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COEXISTENCE STATES FOR A PREY-PREDATOR MODEL WITH CROSS-DIFFUSION

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Abstract. This paper discusses a prey-predator system with cross-diffusion. We can prove that the set of coexistence steady-states of this system contains an S or ⊃-shaped branch with respect to a bifurcation parameter in a large cross-diffusion case. We give also some criteria on the stability of these positive steady-states. Furthermore, we find the Hopf bifurcation point on the steady-state solution branch in a certain case.

1. **Introduction.** This paper is concerned with the following Lotka-Volterra prey-predator interaction model with cross-diffusion;

$$\text{(P)}\quad \begin{cases} u_t = \Delta u + u(a-u-cv) & \text{in } \ \Omega\times(0,\infty), \\ \sigma v_t = \Delta[(1+\beta u)v] + v(b+du-v) & \text{in } \ \Omega\times(0,\infty), \\ u=v=0 & \text{on } \ \partial\Omega\times(0,\infty), \\ u(\,\cdot\,,0)=u_0\ge 0, \quad v(\,\cdot\,,0)=v_0\ge 0 & \text{in } \ \Omega, \end{cases}$$

where Ω is a bounded domain in $\boldsymbol{R}^N$ $(N\ge 1)$ with smooth boundary $\partial\Omega$; σ, a, b, c, d are positive constants and $\beta\ge 0$ is the *cross-diffusion* coefficient. In (P), unknown functions u and v represent the population densities of prey and predator species, respectively, which are interacting and migrating in the same habitat Ω. This system is concerned with an ecological situation such that the population pressure due to the high density of prey induces the diffusion of the form $\beta\Delta(uv)$ in the second equation. See also the monograph of Okubo and Levin [16] for the ecological background. The time local solvability of (P) has been established by Amann [1], where a wide class of quasilinear parabolic systems is discussed. According to his result, (P) has a unique local solution (u,v) provided $(u_0,v_0)\in W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ for $p>N$. Recently, Le Dung [5] has found the global attractor for a class of triangular cross diffusion systems involving (P).

System (P) originates from the competition population model with cross-diffusion proposed by Shigesada, Kawasaki and Teramoto [19]. Since their pioneer work, many mathematicians have discussed such cross-diffusion systems. We refer to [3],[5],[6] and references therein for a recent progress on the global solvability of time-depending solutions. See e.g., [7],[12],[13],[14],[15],[18] about steady-state problems.

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Despite all their works concerning cross-diffusion systems, many problems still remain open. In particular, it is very difficult to know the complete structure of the steady-state solution set (e.g., the number, the stability or the shape of steady-states) to cross-diffusion systems such as (P).

We are interested in the global bifurcation structure of positive steady-state solutions to (P). Regarding a as a bifurcation parameter, we set

$$\mathcal{S} := \{(u, v, a) \,:\, (u, v) \text{ is a positive steady-state solution of (P)}\}.$$

Among other things, we will prove that when (β, b, c, d) belongs to a certain range, $\mathcal{S}$ contains a bounded S or $\supset$-shaped curve with respect to a. So (P) admits two or three positive steady-state solutions if a belongs to suitable ranges. This result implies a great contrast to the linear diffusion case ($\beta = 0$), where the uniqueness of positive steady-states is obtained by López-Gómez and Pardo [11] if the spatial dimension is one. Our method of analysis uses the idea developed by Du and Lou [4] and is based on bifurcation theory and the Lyapunov-Schmidt reduction procedure. If β is large and both of $b-\lambda_1$ and λ_1-d/β are small positive numbers, this reduction enables us to find an approximate limiting problem in a suitable finite dimensional space. Further, we can get the exact solution set of the limiting problem. Making use of the perturbation theory developed in [4], we will depict an S or $\supset$-shaped curve of $\mathcal{S}$ near the limiting solution set.

In Section 2. we will discuss such multiple existence of steady-state solutions. In Section 3, we will give some criteria on the stability of the positive steady-states. Furthermore, we will find a Hopf bifurcation point on the S or $\supset$-shaped solution set if σ is sufficiently large. Throughout the paper, the usual norms of the spaces $L^p(\Omega)$ for $p \in [1, \infty)$ and $C(\overline{\Omega})$ are defined by

$$\|u\|_p := \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p} \quad \text{and} \quad \|u\|_\infty := \max_{x \in \overline{\Omega}} |u(x)|.$$

In particular, we simply write $\|u\|$ instead of $\|u\|_2$. Furthermore, we will denote by Φ a unique positive solution of

$$-\Delta \Phi = \lambda_1 \Phi \ \text{ in } \ \Omega, \quad \Phi = 0 \ \text{ on } \ \partial\Omega, \quad \|\Phi\| = 1,$$

where λ_1 is the least eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary condition on $\partial\Omega$.

2. **Bifurcation branch of positive steady-states.**

2.1. **Main Result.** It is well known that the following elliptic boundary value problem

$$\Delta u + u(a - u) = 0 \ \text{ in } \ \Omega, \qquad u = 0 \ \text{ on } \ \partial\Omega$$

has a unique positive solution θ_a if $a > \lambda_1$; moreover, $a \in [\lambda_1, \infty) \to \theta_a \in C(\overline{\Omega})$ is continuous and strictly increasing function. It is easily verified that (P) has two semitrivial steady-state solutions

$$(u, v) = (\theta_a, 0) \ \text{ for } \ a > \lambda_1 \quad \text{ and } \quad (u, v) = (0, \theta_b) \ \text{ for } \ b > \lambda_1$$

in addition to the trivial solution $(u, v) = (0, 0)$.

Theorem 2.1. *Suppose that $\beta b > \beta\lambda_1 > d$. For any $c > 0$, there exist a large number M and an open set*

$$O = O(c) \subset \{(\beta, b, d) \,:\, \beta \geq M, \, 0 < \lambda_1 - d/\beta, b - \lambda_1 \leq M^{-1}\}$$

such that if $(\beta, b, d) \in O$*, then* $\mathcal{S}$ *contains a bounded smooth curve*

$$\Gamma = \{(u(r), v(r), a(r)) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times (\lambda_1, \infty)\,, r \in (0, C)\},$$

which possesses the following properties,

(i) $(u(0), v(0)) = (0, \theta_b), \quad a(0) > \lambda_1, \quad a'(0) > 0$;

(ii) $(u(C), v(C)) = (\theta_{a(C)}, 0), \quad a(C) > \lambda_1$;

(iii) $a(r)$ *attains a strict local maximum in* $(0, C)$*. Additionally, there exists an open set* $O' \subset O$ *such that, if* $(\beta, b, d) \in O'$*, then* $a(r)$ *attains a strict local minimum in* $(0, C)$*.*

Our result asserts that $\mathcal{S}$ contains a bounded S or $\supset$-shaped branch, which connects the above two semitrivial solutions, in a certain case. We can also find an *unbounded* S-shaped branch of $\mathcal{S}$, under another coefficient assumption [10, Theorem 1.2].

2.2. **Outline of the proof of Theorem 2.1.** In (P), we employ the following change of variables;

$$a = \lambda_1 + \varepsilon a_1,\ b = \lambda_1 + \varepsilon b_1,\ d/\beta = \lambda_1 - \varepsilon\tau,\ \beta = \gamma/\varepsilon,\ u = \varepsilon w,\ (1+\beta u)v = \varepsilon z. \quad (2.1)$$

Here a_1, b_1, τ are positive constants. Furthermore, ε is a small positive constant, thus γ is also a positive constant. In what follows, we will mainly discuss the case when β is large and both of $b - \lambda_1$ and $\lambda_1 - d/\beta$ are small positives. We note that a_1 plays a role of a bifurcation parameter. By (2.1), a pair of new unknown functions (w, z) satisfies

$$\text{(PP)}\quad \begin{cases} w_t = \Delta w + \lambda_1 w + \varepsilon f(w, z, a_1) & \text{in } \Omega \times (0, \infty), \\ \sigma\left[-\dfrac{\gamma z}{(1+\gamma w)^2} w_t + \dfrac{z_t}{1+\gamma w}\right] = \Delta z + \lambda_1 z + \varepsilon g(w, z) & \text{in } \Omega \times (0, \infty), \\ w = z = 0 & \text{on } \partial\Omega \times (0, \infty), \\ w(\,\cdot\,, 0) = u_0/\varepsilon, \quad z(\,\cdot\,, 0) = (1 + \beta u_0)v_0/\varepsilon & \text{in } \Omega, \end{cases}$$

where

$$f(w, z, a_1) := w\left(a_1 - w - \frac{cz}{1+\gamma w}\right), \quad g(w, z) := \frac{z}{1+\gamma w}\left(b_1 - \tau\gamma w - \frac{z}{1+\gamma w}\right).$$

The steady-state problem associated with (PP) is reduced to the following semilinear elliptic equations;

$$\begin{cases} \Delta w + \lambda_1 w + \varepsilon f(w, z, a_1) = 0 & \text{in } \Omega, \\ \Delta z + \lambda_1 z + \varepsilon g(w, z) = 0 & \text{in } \Omega, \\ w = z = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.2)$$

By virtue of (2.1), it is easy to see that (2.2) has two semitrivial solutions

$$(w, z) = (\varepsilon^{-1}\theta_{\lambda_1 + \varepsilon a_1}, 0), \quad (w, z) = (0, \varepsilon^{-1}\theta_{\lambda_1 + \varepsilon b_1})$$

in addition to the trivial solution. For the Lyapunov-Schmidt reduction, we will give a similar framework to that of Du and Lou [4]. For $p > N$, we prepare two Banach spaces

$$X := [\,W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\,] \times [\,W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\,], \quad Y := L^p(\Omega) \times L^p(\Omega).$$

We note that $X \subset C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ by the Sobolev embedding theorem. Define mappings $H : X \to Y$ and $B : X \times \boldsymbol{R} \to Y$ by

$$H(w, z) := (\Delta w + \lambda_1 w, \Delta z + \lambda_1 z), \quad B(w, z, a_1) := (f(w, z, a_1), g(w, z)). \quad (2.3)$$

Then (2.2) is equivalent to the following equation

$$H(w,z)+\varepsilon B(w,z,a_1)=0. \tag{2.4}$$

Let X_1 and Y_1 be the L^2-orthogonal complements of span $\{(\Phi,0),(0,\Phi)\}$ in X and Y, respectively. Let $P: X\to X_1$ and $Q: Y\to Y_1$ represent L^2-orthogonal projections. Thus a pair of unknown functions $(w,z)\in X$ is decomposed as

$$(w,z)=(r,s)\,\Phi+\boldsymbol{u},\quad \boldsymbol{u}=P(w,z).$$

Since $H((r,s)\Phi)=0$ and $(I-Q)H(X_1)=0$, (2.4) is consequently reduced to

$$QH(\boldsymbol{u})+\varepsilon QB((r,s)\,\Phi+\boldsymbol{u},a_1)=0 \tag{2.5}$$

and

$$(I-Q)B((r,s)\,\Phi+\boldsymbol{u},a_1)=0.$$

The Lyapunov-Schmidt reduction procedure leads us to the next lemma:

Lemma 2.1. *For any $C>0$, there exist a neighborhood N_0 of the set*

$$\{(w,z,a_1,\varepsilon)=(r\Phi,s\Phi,a_1,0)\in X\times\boldsymbol{R}^2:|r|,|s|,|a_1|\le C\}$$

and a positive constant ε_0 such that all solutions of (2.5) *in N_0 are given by*

$$\{((r,s)\,\Phi+\varepsilon\boldsymbol{U}(r,s,a_1,\varepsilon),a_1,\varepsilon):|r|,|s|,|a_1|\le C+\varepsilon_0,|\varepsilon|\le\varepsilon_0\}$$

with a smooth X_1-valued function $\boldsymbol{U}$. Then

$$(w,z,a_1,\varepsilon)=((r,s)\Phi+\varepsilon\boldsymbol{U}(r,s,a_1,\varepsilon),a_1,\varepsilon)$$

becomes a solution of (2.4)*, or equivalently* (2.2)*, in N_0 if and only if*

$$F^{\varepsilon}(r,s,a_1)\Phi:=(I-Q)B((r,s)\Phi+\varepsilon\boldsymbol{U}(r,s,a_1,\varepsilon),a_1)=0.$$

See [10] for the proof of Lemma 2.1. Since $(I-Q)(u,v)=(\int_\Omega u\Phi dx,\int_\Omega v\Phi dx)\Phi$, it follows from (2.3) that

$$\begin{aligned}F^0(r,s,a_1)&=\left(\int_\Omega f(r\Phi,s\Phi,a_1)\Phi,\int_\Omega g(r\Phi,s\Phi)\Phi\right)\\&={}^t\!\begin{pmatrix} r\left(a_1-r\|\Phi\|_3^3-cs\displaystyle\int_\Omega\frac{\Phi^3}{1+\gamma r\Phi}\right)\\ s\left\{b_1-(b_1+\tau)\gamma r\displaystyle\int_\Omega\frac{\Phi^3}{1+\gamma r\Phi}-s\int_\Omega\frac{\Phi^3}{(1+\gamma r\Phi)^2}\right\}\end{pmatrix}.\end{aligned} \tag{2.6}$$

Thus $\operatorname{Ker}F^0$ is a union of the following four sets;

$$\mathcal{L}_0=\{(0,0,a_1)\,:\,a_1\in\boldsymbol{R}\},\quad \mathcal{L}_1=\{(a_1/\|\phi_1\|_3^3,0,a_1)\,:\,a_1\in\boldsymbol{R}\},$$
$$\mathcal{L}_2=\{(0,b_1/\|\phi_1\|_3^3,a_1)\,:\,a_1\in\boldsymbol{R}\},\quad \mathcal{L}_p=\{(r,\varphi(\gamma r),\psi(r))\,:\,r\in\boldsymbol{R}\},$$

where

$$\begin{cases}\varphi(r)=\left[b_1-(b_1+\tau)r\displaystyle\int_\Omega\frac{\Phi^3}{1+r\,\Phi}\right]\left(\int_\Omega\frac{\Phi^3}{(1+r\,\Phi)^2}\right)^{-1},\\ \psi(r)=r\|\Phi\|_3^3+c\varphi(\gamma r)\displaystyle\int_\Omega\frac{\Phi^3}{1+\gamma r\Phi}.\end{cases} \tag{2.7}$$

We note that $\mathcal{L}_p\cap\overline{\boldsymbol{R}_+}^3$ is identical with the limiting set of positive solutions of (2.2) as $\varepsilon\to 0$. Indeed the following proposition holds true:

Proposition 2.1. *For a sufficiently large $A_1 > 0$, there exist $\varepsilon_0 > 0$ and a family of smooth curves*

$$\{(r(\xi,\varepsilon), s(\xi,\varepsilon), a_1(\xi,\varepsilon)) \in \boldsymbol{R}^3_+ : (\xi,\varepsilon) \in (0, C_\varepsilon) \times (0,\varepsilon_0)\}$$

such that for each fixed $\varepsilon \in (0,\varepsilon_0\,]$, all positive solutions of (2.2) *with $a_1 \in (0, A_1\,]$ can be parameterized as*

$$\begin{aligned}\Gamma^\varepsilon = \{(w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon)) = ((r,s)\Phi + \varepsilon \boldsymbol{U}(r,s,a_1,\varepsilon), a_1) \,:\, \\ (r,s,a_1) = (r(\xi,\varepsilon), s(\xi,\varepsilon), a_1(\xi,\varepsilon)) \ \ \text{for} \ \ \xi \in (0, C_\varepsilon)\}\end{aligned}$$

and $(r(\xi,0), s(\xi,0), a_1(\xi,0)) = (\xi, \varphi(\gamma\xi), \psi(\xi))$, $r(0,\varepsilon) = 0$. Here $C_\varepsilon > 0$ depends continuously on $\varepsilon \in [\,0,\varepsilon_0]$. Furthermore, $w(C_\varepsilon,\varepsilon) > 0$ in Ω and $z(C_\varepsilon,\varepsilon) \equiv 0$.

The above proposition implies that if $\varepsilon > 0$ is sufficiently small, then Γ^ε forms a positive solution branch near the curve $\{(r\Phi, \varphi(\gamma r)\Phi, \psi(r)) : 0 < r < C\}$. So it is important to study the profile of $\mathcal{L}_p$. By virtue of (2.7), $(0, \varphi(0), \psi(0)) = (0, b_1/\|\Phi\|_3^3, cb_1) \in \mathcal{L}_2$. It is easy to find a positive constant $r_0 = r_0(\tau/b_1)$ such that $\varphi(r) > 0$ for $r \in [\,0, r_0)$ and $\varphi(r) < 0$ for $r \in (r_0, \infty)$. Thus it follows that

$$(r_0/\gamma, \varphi(r_0), \psi(r_0/\gamma)) = (r_0/\gamma, 0, r_0\|\Phi\|_3^3/\gamma) \in \mathcal{L}_1.$$

We note that C_ε stated in Proposition 2.1 satisfies $C_0 = r_0/\gamma$. Additionally the next lemma gives profiles of $\psi(r)$ in the interval of $\{r > 0 : \varphi(\gamma r) > 0\}$ if τ is sufficiently small and γ is sufficiently large.

Lemma 2.2. *There exist positive constants $\tilde\tau = \tilde\tau(c, b_1)$ and $\tilde\gamma = \tilde\gamma(c, b_1)$ such that if $(\tau,\gamma) \in (0,\tilde\tau\,] \times [\,\tilde\gamma, \infty)$, then $\psi'(0) > 0$ and $\psi(r)$ achieves a strict local maximum in $(0, r_0/\gamma)$. Furthermore, there exists a continuous function $\hat\gamma(\tau)$ in $(0,\tilde\tau\,]$ satisfying*

$$\tilde\gamma < \hat\gamma(\tau) \ \ \text{for all} \ \ \tau \in (0,\tilde\tau\,] \ \ \text{and} \ \ \lim_{\tau\downarrow 0} \hat\gamma(\tau) = \infty$$

and that, if $\gamma \in [\,\tilde\gamma, \hat\gamma(\tau))$ for $\tau \in (0,\tilde\tau\,]$, then $\psi(r)$ attains a strict local minimum in $(0, r_0/\gamma)$.

From Proposition 2.1 and Lemma 2.2, one can see the following proposition.

Proposition 2.2. *Suppose that $(\tau,\gamma) \in (0,\tilde\tau\,] \times [\,\tilde\gamma,\infty)$ and that $\varepsilon > 0$ is small enough. Then the positive solution set of* (2.2) *contains a bounded smooth curve*

$$\Gamma^\varepsilon = \{(w(\xi), z(\xi), a_1(\xi)) \in X \times \boldsymbol{R} : \xi \in (0, C_\varepsilon)\},$$

which possesses the following properties;

(i) $(w(0), z(0)) = (0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1})$, $a_1(0) > 0$, $a_1'(0) > 0$;
(ii) $(w(C_\varepsilon), z(C_\varepsilon)) = (\varepsilon^{-1}\theta_{\lambda_1+\varepsilon a_{1*}}, 0)$, $a_{1*} := a_1(C_\varepsilon) > 0$;
(iii) *$a_1(\xi)$ attains a strict local maximum in $(0, C_\varepsilon)$. In particular, if $\gamma \in [\,\tilde\gamma, \hat\gamma(\tau))$ for $\tau \in (0,\tilde\tau\,]$, then $a_1(\xi)$ attains a strict local minimum in $(0, C_\varepsilon)$.*

With use of (2.1), Theorem 2.1 immediately follows from Proposition 2.2. Actually, for small $\varepsilon > 0$, open sets stated in Theorem 2.1 are given as

$$O = \{(\beta, b, d) = (\gamma/\varepsilon, \lambda_1 + \varepsilon b_1, (\lambda_1 + \varepsilon\tau)\gamma/\varepsilon) \,:\, (\tau,\gamma) \in (0,\tilde\tau) \times (\tilde\gamma, \infty)\}\,,$$

$$O' = \{(\beta, b, d) = (\gamma/\varepsilon, \lambda_1 + \varepsilon b_1, (\lambda_1 + \varepsilon\tau)\gamma/\varepsilon) \,:\, (\tau,\gamma) \in (0,\tilde\tau) \times (\tilde\gamma, \hat\gamma(\tau))\}\,.$$

We refer to [10] for the complete proofs.

3. Stability analysis.

3.1. **Main results.** In this section, we will discuss the stability of steady-state solutions on Γ obtained in Theorem 2.1. Before stating our stability results, we need to divide Γ at every turning point with respect to a. In case $(\beta, b, d) \in O$, let

$$0 < r_1 < r_2 < \cdots < r_{k-1} < C$$

be all strict local maximum or minimum points of $a(r)$. Because of $a'(0) > 0$ (see Theorem 2.1), $r_{2j-1}\,(j = 1, 2, \ldots, [k/2])$ are strict local maximum points, and $r_{2j}\,(j = 1, 2, \ldots, [(k-1)/2])$ are strict local minimum points. For each $1 \leq i \leq k$, we set

$$\Gamma_i := \{(u(r), v(r), a(r)) \in \Gamma : r \in (r_{i-1}, r_i)\},$$

where $r_0 := 0$ and $r_k := C$.

We are ready to state stability results. In a case when σ is sufficiently small, we can deduce that the stability of steady-states on Γ changes only at the *turning points*, and moreover, we can know whether each solution on Γ_i is asymptotically stable or not:

Theorem 3.1. *For almost every $(\beta, b, d) \in O$, there exists a small positive constant δ such that if $\sigma \leq \delta$, then all steady-state solutions on $\Gamma_{2j-1}\,(j = 1, 2, \ldots, [(k+1)/2])$ are asymptotically stable in the topology of X, while all steady-state solutions on $\Gamma_{2j}\,(j = 1, 2, \ldots, [k/2])$ are unstable.*

In the above case, we remark that $(u(0), v(0)) = (0, \theta_b)$ and $(u(C), v(C)) = (\theta_{a(C)}, 0)$ by Theorem 2.1. So Theorem 3.1 implies that stable positive steady-states bifurcate from the semitrivial solution $(0, \theta_b)$, the stability on Γ changes at every turning point with respect to a, and moreover Γ connects the other semitrivial solution $(\theta_{a(C)}, 0)$. On the other hand, when σ becomes large enough, we can find a Hopf bifurcation point on Γ_1; so that, time-periodic solutions of (P) appear from the point:

Theorem 3.2. *For any $(\beta, b, d) \in O$, there exists a large positive D such that if $\sigma \geq D$, then the Hopf bifurcation occurs at some point $(u(r^*), v(r^*), a(r^*)) \in \Gamma_1$. In this case, there exists a periodic solution of* (P) *if a lies in a neighborhood of $a(r^*)$ with $a > a(r^*)$.*

3.2. **Outline of the proofs of Theorems 3.1 and 3.2.** By virtue of the regularity of (2.1), the stability of a steady-state (u^*, v^*) of (P) coincides with that of the steady-state $(w^*, z^*) = (u^*/\varepsilon, (1 + \beta u^*)z^*/\varepsilon)$ of (PP). So we will concentrate on the stability analysis for the steady-states on Γ^ε given in Proposition 2.2. By virtue of Proposition 2.1, all positive steady-states of (PP) with $a_1 \in (0, A_1)$ can be parameterized as $\Gamma^\varepsilon = \{(w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon)) : \xi \in (0, C_\varepsilon))\}$ when $\varepsilon > 0$ is sufficiently small. For each $(w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon)) \in \Gamma^\varepsilon$, we define a linear operator $L(\xi, \varepsilon) : X \to Y$ by

$$L(\xi, \varepsilon)\begin{pmatrix} h \\ k \end{pmatrix} := -H\begin{pmatrix} h \\ k \end{pmatrix} - \varepsilon B_{(w,z)}(w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon))\begin{pmatrix} h \\ k \end{pmatrix},$$

where H, B are mappings defined by (2.3) and $B_{(w,z)}$ denotes the Fréchet derivative of B with respect to (w, z). Furthermore, in view of the left hand side of (PP), we set

$$J(\xi, \varepsilon) := \begin{bmatrix} 1 & 0 \\ -\dfrac{\sigma\gamma z(\xi, \varepsilon)}{(1 + \gamma w(\xi, \varepsilon))^2} & \dfrac{\sigma}{1 + \gamma w(\xi, \varepsilon)} \end{bmatrix}.$$

Then the linearized eigenvalue problem associated with $(w(\xi,\varepsilon), z(\xi,\varepsilon))$ is given by

$$L(\xi,\varepsilon)\begin{pmatrix} h \\ k \end{pmatrix} = \mu J(\xi,\varepsilon)\begin{pmatrix} h \\ k \end{pmatrix}. \tag{3.1}$$

In this subsection, we study the linearized stability of steady-states on Γ^ε by the spectral analysis for (3.1). Put

$$\rho(\xi,\varepsilon) := \{\mu \in \boldsymbol{C} \,:\, (3.1) \text{ has no solution except for } h = k = 0\}.$$

We begin with the following lemma.

Lemma 3.1. *Suppose that $\varepsilon > 0$ is sufficiently small. Then there exist positive constants κ_1, ω independent of (ξ,ε) such that $-\rho(\xi,\varepsilon) \supset \{z \in \boldsymbol{C} : |z| \geq \kappa_1$ and $|\arg z| \leq \pi/2 + \omega\}$. On the other hand, all eigenvalues $\{\mu_i(\xi,\varepsilon)\}_{i=1}^{\infty}$ (counting multiplicity) of (3.1) satisfy*

$$\lim_{\varepsilon\downarrow 0}\mu_1(\xi,\varepsilon) = \lim_{\varepsilon\downarrow 0}\mu_2(\xi,\varepsilon) = 0 \tag{3.2}$$

and $\operatorname{Re}\mu_i(\xi,\varepsilon) > \kappa_2$ for all $i \geq 3$ and $\xi \in (0, C_\varepsilon)$ for some positive constant κ_2 independent of (ξ,ε).

The proof of Lemma 3.1 can be established by employing a limiting eigenvalue problem as $\varepsilon \downarrow 0$ in (3.1), and making use of the perturbation theory by T. Kato [8, Chapter 8]. See [9] for details.

We note that all eigenvalues $\{\mu_i(\xi,\varepsilon)\}$ form a symmetric set with respect to the real axis in the complex space $\boldsymbol{C}$. Then $\mu_1(\xi,\varepsilon)$ and $\mu_2(\xi,\varepsilon)$ (with (3.2)) satisfy the following properties (i) or (ii);

(i) both of $\mu_1(\xi,\varepsilon)$ and $\mu_2(\xi,\varepsilon)$ are real numbers;
(ii) $\mu_1(\xi,\varepsilon)$ is a complex conjugate of $\mu_2(\xi,\varepsilon)$.

In what follows, we assume that $\mu_1(\xi,\varepsilon) \leq \mu_2(\xi,\varepsilon)$ in case (i), and $\operatorname{Im}\mu_1(\xi,\varepsilon) \geq \operatorname{Im}\mu_2(\xi,\varepsilon)$ in case (ii).

Definition 3.1. A steady-state $(w(\xi,\varepsilon), z(\xi,\varepsilon))$ of (PP) is called *linearly stable* if $\operatorname{Re}\mu_1(\xi,\varepsilon) > 0$. If $\operatorname{Re}\mu_1(\xi,\varepsilon) < 0$, then it is called *linearly unstable.*

We define matrices $K(r)$ and $M(r)$ by

$$\begin{aligned} K(r) &= \begin{bmatrix} 1 & 0 \\ -\sigma\gamma\varphi(\gamma r)\displaystyle\int_\Omega \frac{\Phi^3}{(1+\gamma r\Phi)^2} & \sigma\displaystyle\int_\Omega \frac{\Phi^2}{1+\gamma r\Phi} \end{bmatrix}, \\ M(r) &= -K(r)^{-1}F^0_{(r,s)}(r,\varphi(\gamma r),\psi(r)) \end{aligned} \tag{3.3}$$

for the mapping F^0 defined by (2.6). To determine the sign of $\operatorname{Re}\mu_1(\xi,\varepsilon)$, the following lemma plays an important role.

Lemma 3.2. *Let $\nu_1(r)$ and $\nu_2(r)$ be eigenvalues of $M(r)$ and satisfy $\operatorname{Re}\nu_1(r) \leq \operatorname{Re}\nu_2(r)$, $\operatorname{Im}\nu_1(r) \geq \operatorname{Im}\nu_2(r)$. Then for any $r \in (0, C_0)$, it holds true that*

$$\lim_{(\xi,\varepsilon)\to(r,0)} \frac{\mu_i(\xi,\varepsilon)}{\varepsilon} = \nu_i(r) \quad \text{for} \quad i = 1, 2. \tag{3.4}$$

Lemma 3.2 can be proved by taking L^2-inner product of (3.1) with Φ and letting $\varepsilon \to 0$. See [9] for details.

Lemma 3.3. *Suppose that $\varepsilon > 0$ is sufficiently small. Suppose further that $\xi \in (0, C_\varepsilon)$. Thus all zeros of $\mu_1(\xi,\varepsilon)$ coincide with all zeros of $\partial_\xi a_1(\xi,\varepsilon)$.*

The above lemma asserts that the degeneracy of steady-states on Γ^ε is equivalent to the criticality of $a_1(\xi,\varepsilon)$ with respect to ξ. We refer the proof of Lemma 3.3 to the perturbation theory for the Fredholm operator developed by Du and Lou [4, Theorem 3.13 and Appendix]

Since ψ is analytic, ψ' possesses at most a finite number of zeros in $(0,C_0)$. Furthermore, by virtue of (2.7), any zero of ψ' must be a strictly critical point of ψ for almost every $(\tau,\gamma)\in(0,\tilde\tau\,]\times[\,\tilde\gamma,\infty)$. For such $(\tau,\gamma)\in(0,\tilde\tau\,]\times[\,\tilde\gamma,\infty)$ and sufficiently small $\varepsilon>0$, all zeros of $\partial_\xi a_1(\xi,\varepsilon)$ are denoted by

$$0<\xi_1(\varepsilon)<\xi_2(\varepsilon)<\cdots<\xi_{k-1}(\varepsilon)<C_\varepsilon.$$

That is,

$$(w_i,z_i,a_1^i):=(w(\xi_i(\varepsilon),\varepsilon),z(\xi_i(\varepsilon),\varepsilon),a_1(\xi_i(\varepsilon),\varepsilon))\in\Gamma^\varepsilon\quad(i=1,2,\ldots,k-1)$$

are all turning points on Γ^ε with respect to a_1. Here we remark that $\lim_{\varepsilon\downarrow 0}a_1(\,\cdot\,,\varepsilon)=\psi$ in $C^2([\,0,C_0\,])$ by Proposition 2.1 (see also the proof of [10, Lemma 5.3]). Additionally, for each $1\le i\le k$ we set

$$\Gamma_i^\varepsilon:=\{((w(\xi,\varepsilon),z(\xi,\varepsilon),a_1(\xi,\varepsilon))\,:\,\xi\in(\xi_{i-1}(\varepsilon),\xi_i(\varepsilon))\},$$

where $\xi_0(\varepsilon):=0$ and $\xi_k(\varepsilon)=C_\varepsilon$. This implies $\bigcup_{i=1}^{k}\Gamma_i^\varepsilon=\Gamma^\varepsilon\setminus\bigcup_{i=1}^{k-1}\{(w_i,z_i,a_1^i)\}$.

Lemma 3.4. *For almost every $(\tau,\gamma)\in(0,\tilde\tau\,]\times[\,\tilde\gamma,\infty)$, there exist small positive constants δ,ε_0 such that if $\sigma\le\delta$ and $\varepsilon\le\varepsilon_0$, then all steady-state solutions on $\Gamma^\varepsilon_{2j-1}$ $(j=1,2,\ldots,[(k+1)/2])$ are linearly stable, while all steady-state solutions on Γ^ε_{2j} $(j=1,2,\ldots,[k/2])$ are linearly unstable.*

Proof. Taking the trace of $M(r)$, one can see

$$\begin{aligned}\nu_1(r)+\nu_2(r)=\frac{\varphi(\gamma r)}{\sigma}&\left[\int_\Omega\frac{\Phi^3}{(1+\gamma r\Phi)^2}\left(\int_\Omega\frac{\Phi^2}{1+\gamma r\Phi}\right)^{-1}-\sigma c\gamma r\int_\Omega\frac{\Phi^4}{(1+\gamma r\Phi)^2}\right]\\&+r\|\Phi\|_3^3+c\gamma r\varphi(\gamma r)\int_\Omega\frac{\Phi^3}{1+\gamma r\Phi}\int_\Omega\frac{\Phi^3}{(1+\gamma r\Phi)^2}\left(\int_\Omega\frac{\Phi^2}{1+\gamma r\Phi}\right)^{-1}.\end{aligned}\tag{3.5}$$

We set $y_1(r):=\int_\Omega r\Phi^4/(1+r\Phi)^2$. Since $y_1(0)=0$ and $y_1(r)=O(r^{-1})\,(r\to\infty)$, $y_1(\hat r)=\sup_{r>0}y_1(r)$ for some $\hat r>0$. Then by (3.5),

$$\nu_1(r)+\nu_2(r)>\frac{\varphi(\gamma r)}{\sigma}\left[\int_\Omega\frac{\Phi^3}{(1+\gamma C_0\Phi)^2}-\sigma c y_1(\hat r)\right]+r\|\Phi\|_3^3$$

for all $r\in[\,0,C_0\,]$. Therefore, it follows from $\varphi(\gamma r)>0$ $(r\in[\,0,C_0))$ that, if

$$\sigma<\frac{1}{2cy_1(\hat r)}\int_\Omega\frac{\Phi^3}{(1+\gamma C_0\Phi)^2},$$

then $\nu_1(r)+\nu_2(r)>0$ for all $r\in[\,0,C_0\,]$. Thus we can see by Lemma 3.2 that for sufficiently small $\varepsilon>0$,

$$\mu_1(\xi,\varepsilon)+\mu_2(\xi,\varepsilon)>0\quad\text{for all}\quad\xi\in[\,0,C_\varepsilon\,].\tag{3.6}$$

Hence (3.6) also implies $\operatorname{Re}\mu_2(\xi,\varepsilon)>0$ for all $\xi\in[\,0,C_\varepsilon\,]$. On the other hand, in view of (3.3), (2.6) and (2.7), direct calculations enable us to obtain

$$\nu_1(r)\nu_2(r)=\det M(r)=\frac{r\varphi(\gamma r)\psi'(r)}{\sigma}\int_\Omega\frac{\Phi^3}{(1+\gamma r\Phi)^2}\left(\int_\Omega\frac{\Phi^2}{1+\gamma r\Phi}\right)^{-1}.\tag{3.7}$$

So it holds that $\operatorname{sign}\nu_1(r)\nu_2(r) = \operatorname{sign}\psi'(r)$ for all $r \in (0, C_0)$. Let $r_0 \in (0, C_0)$ be any fixed point. If $\psi'(r_0) > 0$, then Lemma 3.2 implies $\mu_1(\xi,\varepsilon)\mu_2(\xi,\varepsilon) > 0$ if (ξ,ε) is sufficiently near $(r_0, 0)$. Furthermore, together with (3.6), we obtain $\operatorname{Re}\mu_1(\xi,\varepsilon) > 0$. Similarly if $\psi'(r_0) < 0$ and (ξ,ε) is close to $(r_0,0)$, then $\operatorname{Re}\mu_1(\xi,\varepsilon) < 0$. Additionally it follows from Lemma 3.3 that $\mu_1(\xi,\varepsilon) = 0$ if and only if $\xi = \xi_i(\varepsilon)$ for some $1 \le i \le k-1$ provided that $\varepsilon > 0$ is sufficiently small. Since $\operatorname{Re}\mu_2(\xi,\varepsilon) > 0$ for all $\xi \in [\,0, C_\varepsilon\,]$, consequently $\operatorname{Re}\mu_1(\xi,\varepsilon) = 0$ holds if and only if $\xi = \xi_i(\varepsilon)$ for some $1 \le i \le k-1$. We now remark $\psi'(0) > 0$ if $(\tau,\gamma) \in (0,\tilde{\tau}\,] \times [\,\tilde{\gamma},\infty)$ (see [10, Lemma 4.1]). Therefore we obtain

$$\begin{cases} \operatorname{Re}\mu_1(\xi,\varepsilon) > 0 \ \text{ if } \ (w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon)) \in \Gamma^{\varepsilon}_{2j-1}, \\ \operatorname{Re}\mu_1(\xi,\varepsilon) < 0 \ \text{ if } \ (w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon)) \in \Gamma^{\varepsilon}_{2j}. \end{cases}$$

Thus the proof of Lemma 3.4 is complete. □

By virtue of (2.1), we can complete the proof of Theorem 3.1 from Lemma 3.4. It should be noted that we use the linearized stability theory developed by Potier-Ferry [17]. See [9] for details.

Proposition 3.1. *For any $(\tau,\gamma) \in (0,\tilde{\tau}\,] \times [\,\tilde{\gamma},\infty)$, there exist a large $D > 0$ and a small $\varepsilon_0 > 0$ such that if $\sigma \ge D$ and $\varepsilon \le \varepsilon_0$, then the Hopf bifurcation occurs at a certain point on Γ^{ε}_1.*

Proof. To accomplish the proof, it suffices to find small positive numbers ξ^* and ε such that $\mu_1(\xi^*,\varepsilon)$, $\mu_2(\xi^*,\varepsilon)$ form a pure imaginary pair and satisfy $\partial_\xi \operatorname{Re}\mu_i(\xi^*,\varepsilon) < 0$ for $i = 1, 2$. We refer to Amann [2] for the abstract Hopf bifurcation theorem for strongly coupled parabolic equations.

Take $(\tau,\gamma) \in (0,\tilde{\tau}\,] \times [\,\tilde{\gamma},\infty)$. Let $\nu_1(r)$ and $\nu_2(r)$ be eigenvalues of $M(r)$ defined by (3.3). We first remark that by (3.7) and $\psi'(0) > 0$,

$$\nu_1(r)\nu_2(r) > 0 \ \text{ for all } \ r \in (0, r_1) \tag{3.8}$$

with some $r_1 > 0$. If we set

$$y_2(r) := \int_\Omega \frac{\Phi^4}{(1+\gamma r\Phi)^2} - \int_\Omega \frac{\Phi^3}{1+\gamma r\Phi} \int_\Omega \frac{\Phi^3}{(1+\gamma r\Phi)^2} \left(\int_\Omega \frac{\Phi^2}{1+\gamma r\Phi} \right)^{-1} - \frac{\|\Phi\|_3^3}{c\gamma\varphi(\gamma r)}$$

then, (3.5) is rewritten as

$$\nu_1(r) + \nu_2(r) = \frac{\varphi(\gamma r)}{\sigma} \left[\int_\Omega \frac{\Phi^3}{(1+\gamma r\Phi)^2} \left(\int_\Omega \frac{\Phi^2}{1+\gamma r\Phi} \right)^{-1} - \sigma c\gamma r y_2(r) \right].$$

Thus direct calculations imply

$$\nu_1(0) + \nu_2(0) = \frac{b_1}{\sigma}, \quad \nu_1'(0) + \nu_2'(0) = \frac{1}{\sigma} \left(\tilde{C} - \sigma c\gamma y_2(0) \right) \tag{3.9}$$

with some constant $\tilde{C}$ independent of σ. By virtue of Schwarz' inequality and $\|\Phi\| = 1$, we see $\|\Phi\|_4^4 > \|\Phi\|_3^6$. Thus it turns out that $y_2(0) = \|\Phi\|_4^4 - \|\Phi\|_3^6 - \|\Phi\|_3^3 (cb_1\gamma)^{-1} > 0$ if γ is large enough. It follows from (3.9) that if σ is sufficiently large, we can find a small positive number $r_0 \in (0, r_1)$ such that

$$\nu_1(r) + \nu_2(r) > 0 \ \text{ in } \ (0, r_0), \quad \nu_1(r_0) + \nu_2(r_0) = 0 \ \text{ and } \ \nu_1'(r_0) + \nu_2'(r_0) < 0. \tag{3.10}$$

We can find a certain (ξ^*,ε) near $(r_0, 0)$, such that eigenvalues $\mu_1(\xi^*,\varepsilon)$, $\mu_2(\xi^*,\varepsilon)$ are pure imaginary pair and satisfy $\partial_\xi \operatorname{Re}\mu_i(\xi^*,\varepsilon) < 0$ $(i = 1, 2)$. In this part of the proof, we make use of Lemma 3.4 and Lyapunov-Schmidt reduction technique (see

[9]). Therefore the Hopf bifurcation occurs at $(w(\xi^*,\varepsilon), z(\xi^*,\varepsilon),\ a_1(\xi^*,\varepsilon))$, which belongs to Γ_1^ε because ξ^* is sufficiently small. □

By virtue of (2.1), Proposition 3.1 immediately yields Theorem 3.2.

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