DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
doi:10.3934/dcdsb.2017128

POSITIVE SOLUTIONS TO THE UNSTIRRED CHEMOSTAT MODEL WITH CROWLEY-MARTIN FUNCTIONAL RESPONSE

Hai-Xia Li*

Institute of Mathematics and Information Sciences, Baoji University of Arts and Sciences
Baoji, Shaanxi 721013, China

Jian-Hua Wu and Yan-Ling Li

College of Mathematics and Information Science, Shaanxi Normal University
Xi'an, Shaanxi 710062, China

Chun-An Liu

Institute of Mathematics and Information Sciences, Baoji University of Arts and Sciences
Baoji, Shaanxi 721013, China

(Communicated by Sze-Bi Hsu)

Abstract. A food-chain model with Crowley-Martin functional response in the unstirred chemostat is considered. First, the global framework of coexistence solutions is discussed by the maximum principle and bifurcation theory. We obtain the sufficient and necessary conditions for coexistence of steady-state. Second, the stability and uniqueness of coexistence solutions are investigated by means of the combination of the perturbation theory and fixed point index theory. Our results indicate that if the magnitude of interference among predator is sufficiently large, the model has only one unique linearly stable coexistence solution when the maximal growth rate of predator belongs to certain range. Finally, some numerical simulations are carried out to verify and complement the theoretical results.

1. **Introduction.** Chemostat is a laboratory apparatus used for continuous culture of bacteria and plays an important role in microbiology. Practice indicates that chemostat has the advantages that the parameters are measurable, the experiments are reasonable, and the mathematical analysis is feasible. So, chemostat models are improved and popularized continually to describe the natural phenomena. It has been widely applied to waste treatment, microorganisms culture, sewage treatment, biology pharmacy, food processing, control of environment pollution and so on. Therefore, chemostat has an important significance in reality.

The basic chemostat has well-stirred hypothesis which means that the spatially inhomogeneous distribution has not been considered. However, spatial diffusion

2010 *Mathematics Subject Classification.* Primary: 35K57, 35B32, 35B20; Secondary: 92B05.
Key words and phrases. Coexistence solutions, bifurcation, fixed point index, stability, uniqueness, numerical simulations.

The work is supported by the Natural Science Foundation of China (61672021,11401356,11671243), the Natural Science Basic Research Plan in Shaanxi Province of China (2015JM1008), the Foundations of Shaanxi Educational Committee (16JK1046), the Postdoctoral Science Foundation of China (2016M602767) and the Special Fund of Education Department of Shaanxi Province (16JK1710).

* Corresponding author.

is a common phenomenon and a candidate for its explanation is to remove the well-stirred hypothesis, this leads to a type model often referred to as the unstirred chemostat. The unstirred chemostat models have been considered by many scholars in the past decades. One can see [5][7]-[11][13][16]-[27][32][34]-[38] and the references therein. In [11], steady-state of competition model arising from an unstirred chemostat in the one-dimensional situation has been first studied by Hsu and Waltman. The authors gave the conclusion of coexistence for two species. Dung and Smith studied the same model by the bifurcation theory and obtained the sufficient conditions of the existence of positive steady-state solutions in [5]. Then, Wu [34] generalized this model to the N-dimensional case, and got the existence of coexistence solutions and the local stability for bifurcation solutions. Later, the unstirred chemostat models with two resources have been well discussed; see [8][21][35][37]. Recently, Wu and Nie [18]-[20][22]-[24][26][27][36] studied the unstirred chemostat models with inhibitor, plasmid and toxin.

The above papers studied mainly the chemostat models with the Michaelis-Menten functional response. As we know, Michaelis-Menten functional response is classified as one of prey-dependent functional response, it assumes that there is no interference between predators. Nevertheless, mutual interference between species is a common character in the nature. Hence, it is necessary to consider mutual interference between species. In [1], Crowley and Martin proposed the following functional response

$$f(u,v) = u/[(1+ku)(1+mv)],$$

which is called the Crowley-Martin type functional response, where k and m describe the effects of handling time and the magnitude of interference among predators, respectively, on the feeding rate. Compared to the well-known Beddington-DeAngelis functional response, it has an additional term that models mutual interference among predators. Moreover, Crowley-Martin type functional response allows for interference among predators regardless of whether an individual predator is currently handling prey or searching for prey. Thus the biological model with Crowley-Martin type functional response progresses the Michaelis-Menten model and Beddington-DeAngelis model. For more information about the background and applications of the Crowley-Martin type functional response, one may refer to [3][12][14][15][29][31][33].

Considering the above discussions, in the present paper, we remove well-stirred hypothesis and deal with the following chemostat model under Robin boundary conditions:

$$\begin{cases} S_t = \Delta S - aug(S,u), & (x,t) \in \Omega \times (0,\infty), \\ u_t = \Delta u + aug(S,u) - bvh(u,v), & (x,t) \in \Omega \times (0,\infty), \\ v_t = \Delta v + bvh(u,v), & (x,t) \in \Omega \times (0,\infty), \\ \frac{\partial S}{\partial n} + \gamma S = S^0,\ \frac{\partial u}{\partial n} + \gamma u = 0,\ \frac{\partial v}{\partial n} + \gamma v = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ S(x,0) = S_0(x) \geq 0, \not\equiv 0,\ u(x,0) = u_0(x) \geq 0, \not\equiv 0, & x \in \Omega, \\ v(x,0) = v_0(x) \geq 0, \not\equiv 0, & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$. $S(x,t), u(x,t)$ and $v(x,t)$ represent the densities of nutrient and two microorganisms respectively. S^0 is the input concentration of the nutrient, which is assumed to be constant. $a > 0$ and $b > 0$ are the maximal growth rates of u and v respectively. $\gamma > 0$ represents the washout constant. $S_0(x), u_0(x), v_0(x)$ are continuous functions.

$g(S,u) = S/[(1+k_1S)(1+m_1u)], h(u,v) = u/[(1+k_2u)(1+m_2v)]$, here $k_i, m_i, i = 1,2$ are positive constants.

When $m_1 = 0, m_2 = 0$, Liu and Zheng [16] investigated (1) in one-dimensional case. They established the existence of bifurcating positive solutions to steady-state system by the local bifurcation theory and gave the conditions of permanence by the comparison principle. In addition, in [38], the authors discussed multiple food-chain model in an unstirred chemostat and attained the sufficient conditions of the existence of positive steady-state solutions by using of the degree theory.

From the biological significance, steady-state solutions play an important role in understanding the long-time behavior of the corresponding parabolic problem. Therefore, we also concentrate on the following elliptic system:

$$\begin{cases} \Delta S - aug(S,u) = 0, & x \in \Omega, \\ \Delta u + aug(S,u) - bvh(u,v) = 0, & x \in \Omega, \\ \Delta v + bvh(u,v) = 0, & x \in \Omega, \\ \frac{\partial S}{\partial n} + \gamma S = S^0,\ \frac{\partial u}{\partial n} + \gamma u = 0,\ \frac{\partial v}{\partial n} + \gamma v = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

Let $z = S + u + v$. Then z satisfies

$$\Delta z = 0,\ x \in \Omega,\quad \frac{\partial z}{\partial n} + \gamma z = S^0,\ x \in \partial\Omega.$$

Just as for the unstirred chemostat in [20][34][35], the limiting system of (2) takes the form:

$$\begin{cases} \Delta u + aug(z-u-v,u) - bvh(u,v) = 0, & x \in \Omega, \\ \Delta v + bvh(u,v) = 0, & x \in \Omega, \\ \frac{\partial u}{\partial n} + \gamma u = 0,\ \frac{\partial v}{\partial n} + \gamma v = 0, & x \in \partial\Omega. \end{cases} \tag{3}$$

Since only positive solutions of (3) are meaningful, we redefine the response functions as follows:

$$\tilde{g}(S,u) = \begin{cases} g(S,u), & S \geq 0, u \geq 0, \\ 0, & otherwise, \end{cases} \qquad \tilde{h}(u,v) = \begin{cases} h(u,v), & u \geq 0, v \geq 0, \\ 0, & otherwise. \end{cases}$$

For convenience, we denote $\tilde{g}(S,u), \tilde{h}(u,v)$ by $g(S,u), h(u,v)$, respectively.

At current, the competition models in the unstirred chemostat and food-chain models in the well-stirred chemostat have been studied extensively. To the best of our knowledge, little work has been done for (3). The exact number of positive solutions to the food-chain model in the untirred chemostat remains open. Therefore, the aim of this paper is to establish the global framework, the existence, stability and uniqueness of coexistence states to (3) when m_2 is large enough. We first discuss the sufficient and necessary conditions of the existence of coexistence solutions to (3). Then, by using of degree theory and perturbation technique, the uniqueness and stability of coexistence solutions to (3) are determined. The results show that the parameter m_2 has an effect on the stability and uniqueness of coexistence states to (3). We find that all positive solutions to (3) are of only one type when m_2 is large enough.

The paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we establish the existence of coexistence states and the local stability of bifurcating positive solutions to (3). In Section 4, we investigate the stability and uniqueness of coexistence states to (3) when the parameter m_2 is large. Finally, in Section 5, we make some numerical simulations to verify the analytic results in Sections 3 and 4 and to complement the mathematical results.

2. **Preliminaries.** In this section, we first present some essential notations and well-known results which will be used in this paper.

Lemma 2.1. *(See [6].) Let $\lambda_1(q) < \lambda_2(q) \le \lambda_3(q) \le \cdots$ be all eigenvalues of the following problem*

$$-\Delta\varphi + q(x)\varphi = \lambda\varphi, \quad x \in \Omega, \quad \frac{\partial\varphi}{\partial n} + \gamma(x)\varphi = 0, \quad x \in \partial\Omega,$$

where $q(x) \in C(\overline{\Omega}), \gamma(x) \in C(\partial\Omega)$, and $\gamma(x) \ge 0$ on $\partial\Omega$. Then $\lambda_1(q)$ is simple and strictly increasing in the sense that $q_1 \le q_2$ and $q_1 \not\equiv q_2$ implies $\lambda_1(q_1) < \lambda_1(q_2)$.

Lemma 2.2. *(See [6].) Suppose $p(x) \in C(\overline{\Omega}), p(x) > 0$ on $\overline{\Omega}$ and $\gamma(x) \in C(\partial\Omega), \gamma(x) \ge 0$ on $\partial\Omega$. Then all eigenvalues of the problem*

$$\Delta\psi + \delta p(x)\psi = 0, \quad x \in \Omega, \quad \frac{\partial\psi}{\partial n} + \gamma(x)\psi = 0, \quad x \in \partial\Omega$$

can be listed in order $0 < \delta_1(p) < \delta_2(p) \le \cdots \to \infty$ with the corresponding eigenfunctions $\psi_1, \psi_2, \cdots$, where $\psi_1 > 0$ on $\overline{\Omega}$, and the principal eigenvalue $\delta_1(p) = \inf\limits_{\psi} \frac{\int_\Omega |\nabla\psi|^2 dx + \int_{\partial\Omega} \gamma(x)\psi^2 ds}{\int_\Omega p(x)\psi^2 dx}$ is simple. Furthermore, the comparison principle holds: $\delta_j(p_1) > \delta_j(p_2)$ for $j \ge 1$ if $p_1 \le p_2$ and $p_1 \not\equiv p_2$ on $\overline{\Omega}$.

Next, we introduce μ_1 as the principal eigenvalue of the following eigenvalue problem

$$\Delta\phi + \mu g(z,0)\phi = 0, \quad x \in \Omega, \quad \frac{\partial\phi}{\partial n} + \gamma\phi = 0, \quad x \in \partial\Omega,$$

where $\phi_1 > 0$ is the corresponding eigenfunction and $\|\phi_1\|_\infty = 1$.

Setting $v = 0$ in (3), we get the one-species problem

$$\Delta u + aug(z-u,u) = 0, \quad x \in \Omega, \quad \frac{\partial u}{\partial n} + \gamma u = 0, \quad x \in \partial\Omega. \tag{4}$$

One can argue in the similar way as in Lemmas 3.1-3.3 of [34] to conclude that

Lemma 2.3. *If $a \le \mu_1$, then zero is the unique nonnegative solution of (4). If $a > \mu_1$, then (4) has a unique positive solution, denoted by θ, satisfying the following properties:*

(i) $0 < \theta < z(x), x \in \overline{\Omega}$;

(ii) θ is continuously differentiable for $a \in (\mu_1, +\infty)$, and is pointwisely increasing when a increases;

(iii) Let $L = \Delta + a[g(z-\theta,\theta) - \theta g_1'(z-\theta,\theta) + \theta g_2'(z-\theta,\theta)]$ be the linearized operator of (4) at θ. Then L is a Fréchet differentiable operator in $C_B^2(\overline{\Omega}) = \{u \in C^2(\overline{\Omega}) : \frac{\partial u}{\partial n} + \gamma u = 0, x \in \partial\Omega\}$, and all eigenvalues of L are strictly negative.

3. **Existence and local stability of coexistence states.** The main purpose of this section is to study the global structure and local stability of positive solutions for (3). Obviously, (3) has only one semi-trivial solution $(\theta, 0)$ if $a > \mu_1$. Let μ_1^* be the principal eigenvalue of the following eigenvalue problem

$$\Delta\vartheta + \mu h(\theta,0)\vartheta = 0, \quad x \in \Omega, \quad \frac{\partial\vartheta}{\partial n} + \gamma\vartheta = 0, \quad x \in \partial\Omega,$$

$\vartheta_1 > 0$ is the corresponding eigenfunction and $\|\vartheta_1\|_\infty = 1$.

To state the main results, we first give the necessary conditions and a priori estimate for coexistence states to (3). By similar arguments as in Lemmas 4.1 and 4.2 of [34], we have

Lemma 3.1. *Suppose that (u,v) is the non-negative solution of (3) and $u \not\equiv 0, v \not\equiv 0$. Then*

$$(i)\ u > 0,\ v > 0,\ a > \mu_1,\ b > \mu_1^*; \quad (ii)\ u + v < z, \quad x \in \overline{\Omega}; \quad (iii)\ u \le \theta.$$

Remark 1. Lemma 3.1 indicates that (3) has no positive solution when $a \le \mu_1$ or $b \le \mu_1^*$.

Let $a > \mu_1$ be fixed, we take b as the bifurcation parameter to discuss the bifurcation solutions of (3) which are relative to the semi-trivial solution $(\theta, 0)$. In order to apply the local and global bifurcation theory, we introduce the following spaces:

$$C_B^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : \tfrac{\partial u}{\partial n} + \gamma u = 0, x \in \partial\Omega\}, \quad E = C_B^1(\overline{\Omega}) \times C_B^1(\overline{\Omega}),$$
$$P = \{(b, u, v) \in \mathrm{R}^+ \times E : u, v > 0, x \in \overline{\Omega}\}.$$

Set $U = \theta - u, V = v$. Then $0 \le U \le \theta, V \ge 0$ and U, V satisfy

$$\begin{cases} -\Delta U = a[g(z-\theta,\theta) - \theta g_1'(z-\theta,\theta) + \theta g_2'(z-\theta,\theta)]U \\ \qquad\qquad +[a\theta g_1'(z-\theta,\theta) + bh(\theta,0)]V + G_1(U,V), & x \in \Omega, \\ -\Delta V = bh(\theta,0)V + G_2(U,V), & x \in \Omega, \\ \frac{\partial U}{\partial n} + \gamma U = \frac{\partial V}{\partial n} + \gamma V = 0, & x \in \partial\Omega, \end{cases} \tag{5}$$

where

$$\begin{aligned} G_1(U,V) =& aU[g(z-\theta+U-V,\theta-U) - g(z-\theta,\theta)] - a\theta[g(z-\theta+U-V,\theta-U) \\ & - g(z-\theta,\theta) + g_2'(z-\theta,\theta)U - g_1'(z-\theta,\theta)(U-V)] \\ & + bV[h(\theta-U,V) - h(\theta,0)], \\ G_2(U,V) =& bV[h(\theta-U,V) - h(\theta,0)]. \end{aligned}$$

Let $G(U,V) = (G_1(U,V), G_2(U,V))$. Obviously, $G(0,0) = 0$, G is continuous and the Fréchet derivative $D_{(U,V)}G(0,0) = 0$. Let K be the inverse of $-\Delta$ with homogeneous Robin boundary conditions. Then (5) becomes

$$\begin{cases} U = aK[g(z-\theta,\theta) - \theta g_1'(z-\theta,\theta) + \theta g_2'(z-\theta,\theta)]U \\ \qquad +K[a\theta g_1'(z-\theta,\theta) + bh(\theta,0)]V + KG_1(U,V), & x \in \Omega, \\ V = bKh(\theta,0)V + KG_2(U,V), & x \in \Omega, \\ \frac{\partial U}{\partial n} + \gamma U = \frac{\partial V}{\partial n} + \gamma V = 0, & x \in \partial\Omega. \end{cases} \tag{6}$$

Define $T : \mathrm{R}^+ \times E \to E$ by

$$\begin{aligned} T(b,U,V) =& (aK[g(z-\theta,\theta) - \theta g_1'(z-\theta,\theta) + \theta g_2'(z-\theta,\theta)]U + K[a\theta g_1'(z-\theta,\theta) \\ & + bh(\theta,0)]V + KG_1(U,V), bKh(\theta,0)V + KG_2(U,V)). \end{aligned}$$

Then $T(b,U,V)$ is a compact differential operator on E. Let $F = I - T$. Then F is a C^1 function with $F(b,0,0) = 0$. Moreover, $F(b,U,V) = 0$ with $0 \le U \le \theta, V \ge 0$ if and only if $(b, \theta - U, V)$ is a non-negative solution of (3).

Theorem 3.2. *Let $a > \mu_1$ be fixed. Then $(\mu_1^*, \theta, 0)$ is a bifurcation point for (3) and there exist positive solutions $(u(\varepsilon), v(\varepsilon))$ of (3) on the neighborhood of $(\mu_1^*, \theta, 0)$. Moreover, the set of coexistence states of (3) close to $(\mu_1^*, \theta, 0)$ is a smooth curve*

$$\Upsilon_0 = \{(b(\varepsilon), \theta - \varepsilon(\kappa_1 + \omega(\varepsilon)), \varepsilon(\vartheta_1 + \chi(\varepsilon))) : 0 < \varepsilon < \delta\},$$

where $\kappa_1 = -L^{-1}([a\theta g_1'(z-\theta,\theta) + bh(\theta,0)]\vartheta_1), b(0) = \mu_1^*, \omega(0) = \chi(0) = 0, (\omega,\chi) \in Z, Z \oplus \mathrm{spans}\{(\kappa_1, \vartheta_1)\} = E$.

Next, we study the stability of trivial solution $(0,0)$, semi-trivial solution $(\theta, 0)$ and positive solutions $(u(\varepsilon), v(\varepsilon))$ are given by Theorem 3.2.

Theorem 3.3. *(i) If $a < \mu_1$, then trivial solution $(0,0)$ is stable; if $a > \mu_1$, then trivial solution $(0,0)$ is unstable;*

(ii) Suppose $a > \mu_1$ is fixed. If $b < \mu_1^*$, then semi-trivial solution $(\theta, 0)$ is stable; if $b > \mu_1^*$, then semi-trivial solution $(\theta, 0)$ is unstable.

Proof. (i) can be shown similarly as in the proof of (ii), and so we only prove (ii). Let L_0 be the linearized operator of (3) at $(\theta, 0)$. Then

$$L_0 = \begin{pmatrix} L & -a\theta g_1'(z-\theta,\theta) - bh(\theta,0) \\ 0 & \Delta + bh(\theta,0) \end{pmatrix}.$$

It follows from the Riesz-Schauder theory that the spectrum of L_0 consists of real eigenvalues and $\sigma(L_0) = \sigma(L) \cup \sigma(\Delta + bh(\theta,0))$. If $b < \mu_1^*$, then $\lambda_1(\Delta + bh(\theta,0)) < 0$. Hence, $\sigma(L_0) < 0$. This implies that semi-trivial solution $(\theta, 0)$ is stable. Similarly, if $b > \mu_1^*$, then $\lambda_1(\Delta + bh(\theta,0)) > 0$, which implies that semi-trivial solution $(\theta,0)$ is unstable. □

Lemma 3.4. *The differential of $b(\varepsilon)$ at $\varepsilon = 0$ satisfies*

$$b'(0)\int_\Omega h(\theta,0)\vartheta_1^2 dx = \int_\Omega \mu_1^*[h_1'(\theta,0)\kappa_1 - h_2'(\theta,0)\vartheta_1]\vartheta_1^2 dx. \tag{7}$$

Proof. Substituting $(b(\varepsilon), u(\varepsilon), v(\varepsilon)) = (b(\varepsilon), \theta - \varepsilon(\kappa_1 + \omega(\varepsilon)), \varepsilon(\vartheta_1 + \chi(\varepsilon)))$ into the second equation of (3), dividing by ε, differentiating with respect to ε, and setting $\varepsilon = 0$, we have

$$\Delta\chi'(0) + b'(0)h(\theta,0)\vartheta_1 + \mu_1^*\chi'(0)h(\theta,0) + \mu_1^* h_2'(\theta,0)\vartheta_1^2 = \mu_1^* h_1'(\theta,0)\kappa_1\vartheta_1.$$

Multiplying the above equation by ϑ_1 and integrating over Ω, we get (7). □

Let $E_1 = [C^{2,\alpha}(\overline{\Omega})]^2 \cap E, \mathbf{F} = [C^\alpha(\overline{\Omega})]^2$, $i : E_1 \to \mathbf{F}$ be the inclusion mapping, and $L(b,\theta,0)$ be the linearized operator of (3) at $(b,\theta,0)$, where $0 < \alpha < 1$. According to the definition of i-simple eigenvalue and the proof of Theorem 3.2, it follows that

Lemma 3.5. *0 is an i-simple eigenvalue of $L(\mu_1^*,\theta,0)$, and all the other eigenvalues of $L(\mu_1^*,\theta,0)$ lie in the left half complex plane.*

Let $L(b(\varepsilon), u(\varepsilon), v(\varepsilon))$ be the linearized operator of (3) at $(b(\varepsilon), u(\varepsilon), v(\varepsilon))$. It follows from [30] that

$$\lim_{\varepsilon\to 0} \frac{\varepsilon b'(\varepsilon)\tau'(\mu_1^*)}{\eta(\varepsilon)} = -1, \tag{8}$$

where $\tau(b)$ and $\eta(\varepsilon)$ are the principal eigenvalues of $L(b,\theta,0)$ and $L(b(\varepsilon),u(\varepsilon),v(\varepsilon))$, respectively. On the other hand, from Lemma 2.1, that is, the monotonicity of $\tau(b)$ with respect to b, we get $\tau'(\mu_1^*) > 0$. Since $h_1'(\theta,0) > 0, h_2'(\theta,0) < 0, \kappa_1 > 0, \vartheta_1 > 0$, it follows that $b'(0) > 0$. So, we have the following theorem.

Theorem 3.6. *Suppose $a > \mu_1$. If $0 < \varepsilon \ll 1$, then the coexistence solutions $(u(\varepsilon), v(\varepsilon))$ of (3) in a neighborhood of $(\mu_1^*,\theta,0)$ are stable.*

In the following, we shall extend the local bifurcation Υ_0 to the global bifurcation Υ. Let $b_i(\mu)(\mu \geq 1)$ be eigenvalue of the following problem

$$\mu\Delta V + b_i(\mu)h(\theta,0)V = 0, \quad x \in \Omega, \quad \frac{\partial V}{\partial n} + \gamma V = 0, \quad x \in \partial\Omega.$$

Then $b_i(\mu)$ is increasing with respect to $\mu \geq 1$ and can be ordered as

$$0 < b_1(\mu) < b_2(\mu) \leq \cdots \to \infty.$$

Moreover, $b_1(1) = \mu_1^*$. Using the same arguments as in [34], we get $i(T(b,\cdot),0) = 1$ for $b < \mu_1^*$ and $i(T(b,\cdot),0) = -1$ for $\mu_1^* < b < b_2(1)$. Therefore, by virtue of the global bifurcation theorem of Rabinowitz in [28], there exists a continuum Υ of (3) in $R \times E$ bifurcating from $(\mu_1^*, \theta, 0)$. Clearly, $\Upsilon \supset \Upsilon_0$. Furthermore, $\Upsilon - \{(\mu_1^*, \theta, 0)\}$ satisfies one of the alternatives:

(i) joins with a bifurcation point $(\bar{b}, \theta, 0)$, where $\bar{b} \neq \mu_1^*$;

(ii) joins to ∞ in $\mathrm{R}^+ \times E$;

(iii) consists of points $(b, \theta - u, v)$ and $(b, \theta + u, -v)$, where $(u, v) \neq (0, 0)$.

On the basis of the above discussions, we have the following global bifurcation results.

Theorem 3.7. *Let $a > \mu_1$ be fixed. Then*

(a) $\Upsilon - \{(\mu_1^*, \theta, 0)\}$ *joins* $(\mu_1^*, \theta, 0)$ *to* ∞ *in* P;

(b) $\{b : (b, u, v) \in \Upsilon\} = (\mu_1^*, +\infty)$.

Proof. (a) First, we show that $\Upsilon - \{(\mu_1^*, \theta, 0)\} \subset P$. We argue by contradiction. Assume that $\Upsilon - \{(\mu_1^*, \theta, 0)\} \not\subseteq P$. Then there exists a point $(\bar{b}, \bar{u}, \bar{v}) \in \Upsilon - \{(\mu_1^*, \theta, 0)\} \cap \partial P$, which is the limit of a sequence of $\{(b_i, u_i, v_i)\} \subset \Upsilon \cap P$. As $(\bar{b}, \bar{u}, \bar{v}) \in \partial P$, we see that either $\bar{u} \geq 0$, $\bar{u}(x_0) = 0$ for some $x_0 \in \overline{\Omega}$ or $\bar{v} \geq 0$, $\bar{v}(x_1) = 0$ for some $x_1 \in \overline{\Omega}$. Since

$$\Delta \bar{u} + a\bar{u}g(z - \bar{u} - \bar{v}, \bar{u}) - b\bar{v}h(\bar{u}, \bar{v}) = 0, \quad x \in \Omega, \quad \tfrac{\partial \bar{u}}{\partial n} + \gamma \bar{u} = 0, \quad x \in \partial\Omega,$$

by the strong maximum principle and Hopf lemma, we get $\bar{u} \equiv 0$. Similarly, we have $\bar{v} \equiv 0$. Obviously, $\bar{u} \equiv 0$ implies $\bar{v} \equiv 0$. We consider the following two cases:

(A) Suppose that $\bar{u} \equiv 0$ and $\bar{v} \equiv 0$. Then $(b_i, u_i, v_i) \to (\bar{b}, 0, 0)$. Let $\tilde{u}_i = \frac{u_i}{\|u_i\|_\infty}$. Then $\tilde{u}_i$ satisfies

$$\begin{cases} \Delta \tilde{u}_i + ag(z - u_i - v_i, u_i)\tilde{u}_i - \frac{b_i v_i}{(1+k_2 u_i)(1+m_2 v_i)} \tilde{u}_i = 0, & x \in \Omega, \\ \frac{\partial \tilde{u}_i}{\partial n} + \gamma \tilde{u}_i = 0, & x \in \partial\Omega. \end{cases}$$

By L^p estimates and the Sobolev embedding theorem, there exists $\tilde{u} \in C_B^1(\overline{\Omega})$ satisfying $\tilde{u} \geq 0, \not\equiv 0$ and

$$\Delta \tilde{u} + ag(z, 0)\tilde{u} = 0, \quad x \in \Omega, \quad \tfrac{\partial \tilde{u}}{\partial n} + \gamma \tilde{u} = 0, \quad x \in \partial\Omega.$$

The maximum principle implies that $\tilde{u} > 0$ on $\overline{\Omega}$. Thus, $a = \mu_1$, contrary to the assumption.

(B) Suppose that $\bar{u} \not\equiv 0$ and $\bar{v} \equiv 0$. In this case, $(b_i, u_i, v_i) \to (\bar{b}, \theta, 0)$. Then $\tilde{v}_i = \frac{v_i}{\|v_i\|_\infty}$ satisfies

$$\Delta \tilde{v}_i + b_i h(u_i, v_i)\tilde{v}_i = 0, \quad x \in \Omega, \quad \tfrac{\partial \tilde{v}_i}{\partial n} + \gamma \tilde{v}_i = 0, \quad x \in \partial\Omega.$$

Similarly, we have

$$\Delta \tilde{v} + \bar{b}h(\theta, 0)\tilde{v} = 0, \quad x \in \Omega, \quad \tfrac{\partial \tilde{v}}{\partial n} + \gamma \tilde{v} = 0, \quad x \in \partial\Omega.$$

In view of $\tilde{v} > 0$, we have $\bar{b} = \mu_1^*$, a contradiction. Consequently, we have $\Upsilon - \{(\mu_1^*, \theta, 0)\} \subset P$. So, it follows from [28] that the global bifurcation Υ extends to ∞ in P. This completes the proof of (a).

(b) By L^p estimates and the Sobolev embedding theorem we get $\|u\| \leq M$ and $\|v\| \leq M$. Hence, the only pattern for Υ to approach ∞ in P is to let b increase to ∞. Finally, by virtue of Lemma 3.1, we know that $\{b : (b, u, v) \in \Upsilon\} = (\mu_1^*, +\infty)$. The proof is completed. □

Using Lemma 3.1 and Theorem 3.7, we get the following result.

Theorem 3.8. *Assume $a > \mu_1$. Then (3) has a positive solution if and only if $b > \mu_1^*$.*

4. **Stability and uniqueness of coexistence states.** In this section, we focus on the effect of the parameter m_2 on the stability and uniqueness of coexistence states to (3).

Let $C_B(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : \frac{\partial u}{\partial n} + \gamma u = 0,\ x \in \partial\Omega\}$, $X = C_B(\overline{\Omega}) \times C_B(\overline{\Omega})$, $W = \{(u,v) \in X : u \geq 0, v \geq 0,\ x \in \overline{\Omega}\}$, $D = \{(u,v) \in W : u \leq \theta + 1, v \leq z,\ x \in \overline{\Omega}\}$. Define $A_\tau : D \to W$ by

$$A_\tau(u,v) = (-\Delta + q)^{-1} \begin{pmatrix} \tau aug(z-u-v,u) - \tau bvh(u,v) + qu \\ \tau bvh(u,v) + qv \end{pmatrix},$$

where $\tau \in [0,1]$ and q is sufficiently large positive constant such that $\tau ag(z-u-v,u) - \tau \frac{bv}{(1+k_2u)(1+m_2v)} + q > 0$ and $\tau bh(u,v) + q > 0$ for any $(u,v) \in D$. Clearly, A_τ is a compact operator. Let $A = A_1$. Then (3) has a positive solution in W if and only if A has a positive fixed point in D. Using Theorem 1 in [2], one can easily get the following lemma. The proof is fundamental, so we omit it.

Lemma 4.1. *Suppose $a > \mu_1$. Then*
(i) $\text{index}_W(A, D) = 1$;
(ii) $\text{index}_W(A, (0,0)) = 0$;
(iii) $\text{index}_W(A, (\theta,0)) = 0$ *if* $b > \mu_1^*$; $\text{index}_W(A, (\theta,0)) = 1$ *if* $b < \mu_1^*$.

Next, we discuss the asymptotic behavior of coexistence states to (3) when m_2 is large. First of all, we observe that if (u,v) is any positive solution of (3), then for large m_2, (u, m_2v) is close to the positive solution of the following problem:

$$\begin{cases} \Delta u + aug(z-u,u) = 0, & x \in \Omega, \\ \Delta w + b\bar{h}(u,w)w = 0, & x \in \Omega, \\ \frac{\partial u}{\partial n} + \gamma u = \frac{\partial w}{\partial n} + \gamma w = 0, & x \in \partial\Omega, \end{cases} \tag{9}$$

where $\bar{h}(u,w) = \frac{u}{(1+k_2u)(1+w)}$.

It is easy to see that if $a > \mu_1$, then (9) is equivalent to the following problem

$$\Delta w + b\bar{h}(\theta,w)w = 0, \quad x \in \Omega, \quad \frac{\partial w}{\partial n} + \gamma w = 0, \quad x \in \partial\Omega. \tag{10}$$

Because (10) plays an important role in determining the stability and uniqueness of coexistence states to (3), we first discuss positive solutions of (10).

Lemma 4.2. *If $a > \mu_1$, then (10) has a positive solution $\bar{w}$ if and only if $b > \mu_1^*$. Furthermore, the positive solution is unique and linearly stable.*

Proof. Assume w is any positive solution of (10). Then $\bar{h}(\theta,w) < \bar{h}(\theta,0)$. It follows from Lemma 2.2 that

$$b = \delta_1(\bar{h}(\theta,w)) > \delta_1(\bar{h}(\theta,0)) = \mu_1^*.$$

Suppose $b > \mu_1^*$. We first show that there exists $\mathbb{R} > 0$ such that $\|w\|_{C^1} \leq \mathbb{R}$ for any positive solution of (10). We argue by contradiction. Suppose that we can find $b_i \to b \geq \mu_1^*, w_i$ positive solutions of (10) with $b = b_i$ and $\|w_i\|_\infty \to \infty$, $\bar{h}(\theta,w_i) \to \zeta_1$ weakly in L^2. Set $\bar{w}_i = \frac{w_i}{\|w_i\|_\infty}$. From the equation of w_i, it follows that $\bar{w}_i$ satisfies

$$\Delta \bar{w}_i + b_i\bar{h}(\theta,w_i)\bar{w}_i = 0, \quad x \in \Omega, \quad \frac{\partial \bar{w}_i}{\partial n} + \gamma \bar{w}_i = 0, \quad x \in \partial\Omega.$$

By L^p estimates and the Sobolev embedding theorem, we may assume $\bar{w}_i \to \bar{w} \geq 0, \not\equiv 0$ in C^1. Then

$$\Delta \bar{w} + b\zeta_1\bar{w} = 0, \quad x \in \Omega, \quad \frac{\partial \bar{w}}{\partial n} + \gamma \bar{w} = 0, \quad x \in \partial\Omega.$$

By Harnack inequality, we get $\bar{w} > 0$ on $\overline{\Omega}$. Thus, $\zeta_1 = 0$ and

$$\Delta\bar{w} = 0, \quad x \in \Omega, \quad \frac{\partial \bar{w}}{\partial n} + \gamma\bar{w} = 0, \quad x \in \partial\Omega.$$

This implies that $\bar{w} = 0$, which is a contradiction. Therefore, there exists $\mathbb{R} > 0$ such that $\|w\|_{C^1} \leq \mathbb{R}$ for any positive solution of (10).

Next, we prove the existence of positive solutions. Let $\bar{K} = \{w \in C^1(\overline{\Omega}) : w \geq 0,\ \frac{\partial w}{\partial n} + \gamma w = 0,\ x \in \partial\Omega\}, \bar{D} = \{w \in \bar{K} : \|w\|_\infty \leq \mathbb{R} + 1\}$ and define

$$\mathbf{A}_\tau(w) = (-\Delta + p)^{-1}(\tau bw\bar{h}(\theta, w) + pw),$$

where $\tau \in [0, 1]$ and p is sufficiently large positive constant such that $\tau b\bar{h}(\theta, w)+p > 0$ for all $w \in \bar{D}$. It was clear that $\mathbf{A}_\tau : \bar{D} \to \bar{K}$ is compact. Let $\mathbf{A} = \mathbf{A}_1$. Then (10) has a positive solution in $\bar{K}$ if and only if $\mathbf{A}$ has a positive fixed point in $\bar{D}$. On the other hand, it follows from the Theorem 1 of [2] that $\text{index}_{\bar{K}}(\mathbf{A}, 0) = 0, \text{index}_{\bar{K}}(\mathbf{A}, \bar{D}) = 1$, which implies that (10) has at least a positive solution.

Finally, we prove that the uniqueness and stability of positive solution. Suppose w is a positive solution of (10). Then we consider the linearized eigenvalue problem of (10)

$$-\Delta\chi - b[\bar{h}(\theta, w) + w\bar{h}_2'(\theta, w)]\chi = \eta\chi, \quad x \in \Omega, \quad \frac{\partial\chi}{\partial n} + \gamma\chi = 0, \quad x \in \partial\Omega.$$

From Lemma 2.1, we get $\eta_1 = \lambda_1(-b[\bar{h}(\theta, w) + w\bar{h}_2'(\theta, w)]) > \lambda_1(-b\bar{h}(\theta, w)) = 0$. This shows that any positive solution w of (10) is non-degenerate and linearly stable. Thus, $\text{index}_{\bar{K}}(\mathbf{A}, w) = 1$. By the non-degeneracy of any positive solution to (10) and a compactness argument, we know that (10) has at most finitely many positive solutions. Let them be $\{w_i : 1 \leq i \leq n\}$. Then the additivity property of the index indicates that

$$1 = \text{index}_{\bar{K}}(\mathbf{A}, \bar{D}) = \text{index}_{\bar{K}}(\mathbf{A}, 0) + \sum_{1\leq i\leq n} \text{index}_{\bar{K}}(\mathbf{A}, w_i) = n.$$

The uniqueness follows. The proof is completed. □

As mentioned before, the next lemma shows rigorously that all positive solutions of (3) (if exist) are of only one type when m_2 is large. The work of the following lemma is motivated by Du and Lou [4]. However, due to the particularity of model, we need to adopt some proofs which are different from those in [4].

Lemma 4.3. *Suppose $a > \mu_1$. For any $\tilde{b} > \mu_1^*, \sigma > 0$ small, there exists $C = C(\tilde{b}, \sigma) > 0$ large such that if $m_2 \geq C$, $\mu_1^* < b \leq \tilde{b}$, and (u, v) is a positive solution of (3), then $\|u-\theta\|_{C^1} + \|v\|_{C^1} \leq \sigma$. Moreover, by choosing C suitably large, we have $\|m_2 v - w\|_{C^1} \leq \sigma$, where w is a positive solution of (10).*

Proof. We argue indirectly. Suppose that there exist $\tilde{b}_0 > \mu_1^*, \mu_1^* < b_i \leq \tilde{b}_0, m_{2,i} \to \infty$, and positive solutions (u_i, v_i) of (3) with $(b, m_2) = (b_i, m_{2,i})$ such that (u_i, v_i) are bounded away from $(\theta, 0)$. We may suppose that $h(u_i, v_i) \to \zeta_2$ weakly in $L^2, b_i \to b \in [\mu_1^*, \tilde{b}_0]$. In addition, by L^p estimates and the Sobolev embedding theorem, we may assume $u_i \to u, v_i \to v$ in $C^1(\overline{\Omega})$ for some $u, v \in C_B^1(\overline{\Omega})$. So, v satisfies

$$\Delta v + bv\zeta_2 = 0, \quad x \in \Omega, \quad \frac{\partial v}{\partial n} + \gamma v = 0, \quad x \in \partial\Omega.$$

If $v \geq 0, \not\equiv 0$, from the Harnack inequality, we obtain $v > 0$. Then

$$h(u_i, v_i) = \frac{u_i}{(1 + k_2u_i)(1 + m_{2,i}v_i)} \to \zeta_2 = 0.$$

Hence,

$$\Delta v = 0, \quad x \in \Omega, \quad \frac{\partial v}{\partial n} + \gamma v = 0, \quad x \in \partial\Omega,$$

the above equation leads to $v \equiv 0$ on $\overline{\Omega}$, a contradiction. We must have $v \equiv 0$. Then u satisfies

$$\Delta u + aug(z-u,u) = 0, \quad x \in \Omega, \quad \frac{\partial u}{\partial n} + \gamma u = 0, \quad x \in \partial\Omega,$$

which implies $u \equiv 0$ or $u = \theta$. If $u \equiv 0$, then $\tilde{u}_i = \frac{u_i}{\|u_i\|_\infty}$ satisfies

$$\begin{cases} \Delta\tilde{u}_i + a\tilde{u}_i g(z-u_i-v_i,u_i) - \frac{b_i v_i}{(1+k_2 u_i)(1+m_{2,i}v_i)}\tilde{u}_i = 0, & x \in \Omega, \\ \frac{\partial \tilde{u}_i}{\partial n} + \gamma\tilde{u}_i = 0, & x \in \partial\Omega. \end{cases}$$

By L^p estimates and the Sobolev embedding theorem, we may assume that $\tilde{u}_i \to \tilde{u} \geq 0, \not\equiv 0$ in $C^1(\overline{\Omega})$, and $\tilde{u} > 0$ satisfies

$$\Delta\tilde{u} + a\tilde{u}g(z,0) = 0, \quad x \in \Omega, \quad \frac{\partial\tilde{u}}{\partial n} + \gamma\tilde{u} = 0, \quad x \in \partial\Omega. \tag{11}$$

Multiplying (11) by ϕ_1 and integrating by parts, we have

$$\int_\Omega (a-\mu_1)\tilde{u}g(z,0)\phi_1 = 0,$$

which means $\tilde{u} \equiv 0$, a contradiction. Thus, $u = \theta$, which contradicts our assumption that positive solutions (u_i, v_i) of (3) are bounded away from $(\theta, 0)$.

In the following, we show that the second part. We first claim that $m_{2,i}\|v_i\|_\infty$ is uniformly bounded. Suppose that the conclusion is not true. Then there exist $\tilde{b}_0 > \mu_1^*, b_i \to b \in [\mu_1^*, \tilde{b}_0], m_{2,i} \to \infty$, and positive solutions (u_i, v_i) of (3) with $(b, m_2) = (b_i, m_{2,i})$ such that $m_{2,i}\|v_i\|_\infty \to \infty$. Let $\hat{v}_i = \frac{v_i}{\|v_i\|_\infty}$. Then $\hat{v}_i$ satisfies

$$\Delta\hat{v}_i + b_i\hat{v}_i h(u_i, v_i) = 0, \quad x \in \Omega, \quad \frac{\partial\hat{v}_i}{\partial n} + \gamma\hat{v}_i = 0, \quad x \in \partial\Omega. \tag{12}$$

By L^p estimates and the Sobolev embedding theorem again, we may suppose $\hat{v}_i \to \hat{v} \geq 0, \not\equiv 0$ in $C^1(\overline{\Omega})$. Taking the limit in (12), we see that $\hat{v}$ satisfies

$$\Delta\hat{v} + b\hat{v}\zeta_2 = 0, \quad x \in \Omega, \quad \frac{\partial\hat{v}}{\partial n} + \gamma\hat{v} = 0, \quad x \in \partial\Omega.$$

Using the Harnack inequality, we have $\hat{v} > 0$ on $\overline{\Omega}$. Then, $\zeta_2 = 0$, which means $\hat{v} = 0$, a contradiction. Hence, $m_{2,i}\|v_i\|_\infty$ is uniformly bounded. Set $m_{2,i}v_i = w_i$. Then

$$\Delta w_i + b_i w_i \bar{h}(u_i, w_i) = 0, \quad x \in \Omega, \quad \frac{\partial w_i}{\partial n} + \gamma w_i = 0, \quad x \in \partial\Omega, \tag{13}$$

where $\bar{h}(u_i, w_i) = \frac{u_i}{(1+k_2u_i)(1+w_i)}$. Since $\|w_i\|_\infty$ is uniformly bounded, by L^p estimates and the Sobolev embedding theorems again, we may assume $w_i \to w$ in $C^1(\overline{\Omega})$. Letting $i \to \infty$ in (13), we see that w satisfies (10). We divide the discussion into two cases:

Case a. $b = \mu_1^*$. It follows from equation (10) that $m_{2,i}v_i = w_i \to w = 0$. Because any positive solution of (10) is close to 0 as $b_i \to b = \mu_1^*$, it is clear that $m_{2,i}v_i \to w$, where w is a positive solution of (10).

Case b. $b > \mu_1^*$. Suppose $w \equiv 0$, that is, $(u_i, w_i) \to (\theta, 0)$. Let $\hat{w}_i = \frac{w_i}{\|w_i\|_\infty}$. Then $\hat{w}_i$ satisfies

$$\Delta\hat{w}_i + b_i\hat{w}_i\bar{h}(u_i, w_i) = 0, \quad x \in \Omega, \quad \frac{\partial\hat{w}_i}{\partial n} + \gamma\hat{w}_i = 0, \quad x \in \partial\Omega.$$

By L^p estimates and the Sobolev embedding theorem again, we may assume $\hat{w}_i \to \hat{w} \geq 0, \not\equiv 0$ in $C^1(\overline{\Omega})$. Then

$$\Delta\hat{w} + bh(\theta, 0)\hat{w} = 0, \quad x \in \Omega, \quad \frac{\partial\hat{w}}{\partial n} + \gamma\hat{w} = 0, \quad x \in \partial\Omega.$$

In view of the Harnack inequality, we get $\hat{w} > 0$. Thus, $b = \mu_1^*$, a contradiction. Hence, $w > 0$. We complete the proof. □

Based on above lemmas, we present the main results of this section which establish the stability and uniqueness of (3) when m_2 is large.

Theorem 4.4. *Suppose $a > \mu_1$ is fixed. For any $\hat{b} > \mu_1^*$, there exists $C = C(\hat{b}) > 0$ large such that if $m_2 \geq C$ and $\mu_1^* < b \leq \hat{b}$, then (3) has a unique positive solution. Moreover, it is non-degenerate and linearly stable.*

Proof. At first, we claim that any positive solution (u, v) of (3) is non-degenerate and linearly stable. Let $\tilde{v} = m_2 v, \nu = \frac{1}{m_2}$, then

$$\begin{cases} -\Delta u - aug(z - u - \nu\tilde{v}, u) + b\nu\tilde{v}\bar{h}(u, \tilde{v}) = 0, & x \in \Omega, \\ -\Delta\tilde{v} - b\tilde{v}\bar{h}(u, \tilde{v}) = 0, & x \in \Omega, \\ \frac{\partial u}{\partial n} + \gamma u = \frac{\partial \tilde{v}}{\partial n} + \gamma\tilde{v} = 0, & x \in \partial\Omega, \end{cases} \tag{14}$$

where $\bar{h}(u, \tilde{v}) = \frac{u}{(1+k_2 u)(1+\tilde{v})}$. Clearly, (u, v) is positive solution of (3) if and only if $(u, \tilde{v})$ is positive solution of (14). It suffices to show that the corresponding linearized eigenvalue problem of (14) has no eigenvalue μ with $Re(\mu) \leq 0$. To do this, a contradiction argument will be used by assuming that (14) has a coexistence state $(u_i, \tilde{v}_i)$ which is either degenerate or linearly unstable for some $\hat{b}_1 > \mu_1^*, b_i \to b \in [\mu_1^*, \hat{b}_1], m_{2,i} \to \infty$, where $i \geq 1$. Thus there exist $Re(\mu_i) \leq 0$ and $(\omega_i, \chi_i) \neq (0, 0)$ with $\|\omega_i\|_{L^2}^2 + \|\chi_i\|_{L^2}^2 = 1$ such that

$$\begin{cases} -\Delta\omega_i - a[g(z - u_i - \nu_i\tilde{v}_i, u_i) - u_i g_1'(z - u_i - \nu_i\tilde{v}_i, u_i) \\ \qquad + u_i g_2'(z - u_i - \nu_i\tilde{v}_i, u_i)]\omega_i + b_i\nu_i\tilde{v}_i\bar{h}_1'(u_i, \tilde{v}_i)\omega_i + [b_i\nu_i\bar{h}(u_i, \tilde{v}_i) \\ \qquad + \nu_i a u_i g_1'(z - u_i - \nu_i\tilde{v}_i, u_i) + b_i\nu_i\tilde{v}_i\bar{h}_2'(u_i, \tilde{v}_i)]\chi_i = \mu_i\omega_i, & x \in \Omega, \\ -\Delta\chi_i - b_i\tilde{v}_i\bar{h}_1'(u_i, \tilde{v}_i)\omega_i - b_i[\bar{h}(u_i, \tilde{v}_i) + h_2'(u_i, \tilde{v}_i)\tilde{v}_i]\chi_i = \mu_i\chi_i, & x \in \Omega, \\ \frac{\partial \omega_i}{\partial n} + \gamma\omega_i = \frac{\partial \chi_i}{\partial n} + \gamma\chi_i = 0, & x \in \partial\Omega. \end{cases} \tag{15}$$

Let $\bar{\omega}_i$ and $\bar{\chi}_i$ be the complex conjugates of ω_i and χ_i, respectively. Multiplying (15) by $\bar{\omega}_i$ and $\bar{\chi}_i$, integrating by parts and adding the two equalities, we obtain

$$\begin{aligned} &\mu_i \\ &= \int_\Omega |\nabla\omega_i|^2 dx + \gamma\int_{\partial\Omega} |\omega_i|^2 ds - \int_\Omega \{a[g(z - u_i - \nu_i\tilde{v}_i, u_i) - u_i g_1'(z - u_i - \nu_i\tilde{v}_i, u_i) \\ &\quad + u_i g_2'(z - u_i - \nu_i\tilde{v}_i, u_i)]|\omega_i|^2 - b_i\nu_i\tilde{v}_i\bar{h}_1'(u_i, \tilde{v}_i)|\omega_i|^2 - [\nu_i a u_i g_1'(z - u_i - \nu_i\tilde{v}_i, u_i) \\ &\quad + b_i\nu_i\bar{h}(u_i, \tilde{v}_i) + b_i\nu_i\tilde{v}_i\bar{h}_2'(u_i, \tilde{v}_i)]\chi_i\bar{\omega}_i\} dx + \int_\Omega |\nabla\chi_i|^2 dx + \gamma\int_{\partial\Omega} |\chi_i|^2 ds \\ &\quad - \int_\Omega \{b_i\tilde{v}_i\bar{h}_1'(u_i, \tilde{v}_i)\omega_i\bar{\chi}_i + b_i[\bar{h}(u_i, \tilde{v}_i) + h_2'(u_i, \tilde{v}_i)\tilde{v}_i]|\chi_i|^2\} dx. \end{aligned}$$

It is easy to show that $Im(\mu_i)$ and $Re(\mu_i)$ are bounded due to the boundedness of $u_i, \tilde{v}_i, b_i$, and so μ_i is bounded. We may assume $\mu_i \to \mu$ with $Re(\mu) \leq 0$. By L^p estimates, $\|\omega_i\|_{W^{2,2}}, \|\chi_i\|_{W^{2,2}}$ are bounded. Therefore, we can also assume that $\omega_i \to \omega, \chi_i \to \chi$. By passing to the limit in (15), we have

$$\begin{cases} -\Delta\omega - a[g(z - \theta, \theta) - \theta g_1'(z - \theta, \theta) + \theta g_2'(z - \theta, \theta)]\omega = \mu\omega, & x \in \Omega, \\ -\Delta\chi - bw\bar{h}_1'(\theta, w)\omega - b[\bar{h}(\theta, w) + h_2'(\theta, w)w]\chi = \mu\chi, & x \in \Omega, \\ \frac{\partial \omega}{\partial n} + \gamma\omega = \frac{\partial \chi}{\partial n} + \gamma\chi = 0, & x \in \partial\Omega. \end{cases} \tag{16}$$

If $\omega \not\equiv 0$, then μ is real and $\mu = \lambda_1(-L) > 0$ by the first equation of (16). Thus $\omega \equiv 0$ to avoid a contradiction. So, $\chi \not\equiv 0$. From the second equation of (16) and Lemma 2.1, we have

$$\mu \geq \lambda_1(-b[\bar{h}(\theta, w) + h_2'(\theta, w)w]) > \lambda_1(-b\bar{h}(\theta, w)) = 0.$$

This contradiction indicates that any positive solution of (3) is non-degenerate and linearly stable.

Finally, we prove that the uniqueness of positive solution for (3). It follows from Theorem 3.8 that (3) has positive solutions under the conditions of the theorem. Since any positive solution of (3) is non-degenerate, (3) has at most finitely many positive solutions by a compactness argument. Let them be $\{(u_i, v_i) : 1 \leq i \leq n\}$. Let $\mathfrak{L}$ be the Fréchlet derivative of A at (u_i, v_i). Since $\overline{W}_{(u_i,v_i)} = S_{(u_i,v_i)}$, $\mathfrak{L}$ does not have property α on $\overline{W}_{(u_i,v_i)}$. Due to the non-degeneracy and stability of any positive solution for (3), $I - \mathfrak{L}$ is invertible on $\overline{W}_{(u_i,v_i)}$ and $\mathfrak{L}$ has no real eigenvalue greater than one. Thus, we have $\text{index}_W(A, (u_i, v_i)) = 1$ by Theorem 1 in [2]. By virtue of Lemma 4.1 and the additivity property of the index, we get

$$1 = \text{index}_W(A, D) = 0 + \sum_{1 \leq i \leq n} \text{index}_W(A, (u_i, v_i)) = n.$$

Hence (3) has a unique positive solution for sufficiently large m_2. The proof is completed. □

5. **Numerical simulation.** In this section, we present the results of some numerical simulations which verify and complement the analytic results. We use Matlab and finite-difference method to simulate the corresponding parabolic system of (3) in one dimensional interval $\Omega = (0, 1)$:

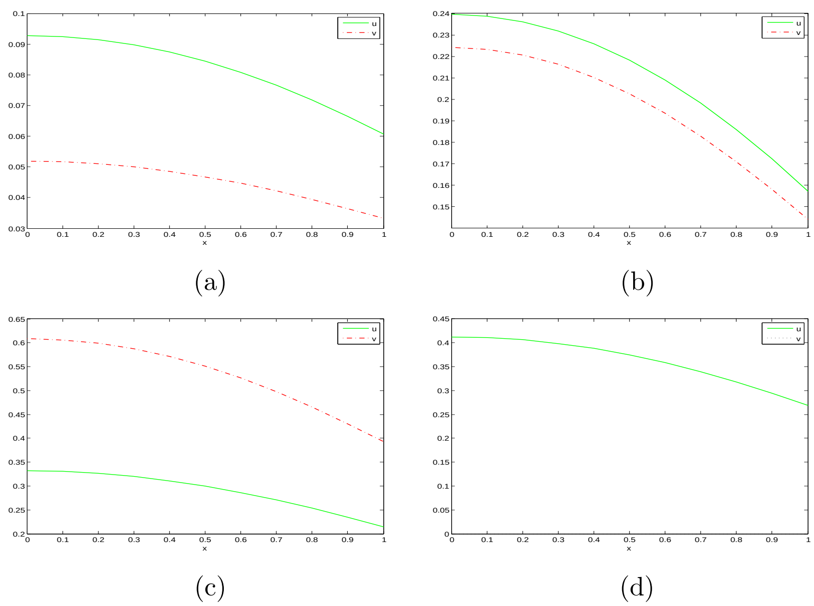
$$\begin{cases} u_t = u_{xx} + aug(z - u - v, u) - bvh(u, v), & x \in (0,1), \quad t > 0, \\ v_t = v_{xx} + bvh(u, v), & x \in (0,1), \quad t > 0, \\ u_x(0,t) = u_x(1,t) + \gamma u(1,t) = 0, & t > 0, \\ v_x(0,t) = v_x(1,t) + \gamma v(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x) \geq 0, \not\equiv 0, \ v(x,0) = v_0(x) \geq 0, \not\equiv 0, & x \in (0,1). \end{cases} \tag{17}$$

It easy to know that $z = \frac{1+\gamma}{\gamma} - x, x \in [0, 1]$. We have chosen a second-order finite-difference scheme to discretize the spatial variables in (17). The temporal variable is approximated by using a Crank-Nicholson method.

(1) The coexistence and non-coexistence of prey u and predator v are plotted in Figure 1. Several parameters values are common: $k_1 = 3, k_2 = 1, m_1 = 2, m_2 = 1, \gamma = 1$. The growth rates a and b of two populations are varied. If v can survive by itself ($b = 10, 5 > \mu_1^*$), then two populations coexist as long as growth rate a of u lies in certain range ($a = 5, 8, 15 > \mu_1 \approx 2.7068$), which is just consistent with Theorem 3.8 (see Figure 1(a)-(c)). If growth rate of u or v is small ($b = 1 < \mu_1^* \approx 2.6054$), then two populations non-coexist, see Figure 1(d). In addition, all simulations indicate that the density of u, v increase as a increases.

(2) The effects of the parameters k_1, k_2, m_1 on the density of two populations are shown in Figure 2. We take $a = 5, b = 4, \gamma = 1, m_2 = 1$. The simulation results show that as k_1 increases, the density of u and v becomes lower and lower. The parameter m_1 has a similar result. Moreover, we observe that as k_2 increases, the density of u increases, while the density of v decreases.

(3) The effect of the parameter m_2 on positive solutions is reported in Figure 3. Take $a = 5, b = 4, \gamma = 1, k_1 = 3, k_2 = 1, m_1 = 2$. Plenty of numerical simulations suggest that there is at most a stable coexistence state for large m_2. In Figure 3(e) and (f), we plot the L^1-norms of u and v versus time t. Furthermore, when m_2 is large, the coexistence state is close to $(\theta, 0)$, see Figure 3(c)-(f).



(a) (b) (c) (d)

FIGURE 1. Existence and non-existence of coexistence states. In (a), $a = 5, b = 10$. In (b) and (c), $a = 8, b = 5$ and $a = 15, b = 5$ respectively. In (d), $a = 5, b = 1$.

Acknowledgments. The authors would like to express their sincere thanks to the anonymous referees for their valuable suggestions for revision of the paper.

REFERENCES

[1] P. H. Crowley and E. K. Martin, Functional responses and interference within and between year classes of a dragonfly population, *Journal of the North American Benthological Society*, **8** (1989), 211–221.

[2] E. N. Dancer, On the indices of fixed points of mapping in cones and applications, *Journal of Mathematical Analysis and Applications*, **91** (1983), 131–151.

[3] Y. Y. Dong, S. B. Li and Y. L. Li, Multiplicity and uniqueness of positive solutions for a predator-prey model with C-M functional response, *Acta Applicandae Mathematicae*, **139** (2015), 187–206.

[4] Y. H. Du and Y. Lou, Some uniqueness and exact multiplicity results for a predator-prey model, *Transactions of The American Mathematical Society*, **349** (1997), 2443–2475.

[5] L. Dung and H. L. Smith, A parabolic system modeling microbial competition in an unmixed bio-reactor, *Journal of Differential Equations*, **130** (1996), 59–91.

[6] D. G. Figueiredo and J. P. Gossez, Strict monotonicity of eigenvalues and unique continuation, *Communications in Partial Differential Equations*, **17** (1992), 339–346.

[7] G. H. Guo, J. H. Wu and Y. E. Wang, Bifurcation from a double eigenvalue in the unstirred chemostat, *Applicable Analysis*, **92** (2013), 1449–1461.

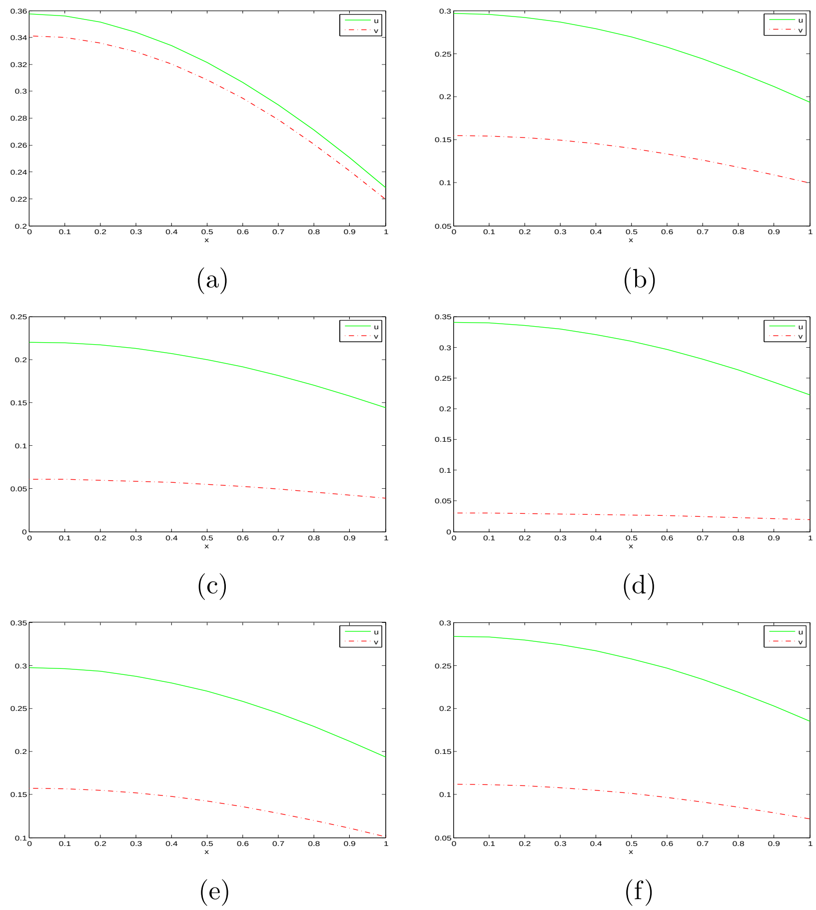


FIGURE 2. Different values of the parameters k_1, k_2, m_1. In (a) and (b), $k_2 = 1, m_1 = 2, k_1 = 1, 2$ respectively. In (c) and (d), $k_1 = 3, m_1 = 2, k_2 = 0.1, 2$ respectively. In (e) and (f), $k_1 = 3, k_2 = 1, m_1 = 0.5, 1$ respectively.

[8] H. J. Guo and S. N. Zheng, A competition model for two resources in un-stirred chemostat, *Applied Mathematics and Computation*, **217** (2011), 6934–6949.

[9] S. B. Hsu, J. F. Jiang and F. B. Wang, On a system of reaction-diffusion equations arising from competition with internal storage in an unstirred chemostat, *Journal of Differential Equations*, **248** (2010), 2470–2496.

[10] S. B. Hsu, J. P. Shi and F. B. Wang, Further studies of a reaction-diffusion system for an unstirred chemostat with internal storage, *Discrete and Continuous Dynamical Systems-Series B*, **19** (2014), 3169–3189.

[11] S. B. Hsu and P. Waltman, On a system of reaction-diffusion equations arising from competition in an unstirred chemostat, *SIAM Journal on Applied Mathematics*, **53** (1993), 1026–1044.

[12] H. X. Li, Asymptotic behavior and multiplicity for a diffusive Leslie-Gower predator-prey system with Crowley-Martin functional response, *Computers and Mathematics with Applications*, **68** (2014), 693–705.

[13] H. X. Li, Existence and multiplicity of positive solutions for an unstirred chemostat model with B-D functional response, *Chinese Journal of Engineering Methematics(China)*, **32** (2015), 369–380.

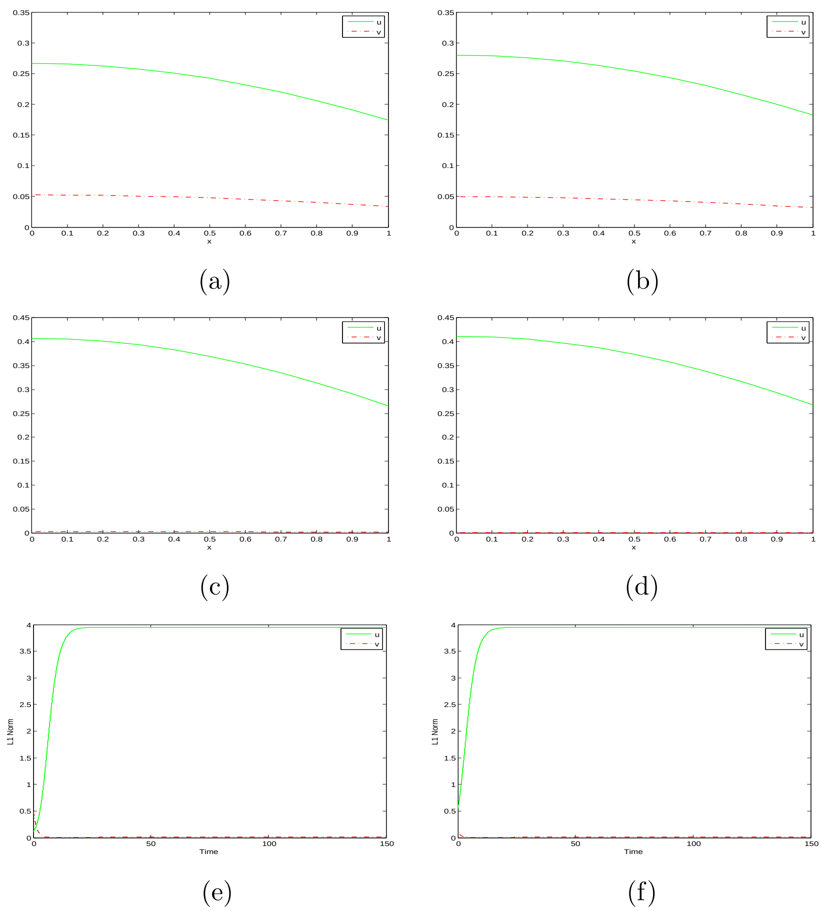


FIGURE 3. Different values of the parameter m_2. In (a)-(d), $m_2 = 1, 2, 200, 1000$ respectively. In (e) and (f), $m_2 = 500$.

[14] S. B. Li and J. H. Wu, Qualitative analysis of a predator-prey model with predator saturation and competition, *Acta Applicandae Mathematicae*, **141** (2016), 165–185.

[15] S. B. Li, J. H. Wu and Y. Y. Dong, Uniqueness and stability of a predator-prey model with C-M functional response, *Computers and Mathematics with Applications*, **69** (2015), 1080–1095.

[16] J. Liu and S. N. Zheng, A reaction-diffusion system arising from food chain in an unstirred Chemostat, *Journal of Biomathematics*, **17** (2002), 263–272.

[17] Y. Z. Lu, J. Liu, X. T. Chen and S. F. Xu, Coexistence of steady states to an annular model in an un-stirred chemostat, *International Journal of Information and Systems Sciences*, **5** (2009), 359–368.

[18] H. Nie, N. Liu and J. H. Wu, Coexistence solutions and their stability of an unstirred chemostat model with toxins, *Nonlinear Analysis: Real World Applications*, **20** (2014), 36–51.

[19] H. Nie and J. H. Wu, A system of reaction-diffusion equations in the unstirred chemostat with an inhibitor, *International Journal of Bifurcation and Chaos*, **16** (2006), 989–1009.

[20] H. Nie and J. H. Wu, Asymptotic behaviour of an unstirred chemostat with internal inhibitor, *Journal of Mathematical Analysis and Applications*, **334** (2007), 889–908.

[21] H. Nie and J. H. Wu, Positive solutions of a competition model for two resources in the unstirred chemostat, *Journal of Mathematical Analysis and Applications*, **355** (2009), 231–242.

[22] H. Nie and J. H. Wu, Coexistence of an unstirred chemostat model with Beddington-DeAngelis functional response and inhibitor, *Nonlinear Analysis: Real World Applications*, **11** (2010), 3639–3652.

[23] H. Nie and J. H. Wu, Exact multiplicity of coexistence solutions to the unstirred chemostat model with the plasmid and internal inhibitor, *Science in China: A*, **41** (2011), 497–516.

[24] H. Nie and J. H. Wu, The effect of toxins on the plasmid-bearing and plasmid-free model in the unstirred chemostat, *Discrete and Continuous Dynamical systems*, **32** (2012), 303–329.

[25] H. Nie and J. H. Wu, Multiplicity results for the unstirred chemostat model with general response functions, *Science in China: A*, **56** (2013), 2035–2050.

[26] H. Nie and J. H. Wu, Multiple coexistence solutions to the unstirred chemostat model with plasmid and toxin, *European Journal of Applied Mathematics*, **25** (2014), 481–510.

[27] H. Nie, H. W. Zhang and J. H. Wu, Characterization of positive solutions of the unstirred chemostat with an inhibitor, *Nonlinear Analysis: Real World Applications*, **9** (2008), 1078–1089.

[28] P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *Journal of Functional Analysis*, **7** (1971), 487–513.

[29] X. Y. Shi, X. Y. Zhou and X. Y. Song, Analysis of a stage-structured predator-prey model with Crowley-Martin function, *Journal of Applied Mathematics and Computing*, **36** (2011), 459–472.

[30] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, 2^{nd} edition, Springer-Verlag, New York, 1994.

[31] M. X. Wang and W. Qiang, Positive solutions of a prey-predator model with predator saturation and competition, *Journal of Mathematical Analysis and Applications*, **345** (2008), 708–718.

[32] Y. F. Wang and J. X. Yin, Predator-prey in an unstirred chemostat with periodical input and washout, *Nonlinear Analysis: Real World Applications*, **3** (2002), 597–610.

[33] M. H. Wei, J. H. Wu and G. H. Guo, The effect of predator competition on positive solutions for a predator-prey model with diffusion, *Nonlinear Analysis*, **75** (2012), 5053–5068.

[34] J. H. Wu, Global bifurcation of coexistence state for the competition model in the chemostat, *Nonlinear Analysis*, **39** (2000), 817–835.

[35] J. H. Wu, H. Nie and G. S. K. Wolkowicz, A mathematical model of competition for two essential resources in the unstirred chemostat, *SIAM Journal on Applied Mathematics*, **65** (2004), 209–229.

[36] J. H. Wu, H. Nie and G. S. K. Wolkowicz, The effect of inhibitor on the plasmid-bearing and plasmid-free model in the unstirred chemostat, *SIAM Journal on Applied Mathematics*, **38** (2007), 1860–1885.

[37] J. H. Wu and G. S. K. Wolkowicz, A system of resource-based growth models with two resources in the unstirred chemostat, *Journal of Differential Equations*, **172** (2001), 300–332.

[38] S. N. Zheng and J. Liu, Coexistence solutions for a reaction-diffusion system of un-stirred chemostat model, *Applied Mathematics and Computation*, **145** (2003), 579–590.

Received March 2016; revised February 2017.

E-mail address: xiami0820@163.com
E-mail address: jianhuaw@snnu.edu.cn
E-mail address: yanlingl@snnu.edu.cn
E-mail address: liu2006@126.com