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NIM-INDUCED DYNAMICAL SYSTEMS OVER $\mathbb{Z}_2$

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Abstract. Winning and losing positions in the well-known two-player game, Nim, are defined recursively as a two symbol sequence depending on a k-parameter set known as the subtraction set. In this paper, we write the recursion as a nonlinear dynamical system defined on the phase space $\mathbb{Z}_2^{s_k}$ with the binary sequence for Nim generated by the appropriate initial conditions. The transient dynamics and Garden of Eden points are completely determined for arbitrary-sized subtraction sets. A characterization of cycle lengths for two parameter subtraction sets is determined. Extensions of the two parameter case to an arbitrary-sized subtraction set are explored.

1. **Introduction.** Nim is a well-known combinatorial two-player game in which each player alternates turns removing any number of tokens from one of possibly many piles. Nim is the canonical example of an impartial game where from any position the same moves are legal for both players: the allowable number of tokens that can be removed from a pile are the same for both players [2]. Using arithmetic over $\mathbb{Z}_2$, Bouton [3] provided a complete analysis of a first player's optimal behavior or winning strategy in multi-pile Nim where players can remove any number of tokens from a pile. A single-pile variant of Nim from [2] requires players to remove only a restricted number of tokens from the pile on their turns. The first player who is not able to remove any more tokens on his/her turn loses the game. The restricted number of tokens is called a *subtraction set*.

Definition 1. The restricted number of tokens that a player can remove on his/her turn is the subtraction set, a subset of $\mathbb{N}$, denoted by $S = \{s_1, s_2, \ldots, s_k\}$ such that $s_1 < s_2 < \cdots < s_k$.

Although there has been research on single-pile Nim with an infinite subtraction set (*e.g.*, [14]), we restrict our attention to finite subtraction sets. For a fixed subtraction set, each position (the number of tokens in the single pile) can be classified as a winning or losing position by using a backward induction argument. In combinatorial game theory, these are referred to as $\mathcal{N}$ (next player to move can force a win) and $\mathcal{P}$ (previous player to move can force a win) positions, respectively. For a general subtraction set, determining the patterns in the sequence of $\mathcal{P}/\mathcal{N}$ positions is the leading unsolved problem in [9]. In order to apply mathematical structure to the Nim problem, we assign the symbol 0 for a $\mathcal{P}$ position and 1 for an $\mathcal{N}$ position. The next example illustrates the backward induction strategy that is used to construct the win/loss sequence.

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Example 1. Let n represent the position or number of tokens in the pile and x_n represent the win/loss state of the first player in position n. That is $x_n = 1$ implies that position n yields a first player win and $x_n = 0$ implies position n is a first player loss. Let the subtraction set be $S = \{1, 4, 7\}$. If there are no tokens in the pile, then Player 1 cannot move and hence Player 2 reduced the pile to zero tokens. Thus, $n = 0$ is a losing position and $x_0 = 0$. Note that $x_0 = 0$ regardless of what the subtraction set is.

If there is one token in the pile, then Player 1 can remove the token forcing Player 2 into the $n = 0$ position. Because we already determined that $x_0 = 0$, this results in a loss for Player 2 and hence, $x_1 = 1$.

In the case where there are two tokens, Player 1 can only remove one token by the restriction defined by the subtraction set. This places Player 2 in the $n = 1$ position, which was determined as $x_1 = 1$. Hence, $x_2 = 0$.

Similarly, Player 1 can still only remove one token in the $n = 3$ case, moving Player 2 to $n = 2$ which was determined as $x_2 = 0$. So, $x_3 = 1$.

For $n = 4$, Player 1 can remove all four tokens because 4 is an element in the subtraction set. Because $x_0 = 0$, $x_4 = 1$. We can continue in this manner to generate $x_5 = 1, x_6 = 1$ and $x_7 = 1$

To demonstrate the backward induction argument when the position is larger than the entries in the subtraction set, consider $n = 8$ and $n = 9$. For $n = 8$, subtracting $1, 4$, or 7 places Player 2 in positions $7, 4$, or 1, respectively. Because at least one of positions 7, 4, or 1 is a losing position (specifically 7), $x_8 = 1$. Similarly, $x_9 = 0$ because subtracting $1, 4$, and 7 from $n = 9$ places Player 2 in the winning positions $8, 5$, and 2, respectively. Applying this method forward, we find that the sequence $\{x_n\}$ is cyclic with period 8 (see Table 1).

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$\cdots$
x_n	0	1	0	1	1	0	1	1	0	1	0	1	1	0	1	1	$\cdots$

TABLE 1. The win/loss sequence for $S = \{1, 4, 7\}$.

Summarizing the strategy implemented in Example 1: Player 1 can win in position n if he/she can force Player 2 into a losing position and Player 1 loses in position n only if all possible subtractions move Player 2 to a winning position. The backward induction argument yields the recursion

$$x_n = \begin{cases} 1 & \text{if } x_{n-s_i} = 0 \text{ for at least one } i,\ 1 \leq i \leq k; \\ 0 & \text{if } x_{n-s_i} = 1 \text{ for all } i,\ 1 \leq i \leq k \end{cases} \tag{1}$$

for $n \geq s_k$, with initial condition for $0 \leq n < s_k$ given by

$$x_n = \begin{cases} 1 & \text{if } x_{n-s_i} = 0 \text{ for some } s_i < n; \\ 0 & \text{if } x_{n-s_i} = 1 \text{ for all } s_i < n \end{cases} \tag{2}$$

where $j = 1, \ldots, k - 1$.

Note that this initial condition describes the traditional Nim game defined in [2]. In [1], Althöfer and Bültermann work with a variation of traditional Nim using an initial condition consisting of all ones or where a player can remove all the tokens off the table whenever $n \leq s_k$. We will prove in Section 2 that the traditional Nim initial condition belongs to the forward orbit of the variation of Nim proposed by

Althöfer and Bültermann. The piecewise defined recursion (1) has appeared in [1, 2]. Because every function defined on a finite dimensional vector space over a finite field can be represented by a polynomial [7], (1) has a polynomial representation. The polynomial representation

$$x_n = \left[\left(\prod_{i=1}^{k} x_{n-s_i}\right) + 1\right] \mod 2. \tag{3}$$

is equivalent to the piecewise definition in (1) because if $x_{n-s_i} = 0$ for some i then the product in (3) is 0, making the sum 1, and if $x_{n-s_i} = 1$ for all i then the product is 1, making the sum 0. The initial condition can also be written in product notation: for $0 \leq n < s_k$,

$$x_n = \begin{cases} 0 & \text{for } 0 \leq n < s_1 \\ \left[\left(\prod_{i=1}^{j} x_{n-s_i}\right) + 1\right] \mod 2 & \text{for } s_j \leq n < s_{j+1} \end{cases} \tag{4}$$

where $j = 1, \ldots, k-1$.

From a dynamical systems perspective it is interesting to consider the recurrence relation, (3), as a dynamical system defined on the phase space $\mathbb{Z}_2^{s_k}$, recalling that the Nim sequence is generated by the particular initial condition (4). In fact, the recursion problem can be viewed as the system

$$\begin{aligned} \mathbf{y}(n+1) &= D_S(\mathbf{y}(n)) \\ \mathbf{y}(0) &= y_0 \in \mathbb{Z}_2^{s_k} \end{aligned} \tag{5}$$

where $\mathbf{y}(n) = (y_1(n), \ldots, y_{s_k}(n))$ and $D_S : \mathbb{Z}_2^{s_k} \rightarrow \mathbb{Z}_2^{s_k}$ is the k-parameter map defined by

$$D_S(\mathbf{x}) = \left(x_2, x_3, \ldots, x_{s_k}, \left[\prod_{j=1}^{k} x_{s_k - s_j + 1}\right] + 1\right) \tag{6}$$

for $\mathbf{x} = (x_1, x_2, \ldots, x_{s_k}) \in \mathbb{Z}_2^{s_k}$ and $S = \{s_1, s_2, \ldots, s_k\}$. *Because we are working solely on the vector space $\mathbb{Z}_2^{s_k}$ all computations are performed modulo 2 throughout this paper.*

The Nim initial condition (4) translates as

$$y_i(0) = \begin{cases} 0 & \text{for } 1 \leq i \leq s_1 \\ \prod_{l=1}^{j} y_{i-s_l}(0) + 1 \mod 2 & \text{for } s_j < i \leq s_{j+1} \end{cases}, \tag{7}$$

for $j = 1, \ldots, k-1$. Notice that D_S is an example of a shift-register system. Shift-register systems have been extensively studied in coding theory [8] and have even been examined from the dynamical systems perspective [6].

It has been stated in the literature that the sequence x_n generated by the Nim initial condition is periodic [1, 9] although no formal proof has been found by the authors. This fact follows immediately from the dynamical system representation. Because $\mathbb{Z}_2^{s_k}$ is a finite set consisting of 2^{s_k} elements, the sequence of images, $D_S^j(\mathbf{x})$ for $j = 1, 2, 3, \ldots$, is eventually periodic. That is, there will be two nonnegative integers, $l \neq m$ such that $D_S^l(\mathbf{x}) = D_S^m(\mathbf{x})$. The number of distinct vectors that appear in the cycle (that is, the period of this cycle) will be called the *cycle length.* We refer to the cycle determined by the Nim initial condition (4) as the Nim cycle. Observe that any cycle length must be bounded by the total number of vectors, 2^{s_k}, in turn providing a bound on the Nim cycle length.

Not all vectors in $\mathbb{Z}_2^{s_k}$ necessarily cycle immediately under iteration of D_S, as they do in Example 1. Vectors in the sequence of images, $D_S^j(\mathbf{x})$ for $j = 1, 2, 3, \ldots$, that do not belong to an eventual cycle are called *preperiodic vectors*. The number of iterations of a preperiodic vector under D_S before it enters a cycle is the *preperiod*. The set of all preperiodic vectors for a particular subtraction set is referred to as the *transient dynamics* of the map.

Linear dynamical systems defined on the vector space $\mathbb{Z}_2^n$ have appeared in a variety of settings ranging from number theory to cellular automata [4, 5, 10, 11, 13, 15]. Nonlinear dynamical systems over $\mathbb{Z}_2^n$ do not occur as often in the literature, although specific polynomial examples in terms of shift register sequences were analyzed in [6, 12].

The purpose of this paper is to characterize the dynamics of the nonlinear Nim-induced dynamical system in (5). Specifically, we are interested in answering the following questions:

1. What are the transient dynamics of the map D_S as a function of the subtraction set S?
2. What are the different possible cycle lengths of D_S as a function of the subtraction set S?
3. In characterizing the global dynamics of D_S, can we answer specific questions about the Nim win/loss sequence?

In Section 2, we examine the transient dynamics along with a complete description of the set of Garden of Eden points for subtraction sets of arbitrary size. These classifications lead to several combinatorial results involving preperiodic behavior. In Section 3, we characterize cycle lengths for 2-parameter subtraction sets. Examples are provided to show these results do not extend to subtraction sets with more elements. The examples lead to several specific questions to be explored as future work. We conclude by considering how this dynamical system approach can be applied to other problems in combinatorial game theory.

2. **The Transient Dynamics of D_S.** Transient dynamics of a map defined on the vector space $\mathbb{Z}_2^{s_k}$ can be thought of as originating from a set of "root vectors" whose forward orbits end at a cycle. Borrowing language from cellular automata theory, we call this set of root vectors, the *Garden of Eden points*, which is formally defined below.

Definition 2. A vector $\mathbf{x} = (x_1, \ldots, x_{s_k}) \in \mathbb{Z}_2^{s_k}$ is a Garden of Eden point of (6) if it has no preimage under the map D_S.

The next example determines the entire dynamics of D_S for the case $S = \{2, 4\}$, focusing on movement from the Garden of Eden points to the cycles.

Example 2. By iterating all 2^4 vectors in $\mathbb{Z}_2^4$, we find that $D_{\{2,4\}}$ yields two different cycles; one of length 6 and the other of length 3. There are four Garden of Eden points, $(0,0,0,0), (1,0,0,0), (0,0,1,0)$, and $(1,0,1,0)$, and iterating these four vectors yields the full dynamics of D_S (see Figure 1).

Motivated by Example 2, we characterize the set of Garden of Eden points for arbitrary subtraction sets and use the characterization to determine the total number of Garden of Eden points as a function of the subtraction set.

Theorem 1. *For $S = \{s_1, \ldots, s_k\}$, a vector $\mathbf{x} = (x_1, \ldots, x_{s_k}) \in \mathbb{Z}_2^{s_k}$ is a Garden of Eden point of (6) if and only if $x_{s_k} = x_{s_k - s_i} = 0$ for some $i < k$.*

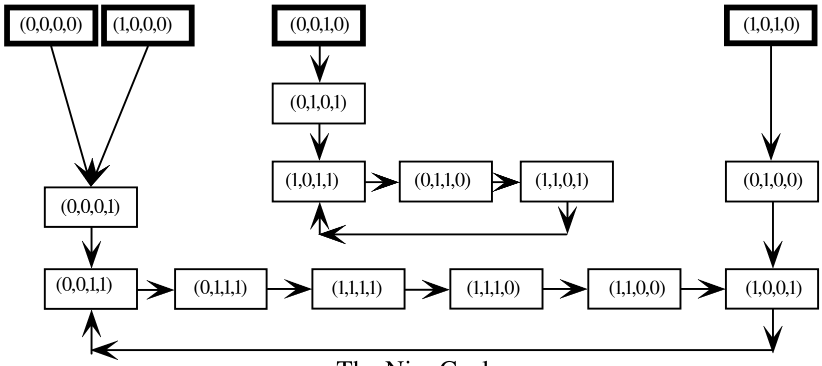

FIGURE 1. Complete dynamics for D_S where $S = \{2, 4\}$. Note that $(0, 0, 1, 1)$ represents the Nim initial condition.

Proof. The vector $\mathbf{x}$ has a preimage $\mathbf{y} = (x_0, x_1, ..., x_{s_k-1})$ if and only if $D_S(\mathbf{y}) = (x_1, x_2, \ldots x_{s_k-1}, x_{s_k})$, where $x_{s_k} = \left(\prod_{i=1}^{k} x_{s_k-s_i}\right) + 1$. If $x_{s_k} = 1$, then $\mathbf{x}$ always has a preimage $\mathbf{y}$ with $x_0 = 0$.

However, if $x_{s_k} = 0$, then $\mathbf{x}$ has a preimage, $\mathbf{y}$, if and only if $\left(\prod_{i=1}^{k} x_{s_k-s_i}\right) = 1$. Therefore, in this case, $\mathbf{x}$ has a preimage if and only if $x_{s_k-s_i} \neq 0$ for $i = 1$ to $k-1$.

Thus, $\mathbf{x}$ has no preimage if and only if $x_{s_k} = 0$ and $x_{s_k-s_i} = 0$ for some $i < k$. □

Corollary 1. *There are $2^{s_k-1} - 2^{s_k-k}$ Garden of Eden points under D_S in $\mathbb{Z}_2^{s_k}$.*

Proof. By Theorem 1, Garden of Eden points of (6) are vectors $\mathbf{x}$ in $\mathbb{Z}_2^{s_k}$ with $x_{s_k} = 0$ and one or more of $x_{s_k-s_i} = 0$ for $i < k$. There are 2^{s_k-1} vectors in $\mathbb{Z}_2^{s_k}$ with $x_{s_k} = 0$. Of these, if any of the $k-1$ terms $x_{s_k-s_i}$ for $i < k$ of $\mathbf{x}$ are 0, then the vector is a Garden of Eden point. Hence, only if $x_{s_k-s_i} = 1$ for all $i < k$ is the vector not a Garden of Eden point. The other $s_k - k$ terms can be either 0 or 1. It follows that 2^{s_k-k} of the 2^{s_k-1} vectors with $x_{s_k} = 0$ are not Garden of Eden points. Therefore, the other $2^{s_k-1} - 2^{s_k-k}$ vectors with $x_{s_k} = 0$ are Garden of Eden points. □

Corollary 1 implies that there are $2^{b-1} - 2^{b-2} = 2^{b-2}$ Garden of Eden points when $S = \{a, b\}$. As a result, for $S = \{2, 4\}$ there are $2^{4-2} = 4$ Garden of Eden points as illustrated in Example 2. In fact we can say more about the set of preperiodic vectors through knowledge of the Garden of Eden points. Define the *left shift map*, $T : \mathbb{Z}_2^{s_k} \to \mathbb{Z}_2^{s_k}$ by

$$T(\mathbf{x}) = (x_2, x_3, \ldots, x_{s_k}, x_1),$$

for $\mathbf{x} = (x_1, x_2, \ldots, x_{s_k}) \in \mathbb{Z}_2^{s_k}$. Repeated applications of the left shift map to the Garden of Eden points in Example 2 yield five distinct vectors

$$(0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (1, 0, 0, 0) \text{ and } (0, 1, 0, 1).$$

Notice that all of these vectors are preperiodic. This fact is actually an extension of Ferguson's Pairing Property [2] from combinatorial game theory.

Corollary 2. *Let $\mathbf{x}$ be a Garden of Eden point of D_S. The forward images of $\mathbf{x}$ under T, $T^m(\mathbf{x})$, $m = 1, 2, \ldots, s_k$, are preperiodic vectors.*

Proof. Let $\mathbf{x}$ be a Garden of Eden point with $x_{s_k} = x_{s_k - s_{i^*}} = 0$. Assume that $\mathbf{y} = T^m(\mathbf{x})$ belongs to a cycle for some $m = 1, \ldots, s_k$. We show that some forward iterate of $\mathbf{y}$ under D_S is a Garden of Eden point, thereby creating a contradiction.

Because T is the left shift, it follows that $T^m(\mathbf{x}) = \mathbf{y}$ has $y_{s_k - m} = 0$ and $y_{(s_k - s_{i^*} - m) \mod s_k} = 0$. Because $\mathbf{y}$ is in a cycle, there exists a $q > m+1$ such that $D_S^q(\mathbf{y}) = \mathbf{y}$. Let $\mathbf{z} = D_S^{q-m}(\mathbf{y})$. It follows that $z_{s_k} = z_{s_k - s_{i^*}} = 0$ so that $\mathbf{z}$ is a Garden of Eden point. But, $D_S^{q-m-1}(\mathbf{y})$ is mapped to $\mathbf{z}$ under D_S, contradicting that $\mathbf{z}$ is a Garden of Eden point. Our assumption led to a contradiction; hence, $T^m(\mathbf{x})$ is a preperiodic vector for every m. □

The converse of Corollary 2 is unfortunately not true. If we consider the subtraction set $S = \{1, 4\}$, the Garden of Eden points are

$$(0,0,0,0), (1,0,0,0), (0,1,0,0) \text{ and } (1,1,0,0)$$

as $x_4 = 0$ and $x_{4-1} = x_3 = 0$. However, $(1,1,1,1)$ is a preperiodic vector that is not a shift of any Garden of Eden point. The total number of distinct vectors of the form $T^m(\mathbf{x})$ where $\mathbf{x}$ is a Garden of Eden point provides a lower bound to the cardinality of the preperiodic vector set.

Notice that in Example 2, the zero vector eventually converged to the Nim cycle. In fact, it is always the case that the zero vector converges to the Nim cycle. We first illustrate why this is important by generating the Nim initial condition using (7) for $S = \{1, 4, 7\}$.

Example 3. As in Example 1, consider the subtraction set $S = \{1, 4, 7\}$. In Example 1, we applied the backward induction process to generate the Nim initial condition; the first seven entries of the sequence x_n in Table 1. The initial condition appears in Table 2.

n	0	1	2	3	4	5	6
x_n	0	1	0	1	1	0	1

Table 2. The win/loss sequence for $S = \{1, 4, 7\}$.

Applying the Nim initial condition recursion (7) yields

$$\begin{aligned}
y_1(0) &= 0 \\
y_2(0) &= y_{2-1}(0) + 1 = 0 + 1 = 1 \\
y_3(0) &= y_{3-1}(0) + 1 = 1 + 1 = 0 \\
y_4(0) &= y_{4-1}(0) + 1 = 0 + 1 = 1 \\
y_5(0) &= y_{5-1}(0) y_{5-4}(0) + 1 = 0 + 1 = 1 \\
y_6(0) &= y_{6-1}(0) y_{6-4}(0) + 1 = 1 + 1 = 0 \\
y_7(0) &= y_{7-1}(0) y_{7-4}(0) + 1 = 0 + 1 = 1
\end{aligned}$$

and hence we arrive at the initial vector $\mathbf{y}(0) = (0,1,0,1,1,0,1)$. Notice that the initial condition generated by the recursion agrees with the bottom row of Table 2.

Because using the recursion formula as in the previous example is computationally complex for large s_k and k requiring different products as s_k increases, it is desirable to arrive at the Nim initial conditions without explicitly calculating them. The next theorem proves that the zero initial condition converges to the Nim cycle and hence provides a simple method to determine the Nim cycle.

Theorem 2.
(a) *The zero vector,* $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{Z}_2^{s_k}$*, converges to the Nim initial condition, (7), in* $2s_k$ *iterations.*

(b) *The one vector,* $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{Z}_2^{s_k}$*, converges to the Nim initial condition, (7), in* s_k *iterations.*

Proof. Define $x_n = 0$ for $n = 0, \ldots, s_k - 1$. Then $\prod_{i=1}^{s_k} x_{n-s_i} = 0$ for $s_k \leq n < 2s_k$ and hence the recursion (3) yields

$$x_n = 1, \; s_k \leq n < 2s_k. \tag{8}$$

Applying the recursion formula again gives

$$x_n = 0, \; 2s_k \leq n < 2s_k + s_1.$$

Because $s_k \leq s_k + s_j + 1 \leq s_k + s_{j+1} + (s_k - s_i) < 2s_k$ for $i \geq j+1$, (8) implies $\prod_{i=j+1}^{k} x_{2s_k+n-s_i} = 1$. Therefore, for $s_j \leq n < s_{j+1}$

$$\begin{aligned} x_{2s_k+n} &= \left(\prod_{i=1}^{k} x_{2s_k+n-s_i} \right) + 1 \\ &= \left(\prod_{i=1}^{j} x_{2s_k+n-s_i} \right) \left(\prod_{i=j+1}^{k} x_{2s_k+n-s_i} \right) + 1 \\ &= \left(\prod_{i=1}^{j} x_{2s_k+n-s_i} \right) + 1 \end{aligned}$$

Thus we have

$$x_n = \begin{cases} 0 & \text{for } 2s_k \leq n < 2s_k + s_1 \\ \left(\prod_{i=1}^{j} x_{2s_k+n-s_i} \right) + 1 & \text{for } 2s_k + s_j \leq n < 2s_k + s_{j+1} \end{cases}$$

which is simply a $2s_k$ translation in the indices of the Nim initial condition defined in (4). □

Theorems 1 and 2 combine to provide a lower bound of $2s_k$ for the length of the longest preperiod for the transient behavior of D_S as the zero vector is a Garden of Eden point. In the combinatorial game theory literature, the term preperiod often refers to the number of elements in the win loss sequence generated by the Nim initial condition that do not belong to the Nim cycle. Althöfer and Bültermann [1] consider an alternate form of Nim in which a player can remove all the tokens to win the game if there remain less than or equal to s_k tokens in the pile. This means that if there are tokens left then a player always has a valid move and the game ends when there are no tokens left. In the game-theoretic terminology, all positions less than or equal to s_k are $\mathcal{N}$ positions. Their version of Nim is equivalent to applying D_S to the initial condition $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{Z}_2^{s_k}$. Because $D_S^{s_k}(\mathbf{0}) = \mathbf{1}$ as seen in the proof of Theorem 2, the sequence of win/loss positions determined by optimal play in the Althöfer-Bültermann variation of Nim converges to the traditional Nim initial condition in s_k iterations.

3. **Characterization of cycle lengths for D_S.** In the case where $S = \{a, b\}$, it is well known that the Nim cycle length is $a + b$ if b is not an odd multiple of a and $2a$ otherwise [1, 2]. We wish to know if there are any other distinct cycle lengths when considering all initial vectors in $\mathbb{Z}_2^b$ and whether the other cycle lengths have any connection to the Nim cycle length.

The full dynamics of the subtraction set $S = \{2, 4\}$ in Example 2 indicate that there may exist other cycles. However, this example leads one to ask if the Nim cycle will always generate the maximal period? From the subtraction set $S = \{1, 5\}$ it is easily seen that the Nim cycle may be a submaximal cycle, that is, a cycle with period strictly smaller than the maximal cycle length. The subtraction set $S = \{1, 5\}$ also yields two different cycles; the Nim cycle of period 2 and another cycle of period 6. However, it is also possible to develop a subtraction set that does not have any submaximal cycles; $S = \{2, 6\}$ has only one cycle of period 4.

Subtraction Set	**Number of Cycles**	**Cycle Lengths**	**Nim Period**
$S = \{2, 4\}$	2	$3, 6$	6
$S = \{1, 5\}$	2	$2, 6$	2
$S = \{2, 6\}$	1	4	4
$S = \{4, 6\}$	2	$5, 10$	10
$S = \{4, 8\}$	2	$3, 12$	12

Table 3. Cycle lengths and periods for some subtraction sets of size two.

Several examples are shown in Table 3. The examples suggest that all periods must divide $a + b$. We state this result in the next theorem.

Theorem 3. *Let q be a cycle length of D_S where $S = \{a, b\}$. Then q divides $m = a + b$.*

Proof. Let the elements of the cycle be denoted by $x_1, x_2, \ldots,$ and x_q. Applying the recursion (3), using properties of the field $\mathbb{Z}_2$, and substituting $x_0 = x_q$ implies that

$$\begin{aligned}
x_m &= x_{m-b}x_{m-a} + 1 \\
&= [x_{m-b-b}x_{m-b-a} + 1]\,[x_{m-a-b}x_{m-a-a} + 1] + 1 \\
&= [x_{a-b}x_0 + 1]\,[x_0x_{b-a} + 1] + 1 \\
&= x_{a-b}x_{b-a}x_0^2 + x_{a-b}x_0 + x_{b-a}x_0 + 1 + 1 \\
&= x_{a-b}x_{b-a}x_q^2 + x_{a-b}x_q + x_{b-a}x_q. \\
&= x_q(x_{a-b}x_{b-a} + x_{a-b} + x_{b-a}).
\end{aligned}$$

If $x_{a-b} \neq x_{b-a}$ or if $x_{a-b} = x_{b-a} = 1$, then $x_{a-b}x_{b-a} + x_{a-b} + x_{b-a} = 1$ and $x_m = x_q$. Likewise if $x_q = 0, x_m = x_q = 0$. The interesting case is when $x_q = 1$ and $x_{a-b} = x_{b-a} = 0$. This results in $x_m = 0$. However, these four values are inconsistent and result in a contradiction. To see this, we use the dynamical system (3),

$$x_b = x_{b-b}x_{b-a} + 1 = x_0x_{b-a} + 1 = x_qx_{b-a} + 1.$$

But, if $x_{b-a} = 0$ then $x_b = 1$. Similarly,

$$x_a = x_{a-b}x_{a-a} + 1 = x_{a-b}x_0 + 1 = x_{a-b}x_q + 1$$

implies that $x_a = 1$.

Because we assume that $x_q = 1$, the equation

$$x_q = x_0 = x_{0-b}x_{0-a} + 1 = x_{-b}x_{-a} + 1$$

indicates that $x_{-b} = 0$ or $x_{-a} = 0$. If $x_{-b} = 0$, then

$$\begin{aligned} x_a &= x_{a-b}x_0 + 1 = [x_{a-2b}x_{-b} + 1]\,[x_{-a}x_{-b} + 1] + 1 \\ &= [x_{a-2b} \cdot 0 + 1]\,[x_{-a} \cdot 0 + 1] + 1 = 1 + 1 = 0. \end{aligned}$$

But, this contradicts $x_a = 1$. Similarly, if $x_{-a} = 0$, the recursion yields an equation such that $x_b = 0$; this is also a contradiction. Hence, $x_{a-b} = x_{b-a} = x_m = 0$ and $x_q = 1$ cannot occur. This assures that x_q and x_m agree. The same argument will show $x_{m+r} = x_{q+r}$ for arbitrary r. Thus, $x_{q+r} = x_{m+r}$ for all r and that $q|m$. □

The natural extension of Theorem 3 to larger subtraction sets would be that all cycle lengths under D_S divide $\sum_{i=1}^{k} s_i$. This is false as indicated by the examples in Table 4. Clearly, all periods divide some nonnegative integer coefficient linear combination of elements from the subtraction set. For example, when $S = \{4, 5, 7\}$ one cycle length, 3 divides $1 \cdot 4 + 1 \cdot 5 + 0 \cdot 7$ and the other cycle length, 11, divides $1 \cdot 4 + 0 \cdot 5 + 1 \cdot 7$. And when $S = \{2, 5, 7\}$, the only cycle length is 21. The period 21 can be viewed in several ways as a linear combination of the elements in S; for example, 21 is $0 \cdot 2 + 0 \cdot 5 + 3 \cdot 7$ and $1 \cdot 2 + 1 \cdot 5 + 2 \cdot 7$. Unfortunately, there does not seem to be a limit on how large the coefficients may become as seen in the example from Observation 4 in [1]; $S = \{2, 8, 25, 33\}$ with Nim period 675 which can be written as $0 \cdot 2 + 0 \cdot 8 + 27 \cdot 25 + 0 \cdot 33$.

Subtraction Set	**Number of Cycles**	**Cycle Lengths**	**Nim Period**
$S = \{1, 4, 5\}$	2	$3, 6$	6
$S = \{2, 5, 7\}$	1	21	21
$S = \{4, 5, 7\}$	2	$3, 11$	12
$S = \{2, 4, 7\}$	1	3	3

TABLE 4. Cycle lengths and periods for some subtraction sets of size three.

These observations lead to the following questions for future work.

1. Characterize the subtraction sets where the period lengths divide a sum of the subtraction set's elements.
2. Determine the number of cycles of the difference equation as a function of the subtraction set.
3. Investigate the asymptotic relationship between k, s_k, and the number of cycles.

4. **Remarks about the Relationship to Other Combinatorial Games.** For all two-player combinatorial games with perfect information where one of the players must win the game, the same recursive relationship can be defined: a position is losing if it can only be changed to a winning position by a player's move and a position is winning if there exists a move that changes the position to losing. This recursion can be adapted to include tie positions to allow analysis of other combinatorial games like Tic-Tac-Toe. Although there are many combinatorial games

(many of them variations of Nim) for which this dynamical systems approach can be applied, a difference equation similar to that defined in (3) may not necessarily extend as positions of the game may not be completely ordered; for example, in Tic-Tac-Toe, board positions only form a partial ordering. There are many extensions of a partial ordering to a linear ordering, rendering preperiod and cycle length a function of the linear extension. Yet, one can examine the dynamics on the directed graph representing the partial ordering.

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