

MATTER-WAVE SOLITONS WITH A MINIMAL NUMBER OF PARTICLES IN A TIME-MODULATED QUASI-PERIODIC POTENTIAL

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ABSTRACT. The two-dimensional (2D) matter-wave soliton families supported by an external potential are systematically studied, in a vicinity of the junction between stable and unstable branches of the families. In this case the norm of the solution attains a minimum, facilitating the creation of such excitation. We study the dynamics and stability boundaries for fundamental solitons in a 2D self-attracting Bose-Einstein condensate (BEC), trapped in an quasiperiodic optical lattice (OL), with the amplitude subject to periodic time modulation.

1. Introduction. Nowadays, solitons have an important role in physics and mathematics. One of the most important studies in solitons has been the study of their different dynamics, see [1], [2] and references therein, especially those that refer to 2D setup. In the last 15 years a challenging subject in the study of dynamic patterns in Bose-Einstein condensates (BECs) is the investigation of matter-wave solitons in multidimensional settings [3]. As known from the previous analyses [4, 5, 6], the dependence between the chemical potential and the norm (which is proportional to the number of atoms in BEC, or total power of the optical beam) for 2D solitons supported by lattice potentials, $\mu(N)$, features two branches, stable and unstable ones [with $d\mu/dN < 0$ and $d\mu/dN > 0$, respectively, according to the Vakhitov-Kolokolov (VK) criterion [7]. The branches merge at a threshold (minimal) value of N , below which the solitons decay due to the delocalization transition [8].

Our aim in this paper is to study the stability in different families of solitons, showing the ranges (e.g. frequency, amplitude, optical potential and time-variation OL) in which the solitons are stable.

It is shown that the existence of stability in solitons 2D is mainly due to the existence and management techniques of optical lattices (OL) [9] in time-periodic modulation, however, this stability could be impeded, causing a collapse.

In our study, we use the two dimensional Gross-Pitaevskii equation (GPE) for mean-field wave function $\Psi(x, y, t)$ with time-modulated OL [3].

$$i \frac{\partial \Psi}{\partial t} = -\frac{1}{2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Psi - g_0 |\Psi|^2 \Psi - V_0 \left[1 + \frac{\varepsilon}{2} \cos(\omega t) \right] V(x, y) \Psi, \quad (1)$$

where ω is the frequency, ε is the amplitude, V_0 is the optical potential, t is the time and (x, y) are coordinates in 2D of the OL. At $g = 1$ the nonlinearity is attractive. Here the quasiperiodic (QP) lattice potential of depth $2V_0$ is taken as [6, 10, 11]

$$V(x, y) = -V_0 \sum_{n=1}^M \cos(\mathbf{k}^{(n)} \mathbf{r}), \quad (2)$$

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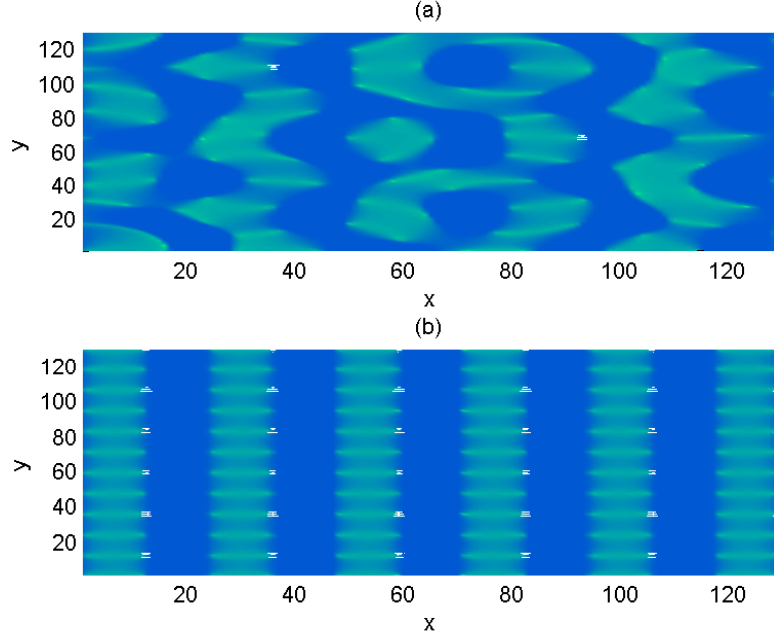


FIGURE 1. (Color on line.) The structure of the optical lattice (OL): (a) quasiperiodic OL, (b) periodic OL.

with the set of wave vectors $\mathbf{k}^{(n)} = k\{\cos(2\pi(n-1)/M), \sin(2\pi(n-1)/M)\}$ and $M = 5$ or $M \geq 7$. Here, following Ref. [6], we focus on the basic case of the Penrose-tiling potential, corresponding to $M = 5$. The 2D profile of such QP potential is displayed below in Fig. 1.

Setting $V_0 > 0$, the center of the 2D soliton will be placed at a local minimum of the potential (2), $x = y = 0$. The solitons will be characterized by the norm, defined as usual: $N = \iint |\Psi(x, y)|^2 dx dy$. The quantity N relates to the actual number of atoms in the condensate, \mathcal{N} , by means of standard rescaling [12]: $\mathcal{N} = (a_\perp/4\pi a_s) N$, where a_\perp (typically, $\sim \mu\text{m}$) and a_s ($\sim 0.1 \text{ nm}$) are the transverse trapping length of the condensate and scattering length of the atomic collisions, respectively. In optics, the same equation (1), with t replaced by the propagation distance, z , governs the transmission of electromagnetic waves with local amplitude $|\Psi|$ in the bulk waveguide with the transverse QP modulation of the refractive index. In the latter case, N is proportional to the beam's total power.

Numerical results: soliton families. Simulations of Eq. (1) are performed on the 2D numerical grid of size 128×128 , starting with the input in the form of an isotropic Gaussian,

$$\Psi(x, y) = A_0 \exp(-q(x^2 + y^2)). \quad (3)$$

Initial amplitude A_0 , along with the OL depth and wavenumber, V_0 and k , were varied, while the initial width was fixed by setting $q = 0.9$ [which is possible by means of rescaling of Eq. (1)].

The first objective is to construct families of localized ground-state modes, in the form of $\Psi(x, y, t) = \exp(-i\mu t)\varphi(x, y)$, with real wave function $\varphi(x, y)$ found by means of the accelerated imaginary-time method [15]. Following the convention commonly adopted in physics literature [4][5][10][13][14][16], we refer to these modes as “solitons”, even though they do not feature the unhindered motion characteristic to “genuine” solitons. The simulations of Eq. (1), rewritten in the imaginary time with a fixed value of μ , quickly converge to the

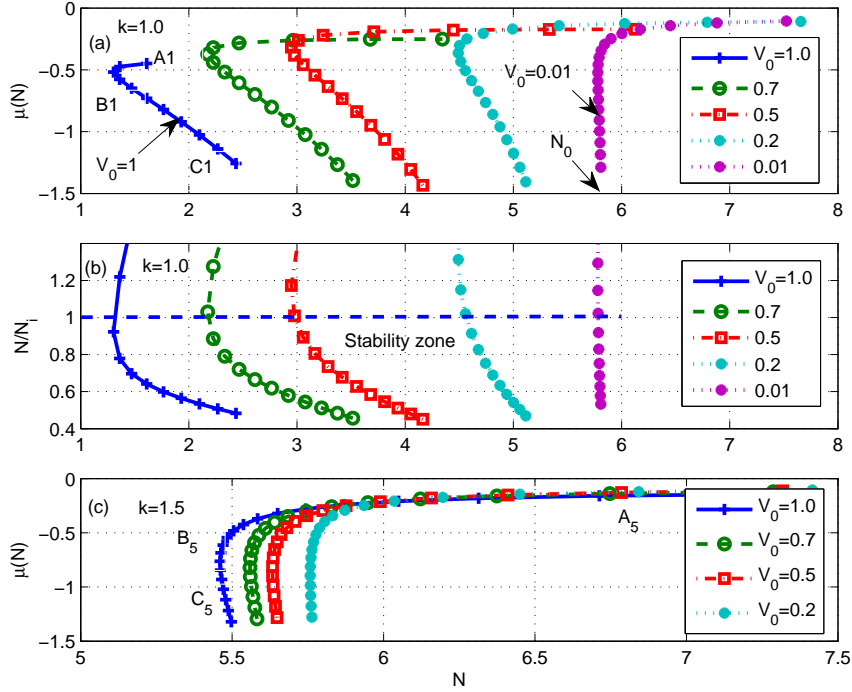


FIGURE 2. (Color online.) Chemical potential μ of the ground-state solitons versus its norm N , at two fixed wavenumbers of the Penrose-tiling potential, $k = 1$. See details in text.

ground state, with about 1000 iterations necessary to reduce the residual error to the level of 10^{-10} .

2. Numerics. In our simulations, a two-step procedure is implemented: (i) generation of the ground state soliton from Eq. (3), and then (ii) the study of the dynamics of such solution in the time-modulated OL. The numerical simulations are performed by means of the split-step method [17].

Fig. 2 shows the chemical potential μ of the ground-state solitons versus its norm N , at two fixed wavenumbers of the Penrose-tiling potential, $k = 1$ (a) and $k = 1.5$ (c) and various values of its depth, V_0 [see Eq. (2)]. Fig. 3 shows $\mu(N)$ for the fixed depth of the lattice potential, and different values of its wavenumber. Labels C_j and A_j ($j = 1, 2, 3, 4$) indicate VK-stable and unstable branches with $d\mu/dN < 0$ and $d\mu/dN > 0$, respectively [18]. Points B_j mark boundaries between the stable and unstable branches, at which $d\mu/dN = 0$ diverges.

To gain some insight into the behavior of such a system we have to verify if the found ground state contains less number of particles than initial one. Since the imaginary-time algorithm[15] of the ground state calculation does not conserve the number of particles, there rises a question on the relation of number of particles of initial condition N_i (see Eq.(3)) to numbers of particles N for resulting ground state.

For convenience one can say that at $N < N_i$ a soliton returns atoms to the bath, while at $N > N_i$ a soliton accepts atoms from the bath and thus it can be only quasistable one. For systems with a minimal number of particles (that is supposed in this paper), the latter

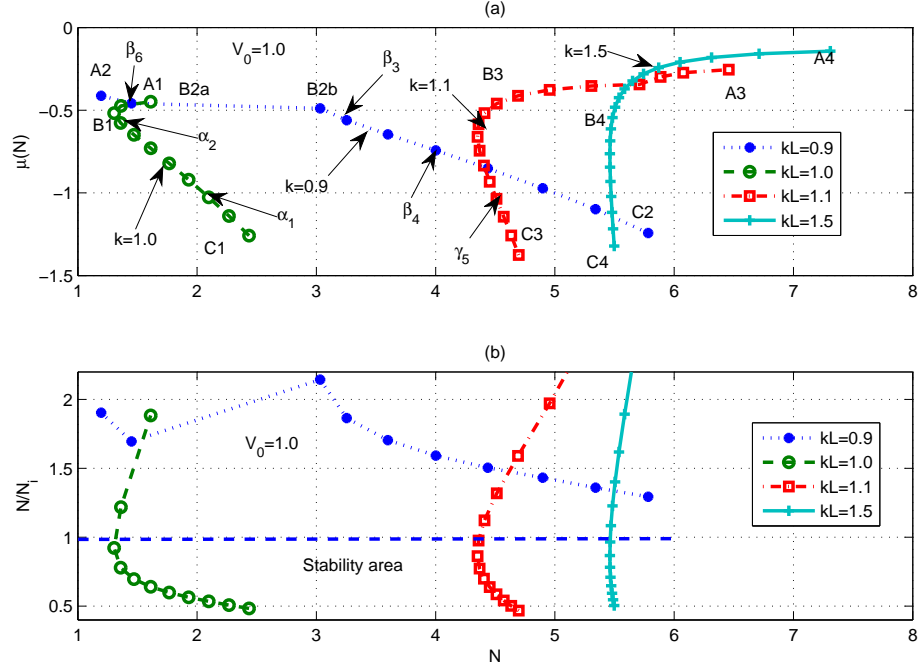


FIGURE 3. (Color on line.) Fixed depth of the OL and different values of the wavenumber.

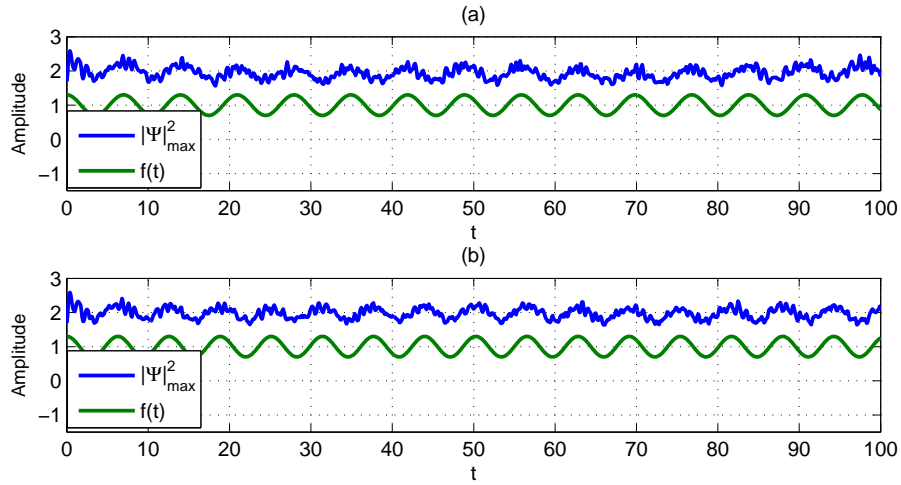


FIGURE 4. (Color on line.) The evolution of the amplitude $|\Psi|_{max}^2$ of the stable 2D soliton for: a) frequency $\omega = 0.9$, $\varepsilon = 0.6$ and the optical potential $V_0 = 2$; b) frequency $\omega = 1.0$, $\varepsilon = 0.6$ and the optical potential $V_0 = 2$. Curve $f(t) = 1 + \frac{\varepsilon}{2} \cos(\omega t)$ shows the modulation function in Eq.1.

it is difficult to realize. So that a soliton state with parameters $p(N) = N/N_i > 1$ is at least a quasi-stable one.

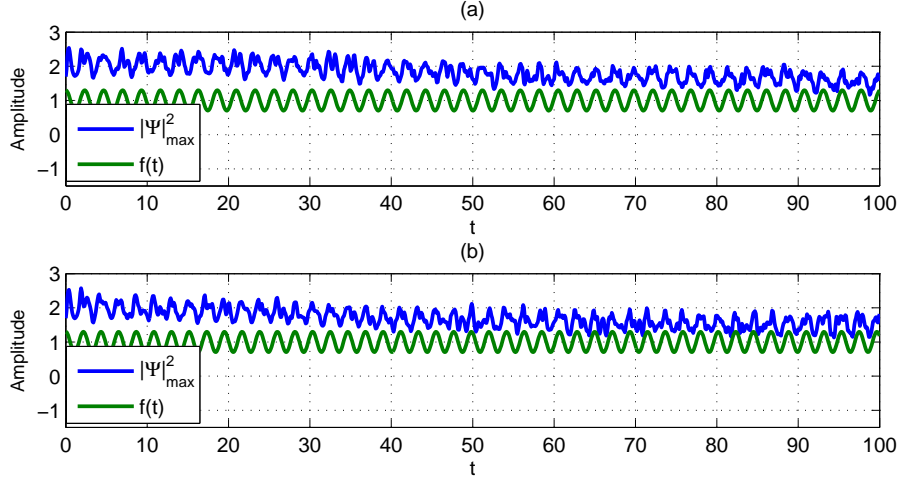


FIGURE 5. (Color on line.) The evolution of the amplitude $|\Psi|_{max}^2$ of the unstable (decaying) solitons for: (a) frequency $\omega < 2.5$, and $\varepsilon = 0.6$, $V_0 = 2$, (b) $\omega = 2.29$, and $\varepsilon = 0.6$, $V_0 = 2$. Curve $f(t)$ shows the same as in Fig. 4.

As said above, the main point in this work is the determination of the stability of the solitons close to the critical points B_j correspond the junctions of stable and unstable solution families, at which $d\mu/dN = \infty$ in the time varying OL. This was done by means of long-time simulations. Fig. 4 shows the evolution of the amplitude $|\Psi|_{max}^2$ of the stable 2D soliton and modulation function $f(t) = 1 + \frac{\varepsilon}{2} \cos(\omega t)$ (see Eq.(1)) for panel (a) frequency $\omega = 0.9$, $\varepsilon = 0.6$ and the optical potential $V_0 = 2.0$, and panel (b) frequency $\omega = 1.0$, $\varepsilon = 0.6$, and the optical potential $V_0 = 2$.

The example of unstable (decaying) soliton is shown in Fig.5. From Fig.5 we observe that (a) if the frequency $\omega < 2.5$, and $\varepsilon = 0.6$, $V_0 = 2$, this soliton is slowly destroyed (by decay), and (b) at $\omega = 2.29$, the decay becomes quicker.

The VK criterion does not guarantee the full stability of solitons, as it does not capture instabilities associated with complex eigenvalues. To test the full stability, we simulated perturbed evolution of the solitons over a sufficiently long interval, typically $t = 1000$ (which covers, roughly, 10 diffraction times of the corresponding localized states), adding small random perturbation to the initial conditions, with a relative amplitude ~ 0.01 .

Fig.6 displays: (a) Stable soliton where $\omega = 0.02$, $\varepsilon = 0.5$ and the optical potential $V_0 = 1.0$, (b) frequency $\omega = 1.0$, $\varepsilon = 0.6$ and the optical potential $V_0 = 1.0$; (c) Stable soliton where $\omega = 0.02$, $\varepsilon = 0.5$ and the optical potential $V_0 = 1.0$, (d) frequency $\omega = 0.2$, $\varepsilon = 0.5$ and the optical potential $V_0 = 1$.

A typical example of unstable (decaying) solitons is shown in Fig. 7. Here the shape $|\Psi|^2$ (panel (a)) at $t = 300$ and solitons amplitude $|\Psi|_{max}^2$ (panel (b)) exhibit a decaying soliton at $\omega = 0.29$, $\varepsilon = 0.5$, and the optical potential $V_0 = 1.0$ respectively. Panels (c) and (d) show the same as in panels (a), (b) but for $\omega = 0.35$, $\varepsilon = 0.5$, and $V_0 = 1$.

3. Conclusions. We have studied the dynamics of 2D matter-wave solitons near the junction points between the stable and unstable branches of curves $\mu(N)$ for the soliton families supported by the interplay of the self-attractive nonlinearity and Penrose-tiling OL potential. These points are interesting to physical applications, as they correspond to the minimal number of atoms which is necessary to build 2D matter-wave solitons, or the minimal total power necessary for the making of spatial optical solitons. Also the dynamics and stability boundaries for fundamental solitons in a two-dimensional (2D) self-attracting Bose-Einstein

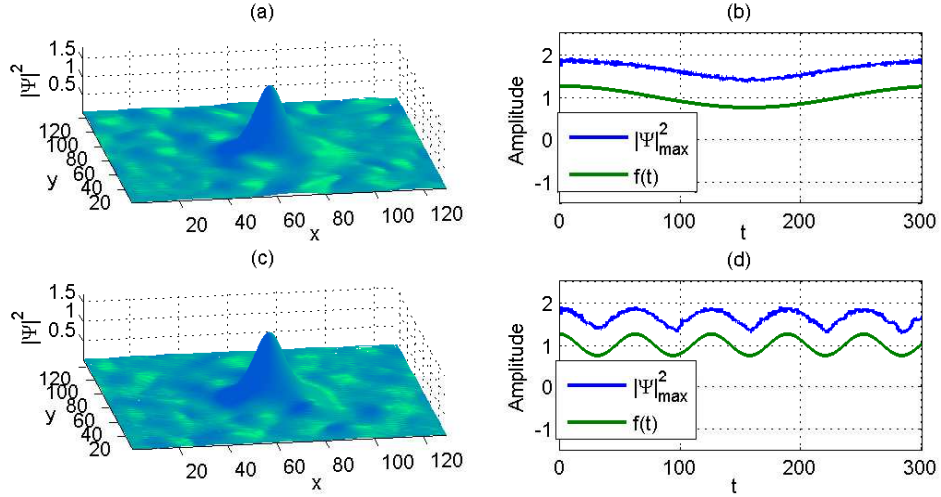


FIGURE 6. (Color on line.) The shape $|\Psi|^2$ at $t = 300$ and evolution of the amplitude $|\Psi|_{max}^2$ for stable soliton at: (a) $\omega = 0.02$, $\varepsilon = 0.5$, and the optical potential $V_0 = 1$; (b) frequency $\omega = 1.0$, $\varepsilon = 0.6$, and the optical potential $V_0 = 1$; (c) $\omega = 0.02$, $\varepsilon = 0.5$, and the optical potential $V_0 = 1$, (d) $\omega = 0.2$, $\varepsilon = 0.5$, and the optical potential $V_0 = 1$.

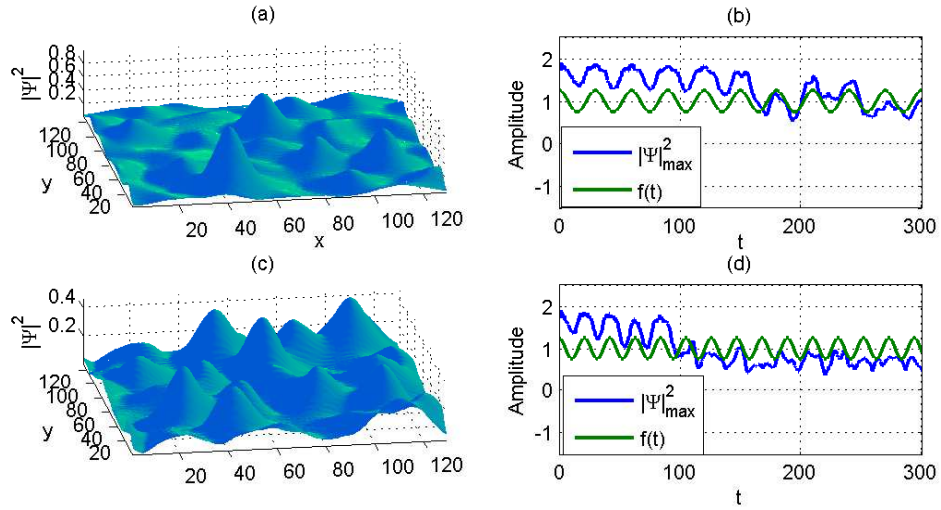


FIGURE 7. (Color on line.) Typical example of the unstable (decaying) solitons: (a) shape $|\Psi|^2$ at $t = 300$ and (b) amplitude $|\Psi|_{max}^2$ of the soliton for $\omega = 0.29$, $\varepsilon = 0.5$, and the optical potential $V_0 = 1$, (c) shape $|\Psi|^2$, and (d) amplitude $|\Psi|_{max}^2$ of the soliton for $\omega = 0.35$, $\varepsilon = 0.5$, and the optical potential $V_0 = 1$.

condensate (BEC), trapped in an quasiperiodic optical lattice (OL), with the amplitude subject to periodic time modulation is investigated. It was found that the shape and stability of such solitons crucially depend on the depth and period of the OL.

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