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STABILITY OF NEUTRAL DELAY DIFFERENTIAL EQUATIONS MODELING WAVE PROPAGATION IN CRACKED MEDIA

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Abstract. Propagation of elastic waves is studied in a 1D medium containing N cracks modeled by nonlinear jump conditions. The case $N = 1$ is fully understood. When $N > 1$, the evolution equations are written as a system of nonlinear neutral delay differential equations, leading to a well-posed Cauchy problem. In the case $N = 2$, some mathematical results about the existence, uniqueness and attractivity of periodic solutions have been obtained in 2012 by the authors, under the assumption of small sources. The difficulty of analysis follows from the fact that the spectrum of the linear operator is asymptotically closed to the imaginary axis. Here we propose a new result of stability in the homogeneous case, based on an energy method. One deduces the asymptotic stability of the zero steady-state. Extension to $N = 3$ cracks is also considered, leading to new results in particular configurations.

1. **Introduction.** Understanding the interactions between ultrasonic waves and contact defects have crucial applications in the field of mechanics, especially as far as the non-destructive testing of materials is concerned. When the cracks are much smaller than the wavelengths, they are usually replaced by interfaces with appropriate jump conditions. Here we consider realistic models describing cracks with finite compressibility, in a 1D geometry (section 2).

The case of $N = 1$ crack, which involves a nonlinear ordinary differential equation, has been completely analysed in [7]. When tackling with $N > 1$ cracks, the analysis becomes much more intricate. The successive reflections of waves between the cracks are described mathematically by a system of N nonlinear neutral-delay differential equations (NDDE) with forcing [5]. The main features of such systems are already contained in the following scalar NDDE:

$$x^{'}(t) + x^{'}(t-1) + f\left(x(t)\right) = s(t), \tag{1}$$

where f is a smooth increasing nonlinear function and s is the forcing. Proving the existence and uniqueness of periodic solutions to (1) is not trivial. The difficulties follow from the weak stability of periodic solutions: if we consider a null forcing $s = 0$ and a linear function $f(x) = x$, then many authors [1, 2, 4, 5, 9, 12] show that 0 is asymptotically stable but not exponentially stable.

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In [8], we have analyzed the existence and uniqueness of periodic solutions in the case of $N = 2$ cracks. Results were obtained in the case of small sources and particular ratios τ/T, where τ is the traveltime between the crack, and T is the period of the forcing. But the stability of periodic solutions was not addressed. In the present paper, we propose new results on that subject, in the case of null forcing. For this purpose, an energy method is followed, extending a technique developped for scalar NDDE [9]. Asymptotic stability is obtained whatever the spacings between the cracks (Section 3.3). Another new result is given for $N = 3$ cracks, but for particular configurations (Section 4).

The paper is organized as follows. The physical motivation of the study is given in Section 2. The case of $N = 2$ cracks is investigated in Section 3. After recalling known properties, a new result of asymptotic stability is given. An extension for $N = 3$ cracks is proved in Section 4.

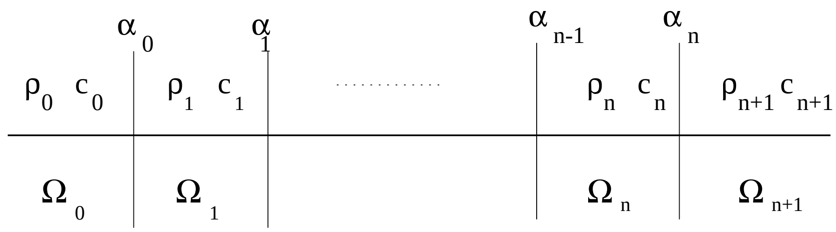

FIGURE 1. Cracks α_k separating elastic media Ω_k and Ω_{k+1}. The physical parameters are the density ρ_k and the sound speed c_k.

2. **Physical motivation.** Let us consider a 1D cracked elastic domain $\Omega = \cup\Omega_k$. The physical parameters of the subdomain Ω_k are the density ρ_k and the elastic speed of the compressional waves c_k. These piecewise constant parameters may be discontinuous across the crack at α_k. Wave propagation is modeled by the 1-D linear elastodynamics

$$\rho \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x}, \qquad \frac{\partial \sigma}{\partial t} = \rho\, c^2 \frac{\partial v}{\partial x}, \tag{2}$$

where $v = \frac{\partial u}{\partial t}$ is the elastic velocity, u is the elastic displacement, and σ is the elastic stress. The magnitude of the source is described by the amplitude v_0 of the elastic velocity.

Two independent jump conditions are required around each crack at $x = \alpha_k$ to obtain a well-posed problem. First, the stress is continuous across each crack:

$$[\sigma(\alpha_k,\, t)] = 0 \;\Rightarrow\; \sigma(\alpha_k^+,\, t) = \sigma(\alpha_k^-,\, t) = \sigma_k^*(t). \tag{3}$$

Secondly, experimental studies have yielded the following conclusions [10]:

- the elastic displacement can be discontinuous across the cracks, depending on the stress applied;
- at small stress levels, a linear model is relevant

$$\sigma_k^*(t) = K_k\, [u(\alpha_k,\, t)]\,, \tag{4}$$

  where $K_k > 0$ is the *interfacial stiffness*;
- the jump in elastic displacement satisfies the inequality

$$[u(\alpha_k,\, t)] \geq -d_k, \tag{5}$$

  where $d_k > 0$ is the *maximum allowable closure*. As the loading increases, the crack tends to become completely closed: $[u(\alpha_k,\, t)] \to -d_k^+$ when $\sigma_k^* \to -\infty$;
- concave stress-closure laws are measured.

The relation

$$\sigma_k^*(t) = K_k\, d_k\, \mathcal{F}_k\left([u(\alpha_k,\, t)]/d_k\right) \tag{6}$$

satisfies these requirements, where $\mathcal{F}_k$ is a smoothly increasing concave function

$$\begin{aligned} &\mathcal{F}_k :]-1,\, +\infty[\rightarrow]-\infty,\, \mathcal{F}_{k\,\max}[, \quad \lim_{X\to -1} \mathcal{F}_k(X) = -\infty, \quad 0 < \mathcal{F}_{k\,\max} \leq +\infty, \\ &\mathcal{F}_k(0) = 0, \quad \mathcal{F}_k^{'}(0) = 1, \quad \mathcal{F}_k^{''} < 0 < \mathcal{F}_k^{'}. \end{aligned} \tag{7}$$

Two models illustrate the nonlinear relation (6): the *model 1* proposed writes

$$\sigma_k^*(t) = \frac{K_k\ [u(\alpha_k,\, t)]}{1 + [u(\alpha_k,\, t)]\,/d_k} \Leftrightarrow \mathcal{F}_k(X) = \frac{X}{1+X}, \quad \mathcal{F}_{k\,\max} = 1, \tag{8}$$

and the *model 2* writes

$$\sigma_k^*(t) = K_k\, d_k\, \ln\left(1 + [u(\alpha_k,\, t)]/d_k\right) \Leftrightarrow \mathcal{F}_k(X) = \ln(1+X), \quad \mathcal{F}_{k\,\max} = +\infty. \tag{9}$$

These two models are sketched in figure 2. The straight line with a slope K tangential to the curves at the origin gives the linear jump conditions (4).

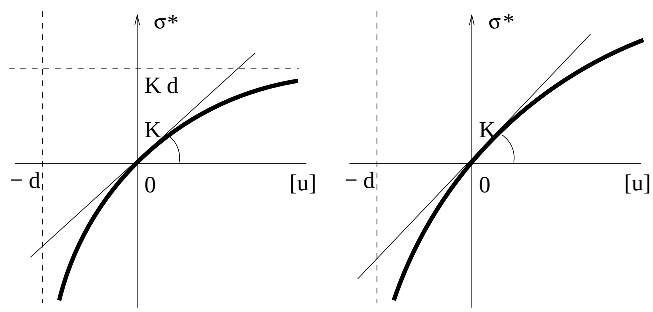

FIGURE 2. Jump conditions (6). Left: model 1 (8), right: model 2 (9).

3. Mathematical results for $N = 2$ cracks.

3.1. System of NDDE.

Wave propagation in the configuration of figure 1 involves a linear system of partial differential equations (2), and two jump conditions (3)-(6) at each crack. Based on the method of characteristics, it can be transformed into a system of two NDDE [8]. Setting

$$\begin{aligned} &y_1(t) = \frac{[u(\alpha_1,\, t)]}{d_1}, \quad y_2(t) = \frac{[u(\alpha_2,\, t)]}{d_1}, \quad r = \frac{d_2}{d_1} > 0, \quad \tau = \frac{\alpha_2 - \alpha_1}{c_1} \\ &f_1(y) = -\mathcal{F}_1(y),\, f_2(y) = -r\,\mathcal{F}_2\left(\frac{y}{r}\right),\, f_{k\,\min} = -\mathcal{F}_{k\,\max} < 0,\, y_{1\,\min} = -1,\, y_{2\,\min} = -r, \end{aligned} \tag{10}$$

one obtains the Cauchy problem

$$\left\{ \begin{aligned} &y_1^{'}(t) + y_2^{'}(t-\tau) = \beta_1\, f_1(y_1(t)) + \gamma_2\, f_2(y_2(t-\tau)) + s(t), \qquad t > 0, &(11a)\\ &y_2^{'}(t) + y_1^{'}(t-\tau) = \beta_2\, f_2(y_2(t)) + \gamma_1\, f_1(y_1(t-\tau)) + s(t-\tau), \quad t > 0, &(11b)\\ &y_k(t) = \phi_k(t) \in C^1\left([-\tau,\, 0],\,]y_{k\,\min},\, +\infty[\right), \quad -\tau \leq t \leq 0. &(11c) \end{aligned} \right.$$

The constant delay τ is the traveltime between the cracks. The coefficients β_k and γ_k depend on the elastic properties of the media Ω_k. The assumptions are

$$\begin{aligned} &\beta_k > 0, \quad 0 \leq |\gamma_k| < \beta_k, \quad y_{k\,\min} < 0,\\ &f_k \in C^2\left(]y_{k\,\min},\, +\infty[\rightarrow]f_{k\,\min},\, +\infty[\right), \qquad \lim_{y \to y_{k\,\min}} f_k(y) = +\infty, \quad -\infty \leq f_{k\,\min} < 0,\\ &f_k(0) = 0, \quad f_k^{'}(0) = -1, \quad q_k = \frac{f_k^{''}(0)}{2} > 0, \quad f_k^{'}(y) < 0 < f_k^{''}(y) \end{aligned} \tag{12}$$

3.2. **Known results.** One recalls the main theoretical results obtained in [8]. First, existence and uniqueness of global solutions of the NDDE (11) is proven.

Proposition 1. *There exists a unique solution* $\mathbf{y} = (y_1,\, y_2)^T$ *to (11)-(12), with* $y_k \in C^1\left([0,\, +\infty[,\,]y_{k\,\min},\, +\infty[\right)$*, except at instants* $t = k\,\tau$*,* $k \in \mathbb{N}$*, where the derivatives may be discontinuous.*

Second, let us assume that periodic solutions exist. The mean value of the solution during one period is denoted

$$\overline{y_k} = \frac{1}{T}\int_0^T y_k(t)\, dt, \qquad k = 1, 2. \tag{13}$$

Proposition 2. *The mean values of periodic solution* $\mathbf{y} = (y_1,\, y_2)^T$ *to (11)-(12) are strictly positive:*

$$\overline{y_k} > 0, \qquad k = 1, 2. \tag{14}$$

Proposition 2 states that a positive mean jump of the elastic displacement occurs across each crack: in other words, a perioding forcing generates a mean dilatation of the cracks. This phenomenon is purely induced by the nonlinear jump condition (6). For linear conditions, or equivalently for infinitesimal forcing, one has $\overline{y_k} = 0$. Figure 3 illustrates this property. The mean spatial values of the displacements in each of the subdomains are denoted by horizontal dotted lines.

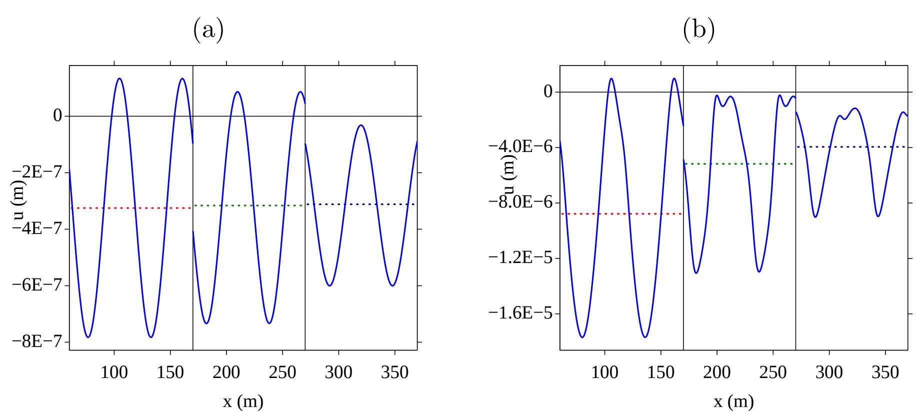

FIGURE 3. Snapshots of the elastic displacement u obtained with model 1 (8) for various amplitudes v_0 of the incident elastic velocity: 10^{-4} m/s (a), $2\,10^{-3}$ m/s (b). The vertical solid lines denote the locations of the cracks. The red, green and navy dotted horizontal lines denote the mean spatial value $\overline{u}$ in each subdomain.

Concerning the existence and uniqueness of periodic solutions, two cases must be distinguished, depending on the ratio

$$\theta = \frac{\tau}{T} = \frac{\alpha_2 - \alpha_1}{c_1\, T} = \frac{\alpha_2 - \alpha_1}{\lambda_1} > 0, \tag{15}$$

where λ_1 is the wavelength in medium Ω_1. Existence and uniqueness of periodic solution is proven in the two following cases:

- for source of arbitrarily large amplitude if $2\,\theta$ is an integer;
- for small sources otherwise, if θ satisfies also a Diophantine condition.

For rigorous statements of these results, the reader is refered to [8].

3.3. **Stability in the homogeneous case.** Here we propose a new result of stability in the homogeneous case. For the sake of simplicity, the delay is taken $\tau \equiv 1$. The following notations are introduced ($k = 1,\, 2$):

$$\begin{aligned}
&E(t) = E_1(t) + E_2(t), \qquad E_k(t) = \int_{t-1}^{t} \Big(y_k^{'}(\eta) + \beta_k\, f_k(y_k(\eta))\Big)^2\, d\eta \geq 0,\\
&F(t) = F_1(t) + F_2(t), \qquad F_k(t) = 2\,(\beta_k - \gamma_k) \int_0^{y_k(t)} f_k(z)\, dz \geq 0,\\
&G(a,\, b) = G_1(a,\, b) + G_2(a,\, b), \qquad G_k(a,\, b) = (\beta_k^2 - \gamma_k^2) \int_a^b f_k^2(y_k(\eta))\, d\eta \geq 0.
\end{aligned} \tag{16}$$

Theorem 3.1. *Assume that the initial data ϕ_1 and ϕ_2 belong to the Sobolev space $H^1((-1,0),\mathbb{R})$. Let $Y = (y_1,\, y_2)$ be the solution of (11)-(12) without source term: $s \equiv 0$. Then one has*

$$\sup_{t>0} E(t) + \sup_{t>0} F(t) + \sup_{t>0} G(0,t) < +\infty, \tag{17}$$

with E, F and G defined in (16). It follows the asymptotic stability of the origin:

$$\lim_{t\to+\infty} Y(t) = 0. \tag{18}$$

Proof. Taking the square of (11a) yields

$$\begin{aligned}
\Big(y_1^{'}(t) + \beta_1\, f_1(y_1(t))\Big)^2 &= \Big(y_2^{'}(t-1) + \gamma_2\, f_2(y_2(t-1))\Big)^2,\\
&= \Big(y_2^{'}(t-1) + \beta_2\, f_2(y_2(t-1)) - (\beta_2 - \gamma_2)\, f_2(y_2(t-1))\Big)^2,\\
&= \Big(y_2^{'}(t-1) + \beta_2\, f_2(y_2(t-1))\Big)^2 - (\beta_2^2 - \gamma_2^2) f_2^2(y_2(t-1))\\
&\quad -2\,(\beta_2 - \gamma_2)\, y_2^{'}(t-1)\, f_2(y_2(t-1)).
\end{aligned} \tag{19}$$

Integrating (19) on $[T-1,\, T]$ gives

$$E_1(T) = E_2(T-1) - [F_2(t-1)]_{T-1}^{T} - G_2(T-2,\, T-1). \tag{20}$$

Similarly, (11b) provides

$$E_2(T) = E_1(T-1) - [F_1(t-1)]_{T-1}^{T} - G_1(T-2,\, T-1). \tag{21}$$

From (16), (20) and (21), it follows

$$E(T) = E(T-1) - [F(t-1)]_{T-1}^{T} - G(T-2,\, T-1). \tag{22}$$

Summing (22) for $T = 1, \cdots, n$, one obtains

$$E(n) + F(n-1) + G(-1,\, n-1) = E(0) + F(-1) \equiv C_0 > 0. \tag{23}$$

Each term E, F and G in (23) is positive and bounded. From the uniform bound for $E(n)$ with respect to n, we get that $(y_1(n), y_2(n))$ is uniformly bounded. We also know that the sequences $(y_1(n+s), y_2(n+s))$ are uniformly bounded in $H^1((-1,0),\mathbb{R})$ and thus equicontinuous: $F(t)$ is then uniformly continuous. Lastly $G(0,+\infty) \le E(0)$. Since $(y_1(t), y_2(t))$ are uniformly continuous and bounded, hence $((f_1^2(y_1(t)), f_2^2(y_2(t)))$ converge towards zero. It follows that $(y_1(t), y_2(t))$ converges towards zero, which concludes the proof. □

4. **Preliminary results for $N = 3$ cracks.** In the case of $N = 3$ cracks, the method of characteristics yields a system of 3 NDDE with two constant delays $\tau_{12} = (\alpha_2 - \alpha_1)/c_1$ and $\tau_{23} = (\alpha_3 - \alpha_2)/c_2$. Up to now, we have not proven the asymptotic stability of the origin whatever the delays. On the contrary, we focus on the particular case $\tau_{12} = \tau_{23}$ without forcing. Under a suitable change of time variable, the homogeneous system can then be written [6]

$$\begin{cases} y_1'(t) + y_2'(t-1) + y_3'(t-2) + \beta_1 f_1(y_1(t)) = 0, & \text{(24a)}\\ y_2'(t) + y_1'(t-1) + y_3'(t-1) + \beta_2 f_2(y_2(t)) = 0, & \text{(24b)}\\ y_3'(t) + y_1'(t-2) + y_2'(t-1) + \beta_3 f_3(y_3(t)) = 0, & \text{(24c)}\\ y_k(t) = \phi_k(t), \qquad k = 1,2,3, \quad -2 \le t \le 0. & \text{(24d)} \end{cases}$$

The assumptions are the same than in (12).

Theorem 4.1. *Assume that ϕ_1, ϕ_2, ϕ_3 belong to the Sobolev space $H^1((-2,0),\mathbb{R})$. Let $Y = (y_1(t), y_2(t), y_3(t))$ be a solution of (24)-(12). Then $\lim_{t\to+\infty} Y(t) = (0,0,0)$.*

Proof. We rewrite the system as in the proof of Theorem 3.1 to isolate the delayed derivatives:

$$\begin{cases} y_1'(t) + \beta_1 f_1(y_1(t)) = -(y_2'(t-1) + y_3'(t-2)), & \text{(25a)}\\ y_2'(t) + \beta_2 f_2(y_2(t)) = -(y_1'(t-1) + y_3'(t-1)), & \text{(25b)}\\ y_3'(t) + \beta_3 f_3(y_3(t)) = -(y_1'(t-2) + y_2'(t-1)). & \text{(25c)} \end{cases}$$

Taking the square of equation (25a) yields

$$\begin{aligned} \left(y_1'(t) + \beta_1 f_1(y_1(t))\right)^2 &= \left(y_2'(t-1) + y_3'(t-2)\right)^2 \\ &= \left(y_2'(t-1)\right)^2 + \left(y_3'(t-2)\right)^2 + 2\, y_2'(t-1)\, y_3'(t-2). \end{aligned}$$

Similarly, (25c) gives

$$\left(y_3'(t) + \beta_3 f_3(y_3(t))\right)^2 = \left(y_2'(t-1)\right)^2 + \left(y_1'(t-2)\right)^2 + 2\, y_2'(t-1)\, y_1'(t-2).$$

Adding the two previous equalities and finally using (25b) gives

$$\begin{aligned} &\left(y_1'(t) + \beta_1 f_1(y_1(t))\right)^2 + \left(y_3'(t) + \beta_3 f_3(y_3(t))\right)^2 \\ &= \left(y_1'(t-2)\right)^2 + \left(y_3'(t-2)\right)^2 + 2\left(y_2'(t-1)\right)^2 + 2\, y_2'(t-1)(y_1'(t-2) + y_3'(t-2)) \end{aligned}$$

$$= \left(y_1'(t-2)\right)^2 + \left(y_3'(t-2)\right)^2 + 2\,y_2'(t-1)(y_2'(t-1) + y_1'(t-2) + y_3'(t-2))$$
$$= \left(y_1'(t-2)\right)^2 + \left(y_3'(t-2)\right)^2 + 2\,y_2'(t-1)(y_2'(t-1) - (y_2'(t-1) + g_2(y_2(t-1))))$$
$$= \left(y_1'(t-2)\right)^2 + \left(y_3'(t-2)\right)^2 - 2\,y_2'(t-1)\beta_2 f_2(y_2(t-1)),$$

which provides

$$\begin{aligned}&\left(y_1'(t) + \beta_1 f_1(y_1(t))\right)^2 + \left(y_3'(t) + \beta_3 f_3(y_3(t))\right)^2 + 2\,y_2'(t-1)\beta_2 f_2(y_2(t-1))\\ &= \left(y_1'(t-2)\right)^2 + \left(y_3'(t-2)\right)^2.\end{aligned} \tag{26}$$

The following notations are introduced:

$$\begin{aligned}E_k(t) &= \int_{t-2}^{t} y_k'^2(\eta)\,d\eta \geq 0,\\ F_k(t) &= 2\,\beta_k \int_0^{y_k(t)} f_k(z)\,dz \geq 0,\\ G_k(a,\,b) &= \beta_k^2 \int_a^b f_k^2(y_k(\eta))\,d\eta \geq 0.\end{aligned} \tag{27}$$

Integration of (26) on $[T-2, T]$ gives

$$\begin{aligned}&E_1(T) + E_3(T) + [F_1(y_1)]_{T-2}^{T} + [F_3(y_3)]_{T-2}^{T} + [F_2(y_2)]_{T-3}^{T-1}\\ &\qquad + G_1(T-2,T) + G_3(T-2,T) = E_1(T-2) + E_3(T-2).\end{aligned} \tag{28}$$

Summing (28) for $T = 2, 4, \cdots, n$ (n even) gives

$$\begin{aligned}&E_1(n) + E_3(n) + F_1(n) + F_3(n) + F_2(n-1) + G_1(0,n) + G_3(0,n)\\ &\quad = E_1(0) + E_3(0) + F_1(0) + F_3(1) + F_2(-1)\\ &\quad = C_0 < +\infty.\end{aligned} \tag{29}$$

We deduce

$$\sup_{t>0} E_1(t) + \sup_{T>0} E_3(t) + \int_0^{+\infty} \left(\beta_1 f_1(y_1(t))\right)^2 + \left(\beta_3 f_3(y_3(t))\right)^2 dt < +\infty,$$

which yields $\lim\limits_{t\to+\infty} (y_1(t), y_3(t)) = (0, 0)$. Lastly, the bound on F_2 ensures that $y_2 \in L^\infty((0,+\infty),\mathbb{R})$. From (25b) and the uniform $L^2((T-2,T),\mathbb{R})$ bounds of y_1' and y_3', we obtain $\sup\limits_{T>0} E_2(T) < +\infty$. Thus, the sequence $y_{2,n}(s) = y_2(n+s)$ is bounded in $H^1((0,2),\mathbb{R})$. Extracting a subsequence $y_{2,n} \to z$ for the $L^\infty((0,2),\mathbb{R})$ norm. Passing to the limit in (25a) and (25b), z has to satisfy:

$$z' = 0, \qquad z' + \beta_2 f_2(z) = 0.$$

Thus $z = 0$, $y_{2,n}$ has a unique limit and converges towards 0. It amounts to say $\lim\limits_{t\to+\infty} y_2(t) = 0$, which concludes the proof. □

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