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STABILITY AND SYMMETRY BREAKING OF SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. In this article, we prove that there are three unstable positive solutions of a semilinear elliptic equation in a two bumps domain or in a one hole domain in which one is axially symmetric and the other two are nonaxially symmetric.

1. **Introduction.** Let $N \geq 2$ and $2 < p < 2^*$, where $2^* = \frac{2N}{N-2}$ for $N \geq 3$ and $2^* = \infty$ for $N = 2$. Consider the semilinear elliptic equation

$$\begin{cases} -\Delta u + u = |u|^{p-2}u \text{ in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1}$$

where Ω is a domain in $\mathbb{R}^N$ and $H_0^1(\Omega)$ is the Sobolev space in Ω with dual space $H^{-1}(\Omega)$. Associated with equation (1), we consider the energy functionals a, b, and J, for each $u \in H_0^1(\Omega)$, $a(u) = \int_\Omega (|\nabla u|^2 + u^2)$, $b(u) = \int_\Omega |u|^p$,
$J(u) = \dfrac{1}{2}a(u) - \dfrac{1}{p}b(u)$.

Let $z = (x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and Ω be a domain in $\mathbb{R}^N$. Denote the ball $B^N(z_0; s)$, the infinite plate $\mathbb{R}^N_{-\rho,\rho}$, the infinite strip $\mathbf{A}^r$, and a sectional strip $\mathbf{A}^r_{-s,s}$ as follows:

$$\begin{aligned} &B^N(z_0; s) = \{z \in \mathbb{R}^N \mid |z - z_0| < s\}, \\ &\mathbb{R}^N_{-\rho,\rho} = \left\{(x, y) \in \mathbb{R}^N \mid |y| < \rho\right\}, \\ &\mathbf{A}^r = \{(x, y) \in \mathbb{R}^N \mid |x| < r\}, \\ &\mathbf{A}^r_{s,t} = \{(x, y) \in \mathbf{A}^r \mid s < y < t\}. \end{aligned}$$

In section 2, we describe various preliminaries. In section 3, we present the index $\alpha_X(\Omega)$ of a domain Ω for J. In section 4, we describe various compactness results. In section 5, we assert that if Ω is a $y-$symmetric large domain in $\mathbb{R}^N$ separated by a $y-$symmetric bounded domain, then $\alpha(\Omega) < \alpha_s(\Omega)$ and that if Ω is a proper $y-$symmetric large domain in $\mathbf{A}^r$, then $\alpha(\Omega) < \alpha_s(\Omega)$ (see Definitions 1 and 19 for the $y-$symmetric lage domains, and Section 3 for $\alpha(\Omega)$ and $\alpha_s(\Omega)$). Using results in section 5, in section 6-8, we assert our main results in two folds:

(A) We assert that there is an $R_0 > 0$ such that for $R \geq R_0$ the equation (1) on the two bumps domain D_R (see Definition 25) has three unstable positive solutions in which one is $y-$symmetric and other two are not axially symmetric (see Theorem 33.) Byeon [1], Chen-Ni-Zhou [2], and Dancer [4] asserted the existence of three positive solutions of semilinear elliptic equations in a dumbbell domain. Since a

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dumbbell is a two bumps domain, their results are the consequences of our main result.

(B) By the Rellich compactness theorem, there is a positive solution of equation (1) in the sectional strip $\mathbf{A}^r_{-t,t}$ for each $t > 0$. Moreover, by Gidas-Ni-Nirenberg [5] and Chen-Chen-Wang [3], every positive solution of equation (1) in $\mathbf{A}^r_{-t,t}$ for each $t > 0$ is radially symmetric in x and axially symmetric in y. Actually, Dancer [4] proved that the positive solution of equation (1) in $\mathbf{A}^r_{-t,t}$ for each $t > 0$ in $\mathbb{R}^2$ is unique. However, we assert that there exists $t_0 > 0$ such that for $t \geq t_0$, the equation (1) on the one hole domain (see Definition 28) Ω_t has three unstable positive solutions in which one is $y-$symmetric and the other two are not axially symmetric (see Theorem 34.)

For the proofs of several useful lemmas and theorems, see Wang [7].

2. Preliminaries.

Definition 1. (i) *Suppose that $(x, y) \in \Omega$ if and only if $(x, -y) \in \Omega$, then we call Ω a y-symmetric domain;*
(ii) *Let Ω be an $y-$symmetric domain and Θ be a $y-$symmetric bounded domain in $\mathbb{R}^N$. If there exist two disjoint subdomains Ω_1 and Ω_2 of Ω such that*

$$(x, y) \in \Omega_2 \text{ if and only if } (x, -y) \in \Omega_1,$$
$$\Omega \setminus \bar{\Theta} = \Omega_1 \cup \Omega_2,$$

then we call that Ω is separated by Θ;
(iii) *Let Ω be a $y-$symmetric domain in $\mathbb{R}^N$. If a function $u : \Omega \to \mathbb{R}$ satisfies $u(x, y) = u(x, -y)$ for $(x, y) \in \Omega$, then we call u a $y-$symmetric (axially symmetric) function.*

Example 2. *For each $\rho > 0$, let $\Omega = \left(\mathbb{R}^N \setminus \overline{\mathbb{R}^N_{-\rho,\rho}}\right) \cup \mathbf{A}^r$. Then Ω is a y-symmetric domain in $\mathbb{R}^N$ separated by the bounded domain $\mathbf{A}^r_{-\rho,\rho}$.*

Let Ω be a $y-$symmetric domain in $\mathbb{R}^N$ and denote the space $H_s(\Omega)$ by the H^1-closure of the space $\{u \in C_0^\infty(\Omega) \mid u \text{ is } y-\text{symmetric}\}$. Throughout this article, let $X(\Omega)$ be either $H_0^1(\Omega)$ or $H_s(\Omega)$. Let $X^{-1}(\Omega)$ be the dual of $X(\Omega)$

We define the Palais$-$Smale (simply by (PS)) sequences, (PS)$-$values, and (PS)$-$conditions in $X(\Omega)$ for J as follows:

Definition 3. *We define*
(i) *For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a (PS)$_\beta-$sequence in $X(\Omega)$ for J if $J(u_n) = \beta + o(1)$ and $J'(u_n) = o(1)$ strongly in $X^{-1}(\Omega)$ as $n \to \infty$;*
(ii) *$\beta \in \mathbb{R}$ is a (PS)$-$value in $X(\Omega)$ for J if there is a (PS)$_\beta-$sequence in $X(\Omega)$ for J;*
(iii) *J satisfies the (PS)$_\beta-$condition in $X(\Omega)$ if every (PS)$_\beta-$sequence in $X(\Omega)$ for J contains a convergent subsequence;*
(iv) *J satisfies the (PS)$-$condition in $X(\Omega)$ if for every $\beta \in \mathbb{R}$, J satisfies the (PS)$_\beta-$condition in $X(\Omega)$.*

A (PS)$_\beta-$sequence in $X(\Omega)$ for J is a (PS)$_\beta-$sequence in $H_0^1(\Omega)$ for J.

Lemma 4. (i) *For a $\mu \in X^{-1}(\Omega)$, we can extend it to be $\mu \in H^{-1}(\Omega)$ such that $||\mu||_{X^{-1}} = ||\mu||_{H^{-1}}$;*
(ii) *Let $\{u_n\}$ be in $X(\Omega)$ satisfying $J'(u_n) = o(1)$ in $X^{-1}(\Omega)$, then $J'(u_n) = o(1)$*

strongly in $H^{-1}(\Omega)$ *as* $n \to \infty$*;*
(iii) If $J'(u) = 0$ *in* $X^{-1}(\Omega)$*, then* $J'(u) = 0$ *in* $H^{-1}(\Omega)$*.*

For any $\beta \in \mathbb{R}$, a $(\mathrm{PS})_\beta-$sequence in $X(\Omega)$ for J is bounded. Moreover, a $(\mathrm{PS})-$value β should be nonnegative.

Lemma 5. *Let* $\beta \in \mathbb{R}$ *and* $\{u_n\}$ *be a* $(PS)_\beta-$*sequence in* $X(\Omega)$ *for* J*, then there exists a positive sequence* $\{c_n(\beta)\}$ *such that* $\|u_n\|_{H^1} \le c_n(\beta) \le c$ *for each* n *and* $c_n(\beta) = o(1)$ *as* $n \to \infty$ *and* $\beta \to 0$*. Furthermore,*

$$a(u_n) = b(u_n) + o(1) = \frac{2p}{p-2}\beta + o(1)$$

and $\beta \ge 0$*.*

3. **Indices of Domains.** In this section, we study the index of a domain Ω which is important to study the existence of solutions of equation (1) in Ω.

Consider the Nehari minimization problem

$$\alpha_{\mathbf{M}}(\Omega) = \inf_{v \in \mathbf{M}(\Omega)} J(v),$$

where $\mathbf{M}(\Omega) = \{u \in X(\Omega) \backslash \{0\} \mid a(u) = b(u)\}$. Note that $\mathbf{M}(\Omega)$ contains every nonzero solution of equation (1). Consider the unit sphere

$$\mathbf{U}(\Omega) = \{u \in X(\Omega) \mid ||u||_{H^1} = 1\}.$$

$\alpha_{\mathbf{M}}(\Omega) > 0$ is a consequence of the following lemma.

Lemma 6. *There is a bijective* $C^{1,1}$ *map* m *from* $\mathbf{U}(\Omega)$ *to* $\mathbf{M}(\Omega)$*. Moreover,* $\mathbf{M}(\Omega)$ *is path−connected and there exists a constant* $c > 0$ *such that for* $u \in \mathbf{M}(\Omega)$*,* $\|u\|_{H^1} \ge c$ *and* $J(u) \ge c$*.*

We have the following useful lemma.

Lemma 7. *Let* $\beta > 0$ *and* $\{u_n\}$ *in* $X(\Omega) \backslash \{0\}$ *be a sequence for* J *such that* $J(u_n) = \beta + \mathrm{o}(1)$ *and* $a(u_n) = b(u_n) + \mathrm{o}(1)$*. Then there is a sequence* $\{s_n\}$ *in* $\mathbb{R}^+$ *such that* $s_n = 1 + \mathrm{o}(1)$*,* $\{s_n u_n\}$ *is in* $\mathbf{M}(\Omega)$ *and* $J(s_n u_n) = \beta + \mathrm{o}(1)$*. In particular, if* $\{u_n\}$ *is a* $(PS)_\beta-$*sequence in* $X(\Omega)$ *for* J*, then there is a sequence* $\{s_n\}$ *in* $\mathbb{R}^+$ *such that* $\{s_n u_n\}$ *is in* $\mathbf{M}(\Omega)$ *and* $\{s_n u_n\}$ *is a* $(PS)_\beta-$*sequence in* $X(\Omega)$ *for* J*.*

A minimizing sequence $\{u_n\}$ in $\mathbf{M}(\Omega)$ of $\alpha_{\mathbf{M}}(\Omega)$ is a $(\mathrm{PS})_{\alpha_{\mathbf{M}}(\Omega)}-$sequence in $X(\Omega)$ for J . In particular, $\alpha_{\mathbf{M}}(\Omega)$ is a $(\mathrm{PS})_{\alpha_{\mathbf{M}}}-$value in $X(\Omega)$ for J.

Lemma 8. *Let* $\{u_n\}$ *be in* $X(\Omega)$*. Then* $\{u_n\}$ *is a* $(PS)_{\alpha_{\mathbf{M}}(\Omega)}-$*sequence in* $X(\Omega)$ *for* J *if and only if* $J(u_n) = \alpha_{\mathbf{M}}(\Omega) + o(1)$ *and* $a(u_n) = b(u_n) + o(1)$*.*

If u achieves $\alpha_{\mathbf{M}}(\Omega)$, then u is a nonzero solution of equation (1).

Lemma 9. *Let* $u \in \mathbf{M}(\Omega)$ *such that* $J(u) = \min_{v \in \mathbf{M}(\Omega)} J(v)$*. Then* u *is a nonzero solution of equation (1) in* $X(\Omega)$*.*

Definition 10. $\alpha_{\mathbf{M}}(\Omega)$ *is called the index of* J *in* $X(\Omega)$ *and denoted by* $\alpha_X(\Omega)$*. By the definition of* $\alpha_{\mathbf{M}}(\Omega)$*, if* u *is a nonzero solution of equation (1) in* $X(\Omega)$*, then* $u \in \mathbf{M}(\Omega)$*. Thus,* $J(u) \ge \alpha_{\mathbf{M}}(\Omega) = \alpha_X(\Omega)$*. We call that a nonzero solution* u *of equation (1) is a ground state solution in* $X(\Omega)$ *if* $J(u) = \alpha_X(\Omega)$*, and is a higher energy solution in* $X(\Omega)$ *if* $J(u) > \alpha_X(\Omega)$*.*

Denote $\alpha_X(\Omega)$ by $\alpha(\Omega)$ for $X(\Omega) = H_0^1(\Omega)$ and $\alpha_X(\Omega)$ by $\alpha_s(\Omega)$ for $X(\Omega) = H_s(\Omega)$.

4. **Palais−Smale Conditions.** In this section, we present several (PS) − conditions in $X(\Omega)$ for J which is used to prove the existence of solutions of equation (1) in section 5.

As a consequence of Lemma 5, for each $(\mathrm{PS})_\beta$−sequence $\{u_n\}$ in $X(\Omega)$ for J, there is a subsequence $\{u_n\}$ and u in $X(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$. Then u is a solution in $X(\Omega)$ of equation (1). To prove this result, we need the following three lemmas.

Lemma 11. *Let $u_n \rightharpoonup u$ weakly in $X(\Omega)$. Then there exists a subsequence $\{u_n\}$ such that*
(i) $\{u_n\}$ is bounded in $X(\Omega)$ and $||u||_{H^1} \le \liminf_{n\to\infty} ||u_n||_{H^1}$;
(ii) $u_n \rightharpoonup u$, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$, and $u_n \to u$ a.e. in Ω;
(iii) $\|u_n - u\|_{H^1}^2 = \|u_n\|_{H^1}^2 - \|u\|_{H^1}^2 + o(1)$.

Lemma 12. *Let $u_n \rightharpoonup u$ weakly in $X(\mathbb{R}^N)$ and*

$$J'(u_n) = -\Delta u_n + u_n - |u_n|^{p-2} u_n = o(1) \text{ in } X^{-1}(\mathbb{R}^N).$$

Then
(i) $|u_n - u|^{p-2}(u_n - u) - |u_n|^{p-2}u_n + |u|^{p-2}u = o(1)$ in $X^{-1}(\mathbb{R}^N)$;
(ii) $J'(\varphi_n) = -\Delta\varphi_n + \varphi_n - |\varphi_n|^{p-2}\varphi_n = o(1)$ in $X^{-1}(\mathbb{R}^N)$, where $\varphi_n = u_n - u$.

Lemma 13. *Let $u \in X(\Omega)$ be a sign changing solution of equation (1). Then $J(u) > 2\alpha_X(\Omega)$.*

Now we conclude the following lemma.

Lemma 14. *We have*
(i) Let $\{u_n\}$ be a $(PS)_{\alpha_X(\Omega)}$ − sequence in $X(\Omega)$ for J satisfying $u_n \rightharpoonup u$ weakly in $X(\Omega)$. Then u is a solution in $X(\Omega)$ of equation (1);
(ii) Let $\{u_n\}$ be a $(PS)_{\alpha_X(\Omega)}$ − sequence in $X(\Omega)$ for J such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$ and u is nonzero. Then u is a positive ground state solution in $X(\Omega)$ of equation (1) and that $u_n \to u$ strongly in $X(\Omega)$;
(iii) The $(PS)_{\alpha_X(\Omega)}$ − condition holds in $X(\Omega)$ for J if and only if for each $(PS)_{\alpha_X(\Omega)}$ − sequence $\{u_n\}$ in $X(\Omega)$ for J, there is a subsequence $\{u_n\}$and a nonzero u in $X(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $X(\Omega)$.

Let $\Omega^1 \subsetneqq \Omega^2$ and $\alpha_X^i = \alpha_X(\Omega^i)$ for $i = 1, 2$, then clearly $\alpha_X^2 \le \alpha_X^1$. If $\alpha_X^2 = \alpha_X^1$, then we have the following useful results.

Theorem 15. *Let $\Omega^1 \subsetneqq \Omega^2$ and $J : X(\Omega^2) \to \mathbb{R}$ be the energy functional. Suppose that $\alpha_X^2 = \alpha_X^1$. Then*
(i) J does not satisfy the $(PS)_{\alpha_X^1}$−condition;
(ii) α_X^1 does not admit any ground state solution;
(iii) J does not satisfy the $(PS)_{\alpha_X^2}$−condition.

Let Ω be any unbounded domains and let $\xi \in C^{\infty}([0,\infty))$ such that $0 \le \xi \le 1$ and

$$\xi(t) = \begin{cases} 0 & \text{for } t \in [0,1]; \\ 1 & \text{for } t \in [2,\infty). \end{cases}$$

Let

$$\xi_n(z) = \xi(\frac{2|z|}{n}). \tag{2}$$

Then, by routine arguments, we have the following result.

Lemma 16. *Let $\{u_n\}$ be a $(PS)_{\alpha_X(\Omega)}-$sequence in $X(\Omega)$ for J such that*

$$\int_{\Omega_n} |u_n|^p = \mathrm{o}(1),$$

where $\Omega_n = \Omega \cap B^N(0;n)$. Then for some $r \ge 1$ we have
(i) $\int_\Omega \xi_n^r |u_n|^p = \int_\Omega |u_n|^p + \mathrm{o}(1) = \frac{2p}{p-2}\alpha_X(\Omega) + \mathrm{o}(1)$;
(ii) $\int_\Omega \xi_n^r (|\nabla u_n|^2 + u_n^2) = \int_\Omega \xi_n^r |u_n|^p + \mathrm{o}(1) = \dfrac{2p}{p-2}\alpha_X(\Omega) + \mathrm{o}(1)$;
(iii) $\int_\Omega (\xi_n - 1) u_n \varphi = \mathrm{o}(1)\, \|\varphi\|_{H^1}$ *for every* $\varphi \in X(\Omega)$;
(iv) $\left|\int_\Omega (\xi_n^r - 1)\, |u_n|^{p-2} u_n \varphi\right| = \mathrm{o}(1)\, \|\varphi\|_{H^1}$ *for every* $\varphi \in X(\Omega)$;
(v) $\left|\int_\Omega (\xi_n^r - 1)\, \nabla u_n \nabla \varphi\right| = \mathrm{o}(1)\, \|\varphi\|_{H^1}$ *for every* $\varphi \in X(\Omega)$.

Lemma 17. *(i) Suppose that J does not satisfy the $(PS)_{\alpha_X(\Omega)}$ − condition in $X(\Omega)$, then there is a $(PS)_{\alpha_X(\Omega)}$ − sequence $\{u_n\}$ in $X(\Omega)$ for J satisfying $\int_{\Omega_n} |u_n|^p =$ $\mathrm{o}(1)$ as $n \to \infty$;*
(ii) Suppose that $\{u_n\}$ is a $(PS)_{\alpha_X(\Omega)}$ − sequence in $X(\Omega)$ for J satisfying $\int_{\Omega_n} |u_n|^p$ $= \mathrm{o}(1)$ as $n \to \infty$, then $\{\xi_n u_n\}$ is also a $(PS)_{\alpha_X(\Omega)}$ − sequence in $X(\Omega)$ for J.

A bounded domain Ω is nice.

Theorem 18. *Let Ω be a bounded domain in $\mathbb{R}^N$. Then the $(PS)_{\alpha_X(\Omega)}-$condition holds in $X(\Omega)$ for J. In particular, there is a positive ground state solution in $X(\Omega)$ of equation (1).*

5. **Large Domains.** We give the following definitions.

Definition 19. *(i)A domain Ω in $\mathbf{A}^r$ is large if for any $m > 0$ there exists $s < t$ such that $t - s = m$ and $\mathbf{A}^r_{s,t} \subset \Omega$.*
(ii) A domain Ω in $\mathbb{R}^N$ is large if for any $R > 0$ there exists $z \in \Omega$ such that $B^N(z;R) \subset \Omega$.

Example 20. *(i) Let $0 < r_1 < r$ and Ω be $\mathbf{A}^r \backslash B^N(0;r_1)$. Then Ω is a y-symmetric large domain in $\mathbf{A}^r$;*
(ii)For each $\rho > 0$, let $\Omega = \left(\mathbb{R}^N \setminus \overline{\mathbb{R}^N_{-\rho,\rho}}\right) \cup \mathbf{A}^r$. Then Ω is a y-symmetric large domain in $\mathbb{R}^N$ separated by the bounded domain $\mathbf{A}^r_{-\rho,\rho}$.

Theorem 21. *(i) If Ω is a large domain in $\mathbb{R}^N$, then $\alpha(\Omega) = \alpha(\mathbb{R}^N)$;*
(ii) If Ω is a large domain in $\mathbf{A}^r$, then $\alpha(\Omega) = \alpha(\mathbf{A}^r)$.

Theorem 22. *We have that*
(i) $\alpha_s(B^N(0;R)) = \alpha(B^N(0;R))$;
(ii) $\alpha_s(\mathbb{R}^N) = \alpha(\mathbb{R}^N)$;
(iii) $\alpha_s(\mathbf{A}^r_{-t,t}) = \alpha(\mathbf{A}^r_{-t,t})$;
(iv) $\alpha_s(\mathbf{A}^r) = \alpha(\mathbf{A}^r)$.

We have the following useful result.

Theorem 23. *If Ω is a $y-$symmetric large domain in $\mathbb{R}^N$ separated by a $y-$ symmetric bounded domain, then $\alpha(\Omega) < \alpha_s(\Omega)$.*

Similarly, we have the following result for the infinite strip .

Theorem 24. *If Ω is a proper $y-$symmetric large domain in $\mathbf{A}^r$, then $\alpha(\Omega) < \alpha_s(\Omega)$.*

6. Three Solutions in Two Bumps Domains and in One Hole Domains.

Definition 25. *Let Ω^1_R and Ω^2_R be two disjoint bounded domains in $\mathbb{R}^N$ such that Ω^1_R contains a ball of radius $2R$ and $\Omega^2_R = \{(x,y) \mid (x,-y) \in \Omega^1_R\}$. Let Θ be a bounded $y-$symmetric domain in $\mathbb{R}^N$ such that $\Theta \cap \Omega^1_R \neq \emptyset$. Let $D_R = \Omega^1_R \cup \Theta \cup \Omega^2_R$. Then D_R is called a two bumps domain.*

Example 26. *Let $t > R > r > 0$. The bounded dumbbell domain D^1_R is a two bumps domain, where*

$$D^1_R = B^N((0,-t),R) \cup \mathbf{A}^r_{-t,t} \cup B^N((0,t),R).$$

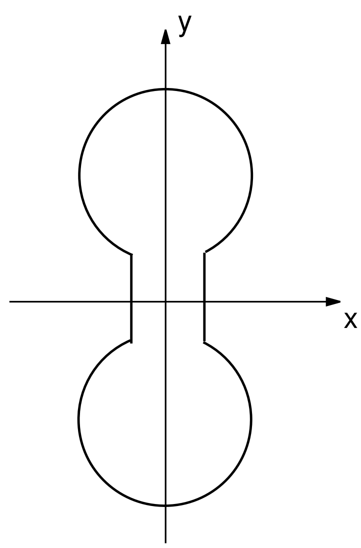

FIGURE 1. bounded dumbbell domain

We obtain the following multiplicity result.

Theorem 27. *There is an $R_0 > 0$ such that for $R > R_0$ the equation (1) in the two bumps domain D_R has three positive solutions in which one is $y-$symmetric and other two are not axially symmetric.*

Definition 28. *Fix x_0 in $\mathbb{R}^{N-1}$ and let $0 < |x_0| + r_1 < r$. Then $\Omega_t = \mathbf{A}^r_{-t,t} \setminus \overline{\Omega_1}$ is called a one hole domain, where Ω_1 is a $y-$symmetric domain such that $\Omega_1 \subset B^N\left((x_0, 0); r_1\right)$.*

Then we have the following multiplicity result.

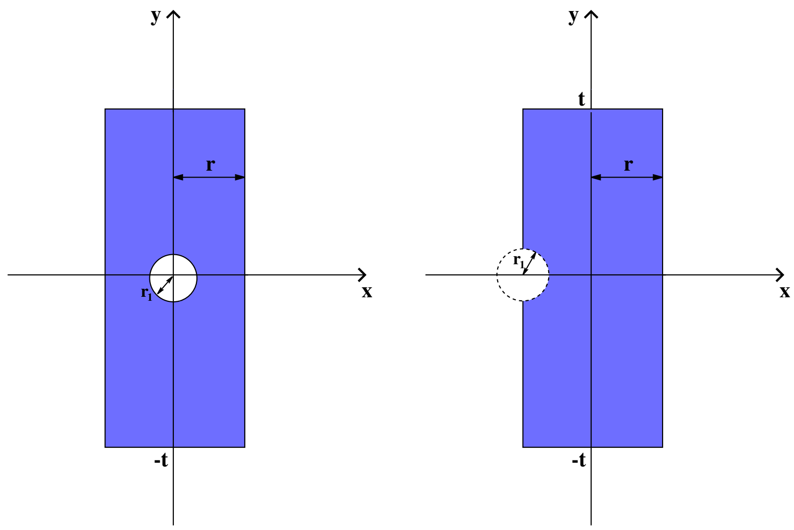

FIGURE 2. one hole domain

Theorem 29. *There exists $t_0 > 0$ such that for $t \geq t_0$, the equation (1) in the one hole domain Ω_t has three positive solutions in which one is $y-$symmetric and the other two are not axially symmetric.*

Proof. We may consider the case $\Omega = \mathbf{A}^r \setminus \overline{B^N\left((x,0); r_1\right)}$, then Ω is a $y-$symmetric large domain in $\mathbf{A}^r$. By Theorem 24, we have $\alpha\left(\mathbf{A}^r\right) < \alpha_s\left(\Omega\right)$. By Lien-Tzeng-Wang [6], equation (1) admits a ground state solution in $\mathbf{A}^r_{0,t}$ and in $\mathbf{A}^r$, we have that $\alpha\left(\mathbf{A}^r_{0,t}\right)$ is strictly decreasing as t is strictly increasing and

$$\alpha\left(\mathbf{A}^r_{0,t}\right) \searrow \alpha\left(\mathbf{A}^r\right) \text{ as } t \to \infty.$$

Take $t_1 > 0$ such that for $t \geq t_1$,

$$\alpha\left(\mathbf{A}^r\right) < \alpha\left(\mathbf{A}^r_{0,t}\right) < \alpha_s\left(\Omega\right). \tag{3}$$

Note that $\mathbf{A}^r_{r_1,t_1+r_1} \subsetneqq \Omega_t \subsetneqq \mathbf{A}^r$ for $t \geq t_0 = t_1 + r_1$. By Lemma 15, we conclude that

$$\alpha\left(\mathbf{A}^r\right) < \alpha\left(\Omega_t\right) < \alpha(\mathbf{A}^r_{r_1,t_1+r_1}). \tag{4}$$

By Lien-Tzeng-Wang [6], if Ω is a domain of $\mathbb{R}^N$, then $\alpha(\Omega)$ is invariant by rigid motions. Thus,

$$\alpha(\mathbf{A}^r_{r_1,t_1+r_1}) = \alpha\left(\mathbf{A}^r_{0,t_1}\right). \tag{5}$$

Therefore, by (3)-(5)

$$\alpha\left(\mathbf{A}^{r}\right)<\alpha\left(\Omega_{t}\right)<\alpha\left(\mathbf{A}_{0,t_{1}}^{r}\right)<\alpha_{s}\left(\Omega\right). \tag{6}$$

Since $\Omega_t \subset \Omega$, we have

$$\alpha_{s}\left(\Omega\right)\leq\alpha_{s}\left(\Omega_{t}\right). \tag{7}$$

By (6) and (7), we obtain

$$\alpha\left(\Omega_{t}\right)<\alpha_{s}\left(\Omega_{t}\right). \tag{8}$$

By Lemma 18, there are an $y-$symmetry solution u_1 and a solution u_2 of equation (1) in domain Ω_t such that

$$J(u_1)=\alpha_{s}\left(\Omega_{t}\right),$$
$$J(u_2)=\alpha\left(\Omega_{t}\right).$$

By Lemma 13, we may take u_1 and u_2 to be positive. Let

$$u_3(x,y)=u_2(x,-y),$$

then u_3 is the third solution. By (8), u_1, u_2 and u_3 are different. Moreover, u_1 is a $y-$symmetric solution while both u_2 and u_3 are nonaxially symmetric solutions of equation (1) in domain Ω_t. □

7. **Eigenvalues.** Let u be a positive solution of equation (1), then by the Schauder, the L^p, and the Kato regularities, $u \in C^2(\overline{\Omega}) \cap W^{2,s}(\Omega)$ for some $s > N$, and

$$\|u\|_{L^{\infty}(\Omega)}\leq\|u\|_{C^{2}(\overline{\Omega})}.$$

Fix $u \in H_0^1(\Omega)$ and let $c > 0$ be such that $q(u) = -(p-1)u^{p-2} + c > 0$. Let

$$a(w,v)=\int_{\Omega}\left(\nabla w\nabla v+wv+q(u)wv\right).$$

Then $a(w, v)$ is a continuous coercive bilinear form on $H_0^1(\Omega)$. By the Lax-Milgram Theorem , for each $k \in L^2(\Omega)$, there is a unique $w \in H_0^1(\Omega)$ such that

$$a(w,v)=(k,v)\quad\text{for } v\in H_{0}^{1}(\Omega),$$

or

$$\int_{\Omega}\nabla w\nabla v+wv+q(u)wv=\int_{\Omega}kv\ \text{ for each } v\in H_{0}^{1}(\Omega),$$

or

$$-\Delta w+w+q(u)w=k.$$

Let $T : L^2(\Omega) \to H_0^1(\Omega)$ be defined by $Tk = w$. Consider $T : L^2(\Omega) \to L^2(\Omega)$ such that $Tk = w$, then we have that T is a compact, self-adjoint, and positive linear operator such that $Ker\ T = \{0\}$. We conclude that $L^2(\Omega)$ admits a basis $\{\phi_n\}$ and $\gamma_n > 0$ for each n, $\gamma_1 < \gamma_2 \leq \gamma_3 \leq \cdots$ and $\gamma_n \to \infty$ as $n \to \infty$ such that

$$-\Delta\phi_{n}+\phi_{n}+q(u)\phi_{n}=\gamma_{n}\phi_{n}.$$

Since $\gamma_1 > 0$ is the least eigenvalue, we may choose $\phi_1 > 0$ in Ω. Thus, we have

Theorem 30. *$L^2(\Omega)$ admits a basis $\{\phi_n\}$ and λ_n in $\mathbb{R}$ for each n and $\lambda_n \to \infty$ as $n \to \infty$ such that ϕ_1 is positive and*

$$-\Delta\phi_{n}+\phi_{n}-(p-1)u^{p-2}\phi_{n}=\lambda_{n}\phi_{n}, \tag{9}$$

where $\lambda_n = (\gamma_n - c)$.

8. **Stability.** Let u be a positive solution of equation (1), then by Theorem 30, $L^2(\Omega)$ admits a basis $\{\phi_n\}$ and λ_n in $\mathbb{R}$ for each n and $\lambda_n \to \infty$ as $n \to \infty$ such that

$$-\Delta\phi_n + \phi_n - (p-1)u^{p-2}\phi_n = \lambda_n\phi_n.$$

Definition 31. *If all the eigenvalues λ_n are positive, then we say that u is stable, otherwise it is unstable.*

We then have

Theorem 32. *Every positive solution of equation (1) in a bounded domain Ω is unstable.*

We now may rewrite our main theorems 27 and 29 as follows

Theorem 33. *There is an $R_0 > 0$ such that for $R \geq R_0$ the equation* (1) *in the two bumps domain D_R has three unstable positive solutions in which one is $y-$symmetric and other two are not axially symmetric.*

Theorem 34. *There exists $t_0 > 0$ such that for $t \geq t_0$, the equation* (1) *in the one hole domain Ω_t has three unstable positive solutions in which one is $y-$symmetric and the other two are not axially symmetric.*

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