angle in the same field may be profitable for anyone interested in the equivariant degree and its applications.

1.6 Potential for Further Research

We would like to point out that the equivariant degree theory is far from being completed. The axiomatic approach seems to be standard for applications of the Brouwer degree. Since any irreducible representation of an abelian group has only one nontrivial orbit type, a similar approach seems to be possible even for the equivariant degree with abelian actions. However, in the case of the equivariant degree with respect to a non-abelian group action, the immense variety of all possible irreducible representations and their isotropy structures creates limitations forcing us to consider the equivariant degree for any particular group as an independent theory. Consequently, it is difficult to imagine that the potential for the equivariant degree could be easily exhausted. On the other hand, the close connection of the equivariant degree with the equivariant algebraic topology provides another alley for a further research. The multiparameter equivariant degree has not yet been properly studied, and there is a whole universe to be discovered in relation to its properties and possible applications. For instance, the results related to the $S^1 \times S^1$-equivariant degree with two parameters (cf. [131]) could be extended to the case of non-abelian actions, by applying the reductions to torus. Moreover, the full power of the equivariant degree lies in all its components, including the secondary ones. Up to now, we were able to effectively use only the *primary* degree in the case of one free parameter, corresponding to relatively large orbits. The secondary degree still waits to be properly studied and applied to meaningful real-life problems.

As the research related to the equivariant degree is clearly dependent on a development of a database containing for each considered group of symmetries the *multiplication tables*, *lattices of the isotropy groups for its irreducible representations*, *degrees of the basic maps associated with these irreducible representations*, we provide some of this information on our web site at `http://krawcewicz.net/degree`. It is our intention to further develop this web site and provide all the researchers interested in application with the computational tools.

1.7 Acknowledgments

Special thanks go to S. Brodsky, A. Kushkuley, and J. Wu for their interest and stimulating discussions. We are also thankful to Y. Rudiak for his suggestions related to the computations of the secondary obstructions, and S. Rybicki for his comments and discussions. We are deeply grateful to P.P. Zabreiko for his encouragement and inspiration. We would like to thank B. Rai for collaboration and interesting suggestions. We are grateful to P. Antonelli, S. Antonyan, H. Freedman and E. Woolgar for their comments and remarks. We would like to acknowledge the contribution of M. Farzamirad, who helped us with the computations of the multiplication tables and some of the examples. The Maple package for the equivariant degree was developed by A. Biglands (during the summer of 2006 he was supported by the NSERC undergraduate research grant), and we are thankful to him for volunteering for this undertaking. We are sincerely grateful to H. Ruan, for her contribution to this book and long hours spent on helping us with the manuscript, improving the text, checking the computations, and her highly appreciated patience. Finally, the authors would like to thank J. Bernstein, H. Petersson and S. Walcher (resp. G. Belitsky) for improving our understanding of the algebraic (resp. dynamical systems) ideas underlying the subject presented in Appendix A2.

The authors would like to acknowledge the financial support from the Alexander von Humboldt Foundation, which in 1997, 1998, 2000, 2003, 2004 and 2006, provided the first two authors with grants allowing us to work on this book together in Munich. The financial support from the Netanya Academic College and Bar-Ilan University given to the first author was an important factor for making these visits possible. We are thankful to the Institute of Mathematics of Ludwig-Maximilians-University for supplying us the facilities and the infrastructure required by our project. The second author was also supported by a grant from the NSERC of Canada. We are also grateful to the PIMS for providing financial support for the first author's visit at the University of Alberta during the year 1996, and the Department of Mathematics of the University of Alberta for the hospitality during his visits in 1998, 2000, 2002 and 2005. The second author was supported by the Harish-Chandra Research Institute in Allahabad, where during the summer 2001, a part of the manuscript was written. In particular, we would like to thank its director H.S. Mani, D. Prasad, Dean Administrator S.N. Rai, for their hospitality. The second author would like also to thank the Netanya Academic College for the hospitality during his two visits in Israel in 2005.

München, October 2006

Z. Balanov
W. Krawcewicz
H. Steinlein

Part I

EQUIVARIANT DEGREE THEORY: TECHNICAL TOOLS

2

Preliminaries

In this chapter we collect some results and facts from the representation theory and equivariant topology, which will be used in Chapter 3 to construct the G-equivariant degree theory with n free parameters for an arbitrary compact Lie group G.

2.1 Basic Definitions

In what follows, G stands for a compact Lie group.

Definition 2.1. A *topological transformation group* is a triple (G, X, φ), where X is a Hausdorff topological space and $\varphi : G \times X \to X$ is a continuous map such that:

(i) $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$ for all g, $h \in G$ and $x \in X$;
(ii) $\varphi(e, x) = x$ for all $x \in X$, where e is the identity element of G.

The map φ is called a *G-action* on X and the space X, together with a given action φ of G, is called a *G-space* (or, more precisely, *left G-action* and *left G-space*, respectively; also, the notions of *right action* and *right G-space* can be defined in a similar way). We shall use the notation $g(x)$, $g \cdot x$ or simply gx, for $\varphi(g, x)$ (or xg in the case of a right G-space). For $K \subset G$ and $A \subset X$, we put $K(A) := \{gx : g \in K, \quad x \in A\}$ and for $g \in G$ we write $gA := \{gx : x \in A\}$. A set $A \subset X$ is said to be *G-invariant*, if $G(A) = A$. Notice that if A is a compact set, $G(A)$ is also compact. Observe that on any Hausdorff topological space X one can define the *trivial action* of G by $gx = x$ for all $g \in G$ and $x \in X$.

For any $x \in X$, the closed subgroup $G_x = \{g \in G : gx = x\}$ of G is called the *isotropy group* of x and the invariant subspace $G(x) := \{gx : g \in G\}$ of X is called the *orbit* of x. It is easy to see that if H is the isotropy group of $x \in X$, then $G(x)$ is homeomorphic to G/H. Denote by X/G the set of all orbits in X and consider the canonical projection $\pi : X \to X/G$ given by $\pi(x) = G(x)$. The space X/G equipped with the quotient topology induced by π is called the *orbit space* of X under the action of G.

A G-action on X is called *effective* if

$$\forall\, g \in G \;\; \exists\, x \in X \;\; s.t. \;\; gx \neq x.$$

Also, we say that a G-action on X is *free* (resp. *semi-free*) if $G_x = \{e\}$ for all $x \in X$ (resp. if either $G_x = \{e\}$ or $G_x = G$ for all $x \in X$).

Recall that two closed subgroups H and K of G are said to be *conjugate* in G, if $H = gKg^{-1}$ for some $g \in G$. Clearly, the conjugacy relation is an equivalence relation; denote by (H) the equivalence class of H and call (H) the *conjugacy class* of H in G. Denote by $\mathcal{O}(G)$ the set of all conjugacy classes. The set $\mathcal{O}(G)$ is partially ordered by the relation $\leq$ defined as follows

$$(H) \leq (K) \;\; \overset{\text{Def}}{\Longleftrightarrow} \;\; \exists_{g \in G} \;\; gHg^{-1} \subset K. \tag{2.1}$$

Obviously, for a G-space X and $x \in X$ one has $G_{gx} = gG_xg^{-1}$. This gives rise to the notion of the *orbit type* of x defined as the conjugacy class (G_x).

Denote by $\mathcal{J}(X)$ the set of all orbit types occuring in X. Let us point out that according to the order relation (2.1), if an orbit type (G_x) is smaller than an orbit type (G_y), then the orbit $G(x)$ is "bigger" than the orbit $G(y)$. Free and semi-free G-actions provide the most simple examples of G-actions with a finite number of orbit types. Moreover, according to the famous result by Mann (cf. [180]), every action of a compact Lie group on a manifold with finitely generated homology groups has a finite number of orbit types.

The following result is known as Gleason Lemma (cf. [49]).

Theorem 2.2. *Let X be a metric G-space such that $\mathcal{J}(X) = \{(H)\}$. Then, the orbit map $\pi : X \to X/G$ is a projection in a locally trivial fiber bundle with fiber G/H.*

Suppose that X is a finite-dimensional G-space and G is a compact Lie group. Using the definition of covering dimension (cf. [81]), the Gleason lemma and the Morita theorem (see [196]), which states that $\dim (K \times [0,1]) = \dim K + 1$ for any metric space K, one can easily prove:

Proposition 2.3. *If X is a (metric) free G-space, then* $\dim (X/G) = \dim X - \dim G$.

For a G-space X and a closed subgroup H of G we adopt the following notations:

$$\begin{aligned} X_H &:= \{x \in X : G_x = H\}, \\ X^H &:= \{x \in X : G_x \supset H\}, \\ X_{(H)} &:= \{x \in X : (G_x) = (H)\}, \\ X^{(H)} &:= \{x \in X : (G_x) \geq (H)\}, \\ X^{[H]} &:= \bigcup_{H \subsetneq K} X^K. \end{aligned}$$

We call X^H the *H-fixed-point subset* of X.

For a closed subgroup H of G, we will use $N(H) := \{g : gHg^{-1} = H\}$ to denote the *normalizer* of H in G and $W(H) := N(H)/H$ to denote the *Weyl group* of H in G. Notice that $N(H)$ is a closed subgroup of G and X^H is $N(H)$-invariant. Consequently, X^H is $W(H)$-invariant, where the $W(H)$-action on X^H is given by $gH \cdot x := gx$ for $g \in N(H)$ and $x \in X^H$. It is also clear that $W(H)$ acts freely on X_H.

Example 2.4. (i) The group $O(2)$ consists of all matrices

$$r_\varphi := \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}, \quad \varphi \in [0, 2\pi).$$

plus the matrices

$$\overline{r}_\varphi := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot r_\varphi = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ -\sin\varphi & -\cos\varphi \end{bmatrix}.$$

The matrix r_φ corresponds to a rotation of the plane $\mathbb{R}^2$ by the angle φ, $\overline{r}_\varphi$ a reflection at the axis $\left\{c \cdot \begin{bmatrix} \cos\left(-\frac{\varphi}{2}\right) \\ \sin\left(-\frac{\varphi}{2}\right) \end{bmatrix} : c \in \mathbb{R}\right\}$. In $\mathbb{C}$, r_φ corresponds to the multiplication with $e^{i\varphi}$, $\kappa := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to the complex conjugation $z \mapsto \overline{z}$.

Let $n \in \mathbb{N}$, $n \geq 2$, $\gamma := r_{\frac{2\pi}{n}}$, and $\varphi_o \in [0, 2\pi)$. The subgroups

$$D_n^{\varphi_o} := \left\{\gamma^k, r_{\varphi_o}\kappa\gamma^k r_{\varphi_o}^{-1} : k = 0, 1, \ldots, n-1\right\}$$

are the conjugate copies of the dihedral group

$$D_n = \{\gamma^k, \kappa\gamma^k : k = 0, 1, \ldots, n-1\}$$

in $O(2)$.

(ii) Consider the dihedral group $G := D_6$, where γ is identified with the rotation $r_{\frac{\pi}{3}}$. Consider the space $V := \mathbb{C}$, where the G-action of the group G is defined by

$$\gamma^k z = \gamma^{2k} \cdot z, \quad k = 0, 1, \dots, 5,$$
$$\kappa z = \overline{z},$$

and $\gamma^{2k} \cdot z$ stands for the usual multiplication of complex numbers. Then, there are three orbit types in V, namely (D_2), $(\mathbb{Z}_2)$ and (G). Indeed, it is clear that a rotation $\gamma^k \in D_6$ satisfies $\gamma^k z = z$ for $z \neq 0$ if and only if $\gamma^{2k} = 1$, i.e. $\gamma^k = \pm 1$. On the other hand, $\kappa\gamma^k z = z$ if and only if $\gamma^{2k} \cdot z = \overline{z}$. If $z = |z|e^{i\theta}$, then we obtain the relation $\frac{2\pi k}{3} + \theta = -\theta + 2\pi l$ for some $l \in \mathbb{Z}$, i.e. $\theta \in \frac{\pi}{3} \cdot \mathbb{Z}$. In particular, we have for $z_j = \pm e^{i\pi j/3}$, $j = 0, 1, 2$, that $G_{z_0} = D_2 = \{1, \gamma^3, \kappa, \kappa\gamma^3\}$, $G_{z_1} = \{1, \gamma^3, \kappa\gamma^2, \kappa\gamma^5\} = \gamma^2 D_2 \gamma^{-2}$, and $G_{z_2} = \{1, \gamma^3, \kappa\gamma, \kappa\gamma^4\} = \gamma D_2 \gamma^{-1}$. The sets $V_{(D_2)}$, $V_{(\mathbb{Z}_2)}$ and $V_{(G)} = \{0\}$ are indicated on Figure 2.1

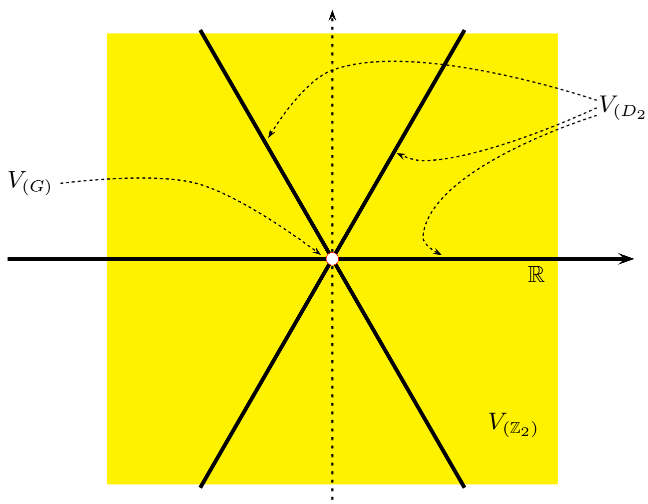

Fig. 2.1. Fixed-point subspaces for the considered D_6-Space

Example 2.5. Let V and G be as in Example 2.4. Consider the G-action on V defined by

$$\gamma^k z = \gamma^{3k} \cdot z, \quad k = 0, 1, \dots, 5,$$
$$\kappa z = \overline{z},$$

Then, there are four orbit types in V, namely (D_3^1), (D_3^2), $(\mathbb{Z}_3)$ and (G), where $D_3^1 = \{1, \gamma^2, \gamma^4, \kappa, \kappa\gamma^2, \kappa\gamma^4\}$ and $D_3^2 = \{1, \gamma^2, \gamma^4, \kappa\gamma, \kappa\gamma^3, \kappa\gamma^5\}$ are two isomorphic copies of D_3 which are not conjugate in G (but are conjugate in D_{12} – see Figure 2.2).

The sets $V_{(D_3^1)}$, $V_{(D_3^2)}$, $V_{(\mathbb{Z}_3)}$ and $V_{(G)} = \{0\}$ are indicated on Figure 2.3.

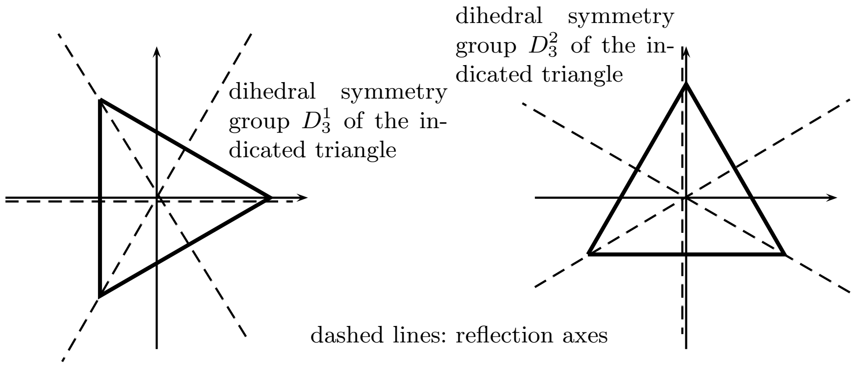

Fig. 2.2. Conjugate subgroups D_3^1 and D_3^2

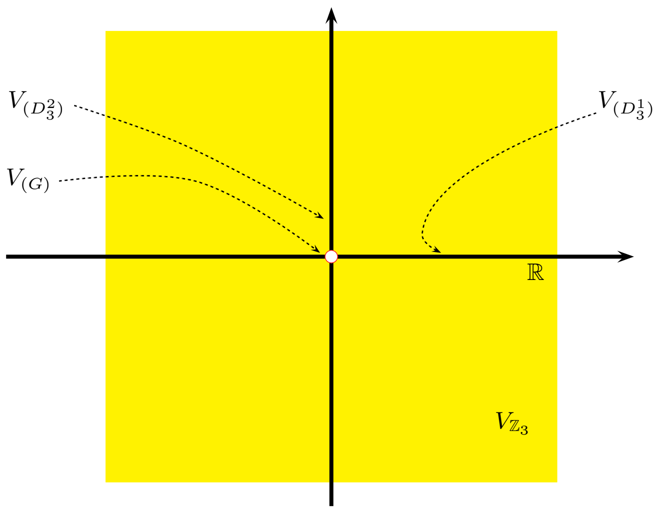

Fig. 2.3. Fixed-point subspaces for the D_6-action on $\mathbb{C}$

Let X and Y be two G-spaces. A continuous map $f : X \to Y$ is called G-*equivariant*, or simply *equivariant*, if

$$\forall\, g \in G \ \ \forall\, x \in X \quad f(gx) = gf(x).$$

If the G-action on Y is trivial, an equivariant map $f : X \to Y$ is called G-*invariant*, or simply *invariant*, i.e.

$$\forall\, g \in G \ \ \forall\, x \in X \quad f(gx) = f(x).$$

Let X and Y be two G-spaces. Since, for an equivariant map $f : X \to Y$, one has $G_{f(x)} \supset G_x$, it follows that $f(X^H) \subset Y^H$ for every subgroup $H \subset G$.

Consequently, the maps $f^H : X^H \to Y^H$ with $f^H := f_{|X^H}$, are well-defined and $W(H)$-equivariant for every subgroup $H \subset G$.

2.2 Elements of Representation Theory

2.2.1 Finite-dimensional G-Representations

We start with the following:

Definition 2.6. Let W be a finite-dimensional real (resp. complex) vector space. We say that W is a real (resp. complex) *representation* of G (in short, G-*representation*), if W is a G-space such that the *translation map* $T_g : W \to W$, defined by $T_g(v) := gv$ for $v \in W$, is an $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear) operator for every $g \in G$.

It is clear that for a G-representation W the map $T : G \to GL(W)$, given by $T(g) = T_g : W \to W$, is a continuous homomorphism. It is convenient to identify the representation W with the homomorphism $T : G \to GL(W)$. For example, a continuous homomorphism $T : G \to GL(n;\mathbb{R})$ (resp. $T : G \to GL(n;\mathbb{C})$) is called a *real* (resp. *complex*) *matrix G-representation*. For two representations W_1 and W_2, if there is an equivariant isomorphism $A : W_1 \to W_2$, we say that W_1 and W_2 are *equivalent* and write $W_1 \cong W_2$. Let W be a real (resp. complex) G-representation. An inner product (resp. Hermitian inner product) $\langle \cdot, \cdot \rangle : W \oplus W \to \mathbb{R}$ (resp. $\langle \cdot, \cdot \rangle : W \oplus W \to \mathbb{C}$) is called *$G$-invariant* if $\langle gu, gv \rangle = \langle u, v \rangle$ for all $g \in G$, u, $v \in W$. A G-representation together with a G-invariant inner product is called an *orthogonal* (resp. *unitary*) *G-representation*. It is well-known that every real (resp. complex) G-representation is equivalent to an orthogonal (resp. unitary) representation $T : G \to O(n)$ (resp. $T : G \to U(n)$).

Notice that any G-representation W is a manifold with trivial homology groups, therefore, there are only finitely many orbit types in W.

An invariant linear subspace $\widetilde{W} \subset W$ is called a *subrepresentation* of W and we say that W is an *irreducible* representation if it has no subrepresentation different from $\{0\}$ and W. Otherwise, W is called *reducible*.

The *complete reducibility theorem* states that every (finite-dimensional) G-representation W is a (not necessarily unique) direct sum of irreducible subrepresentations of W, i.e. there exist irreducible subrepresentations $\mathcal{W}^1$, ..., $\mathcal{W}^m$ of V such that

$$W = \mathcal{W}^1 \oplus \mathcal{W}^2 \oplus \cdots \oplus \mathcal{W}^m.$$

Let W_1 and W_2 be two G-representations. Denote by $L^G(W_1, W_2)$ the space of all linear G-equivariant maps $A : W_1 \to W_2$, and by $GL^G(W_1, W_2)$

its subspace of all (G-equivariant) isomorphisms. Also write $L^G(W) := L^G(W, W)$ and $GL^G(W) := GL^G(W, W)$.

Let $\mathcal{W}^1$ and $\mathcal{W}^2$ be two real or complex irreducible G-representations. Then, Schur's Lemma states that every equivariant linear map $A : \mathcal{W}^1 \to \mathcal{W}^2$ is either an isomorphism or zero. Assume $\mathcal{U}$ is a *complex* irreducible G-representation. Then, it follows from Schur Lemma that every equivariant linear map $A : \mathcal{U} \to \mathcal{U}$ satisfies $A = \lambda \mathrm{Id}$, for some $\lambda \in \mathbb{C}$, i.e. the G-representation $\mathcal{U}$ is *absolutely irreducible* (cf. [52]). Consequently, we have that $\dim_{\mathbb{C}} L^G(\mathcal{U}^1,\mathcal{U}^2) = 1$ or 0 (where $\mathcal{U}^1$ and $\mathcal{U}^2$ are two complex G-representations). Using this fact, it can be easily proved that every complex irreducible G-representation of an abelian compact Lie group G is one-dimensional.

In the case $\mathcal{V}$ is a *real* irreducible G-representation, the set $L^G(\mathcal{V})$ is a finite-dimensional associative division algebra over $\mathbb{R}$, so it is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and we call $\mathcal{V}$ to be of *real*, *complex* or *quaternionic type*, respectively. Observe also that the type of a real irreducible G-representation is closely related to its *complexification*.

Remark 2.7. (Convention of Notation) Let us explain our convention, which we use in connection to the complex and real G-representations. As long as it is possible, we use the letter V to denote a real G-representation, while the letter U is reserved for complex G-representations. In the case the type of a G-representation is not specified, we apply the letter W. Since, for a given compact Lie group G, there are only countably many irreducible G-representations (see Corollary 2.15), we also assume that a complete list, indexed by numbers $n = 0, 1, 2, 3, \dots$, of these irreducible representation is available, and, in the case of real G-representations, we denote them by $\mathcal{V}_0$, $\mathcal{V}_1$, $\mathcal{V}_2$, ... (where $\mathcal{V}_0$ always stands for the trivial irreducible G-representation), in the case of complex G-representations, by $\mathcal{U}_0$, $\mathcal{U}_1$, $\mathcal{U}_2$, ... (where $\mathcal{U}_0$ is the trivial complex irreducible G-representation), and in the case the type of an irreducible G-representation is not clearly specified as real or complex, we denote them by $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$, ... (where again $\mathcal{W}_0$ is the trivial irreducible G-representation). Unspecified irreducible G-representations are denoted as follows: in the case of real representations $\mathcal{V}$, $\mathcal{V}^1$, or $\mathcal{V}^k$, in the case of complex representations $\mathcal{U}$, $\mathcal{U}^1$, or $\mathcal{U}^k$, and if the type is unknown $\mathcal{W}$, $\mathcal{W}^1$, or $\mathcal{W}^k$. We summarize our convention in Table 2.1.

Example 2.8. (i) Let us consider the representation $V := \mathbb{C}$ of the group $G = D_6$, described in Example 2.4(ii). In order to determine the type of this representation we consider a real 2×2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in L^G(V)$. Then, since $\kappa M = M\kappa$, we get $b = c = 0$, and since $\gamma M = M\gamma$, we also get $a = d$, which implies that $M = a\mathrm{Id}$, for some real number a. Consequently V is of real type.

	Real	Complex	Unspecified
G-representation	V, $\mathfrak{V}$	U, $\mathfrak{U}$	W, $\mathfrak{W}$
Irreducible G-representation	$\mathcal{V}$	$\mathcal{U}$	$\mathcal{W}$
List of all irreducible G-representations	$\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \dots$	$\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \dots$	$\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \dots$
Various irreducible G-representations	$\mathcal{V}, \mathcal{V}^1, \mathcal{V}^2, \mathcal{V}^j$	$\mathcal{U}, \mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^j$	$\mathcal{W}, \mathcal{W}^1, \mathcal{W}^2, \mathcal{W}^j$

Table 2.1. Notational convention for real and complex G-representations

(ii) If we consider the space $V := \mathbb{C}$ as the representation of $G = \mathbb{Z}_6$, where the action of $\mathbb{Z}_6$ is the restriction to the action D_6 mentioned in (i). Then, a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in L^G(V)$, satisfies $\gamma M = M\gamma$, which implies that $a = d$ and $c = -b$. Consequently, M can be identified with the complex number $z = a + ic$ acting on V by complex multiplication. That means that the $\mathbb{Z}_6$-representation V is of complex type.

(iii) Let $\mathbb{H}$ denote the algebra of quaternions, which can be identified with $\mathbb{C}^2 = \{z_1 + jz_2 : z_1,\, z_2 \in \mathbb{C}\}$ equipped with the multiplication given by the complex multiplication and the relations $j^2 = -1$, $ji = -ij$. For a fixed $h = u_1 + ju_2 \in \mathbb{H}$, the linear map $T_h : \mathbb{H} \to \mathbb{H}$ given by the quaternionic product $T_h(k) = hk$, $k \in \mathbb{H}$, has the following matrix representation as a linear transformation from $\mathbb{C}^2$ to $\mathbb{C}^2$

$$T_h = \begin{bmatrix} u_1 & -\overline{u}_2 \\ u_2 & \overline{u}_1 \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^2. \tag{2.2}$$

It is clear that the group $G = \mathcal{Q}_8 = \{\pm 1, \pm i, \pm j, \pm ji\}$ of quaternionic units acts on $\mathcal{V} = \mathbb{H}$ and, clearly, $\mathcal{V}$ is an irreducible (real) representation of $\mathcal{Q}_8$. Notice that every (real) matrix $M \in L^G(\mathcal{V})$ can be written as a block matrix of the type

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad A,\, B,\, C,\, D \in L(\mathbb{R}^2).$$

Put $P := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\kappa := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. We have $T_i = \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix}$ and $T_j = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ (cf. (2.2)). It is clear that M commutes with the $\mathcal{Q}_8$-action if and only if $T_iM = MT_i$ and $T_jM = MT_j$. We can use the