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STABILITY ANALYSIS OF A MODEL ON VARYING DOMAIN WITH THE ROBIN BOUNDARY CONDITION

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Abstract. In this paper we develop a non-autonomous reaction-diffusion model with the Robin boundary conditions to describe insect dispersal on an isotropically varying domain. We investigate the stability of the reaction-diffusion model. The stability results of the model describe either insect survival or vanishing.

1. **Introduction.** It is well known that the reaction-diffusion mechanism on a fixed domain is one of the simplest and most elegant pattern formation models [3]. For example, Murray [12] studied the following nonlinear reaction-diffusion equation which comes from insect dispersal model

$$\begin{cases} u_t = d\Delta u + u\left(a - bu^q\right), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where $u = u(x,t)$ represents the population density of insect species, d is dispersal rate, a and b stand for intrinsic growth rate and intra-specific competition rate respectively, q is a positive constant, Ω is an open bounded subset of $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$.

However, the ecological environment is not always the same, the habitats of species usually change due to many reasons, for example, some insects live on a growing leaf, some rabbits live in an shrinking grassland due to environmental degradation. There is a wealth of research focusing on the effect of growing domain on the long time behaviors of solutions to reaction-diffusion equations or systems, see for example, [4, 6, 10, 18] and references cited therein. Recently, Tang and Lin [17] proposed the following reaction-diffusion equation on n-dimensional isotropically

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growing domain with 0-Dirichlet boundary condition

$$\begin{cases} v_t = \frac{d}{\rho^2(t)}\Delta v - \frac{n\rho'(t)}{\rho(t)} v + v\left(a - bv^q\right), & y \in \Omega(0),\ t > 0, \\ v(y,t) = 0, & y \in \partial\Omega(0),\ t > 0, \\ v(y,0) = u_0(y), & y \in \Omega(0), \end{cases}$$

where $\rho(t)$ is domain growth function satisfying $\rho \in C[0,+\infty)$, $\rho(0) = 1$, $\rho'(t) > 0$ and $\lim_{t\to+\infty} \rho(t) = \rho_\infty > 1$. They studied the asymptotic behavior of the solution to the reaction-diffusion problem by constructing upper and lower solutions.

In some cases a more realistic assumption about the region Ω would be that it is surrounded by a region that is unable to sustain a population but which is not so inhospitable that the population density outside Ω is driven to zero immediately. That modelling assumption would lead to boundary conditions of the Robin type; see, for example, [1]. Moreover, the intrinsic growth rate may vary in time, i.e., a can be a function of t. These facts give rise to the following model with the Robin boundary condition on fixed domain

$$\begin{cases} u_t = d\Delta u + u\left(a(t) - bu^q\right), & x \in \Omega,\ t > 0, \\ \frac{\partial u(x,t)}{\partial \nu} + \beta u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $a(t) : (0,+\infty) \to \mathbb{R}$ is a continuous function satisfying $\lim_{t\to+\infty} a(t) = a_\infty > 0$, $\sup_{t>0} a(t) := a_S > 0$, β is a positive constant, ν is the outward normal vector to $\partial\Omega$, $u_0(x)$ is the initial density function satisfies $u_0(x) \le (a_S/b)^{1/q}$ for any $x \in \Omega$.

The aim of this paper is to study the stability of non-autonomous parabolic equations with the Robin boundary conditions arising from the insect dispersal model on an isotropically varying domain. The reason to consider the varying domain is that the habitats of species may expand or shrink due to many reasons, for example, some fishes live in a lake whose size may be varying due to seasonal or environmental influence. We shall identify the cut-off point for either insect survival or vanishing.

The rest of this paper is organized as follows. In Section 2, we construct the insect dispersal model on an isotropically varying domain with the Robin boundary condition and give some preliminary results. Section 3 presents a number of stability results for reaction-diffusion equations with the Robin boundary condition on a varying domain.

2. **Model and preliminaries.**

Firstly, we consider the following autonomous reaction-diffusion equations with the Robin boundary condition on a fixed domain

$$\begin{cases} u_t = d\Delta u + u\left(a_\infty - bu^q\right), & x \in \Omega,\ t > 0, \\ \frac{\partial u(x,t)}{\partial \nu} + \beta u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2}$$

Let λ_1 denote the principal eigenvalue of

$$\begin{cases} -\Delta u = \lambda u, & x \in \Omega, \\ \frac{\partial u(x)}{\partial \nu} + \beta u(x) = 0, & x \in \partial\Omega. \end{cases}$$

Applying Theorems 1.1 and 4.15 of [15] to (2), we can easily obtain the following results.

Lemma 2.1. *(i) if $a_\infty \le d\lambda_1$, then (2) admits only one nonnegative steady state solution $u = 0$, which is globally asymptotically stable, that is, for any nonnegative nontrivial u_0, $\lim_{t\to+\infty} u(x,t) = 0$ uniformly on $\overline{\Omega}$;*

(ii) if $a_\infty > d\lambda_1$, then (2) has only one positive steady state solution $u = u^*(x)$, which is globally asymptotically stable, that is, for any nonnegative nontrivial u_0, $\lim_{t\to\infty} u(x,t) = u^*(x)$ uniformly on $\overline{\Omega}$.

Now, we consider the effect of varying domain with respect to t. Let $\Omega(t) \subset \mathbb{R}^n$ be a simply connected bounded varying domain at time $t \ge 0$ with its varying boundary $\partial\Omega(t)$. By the principle of mass conservation, we get that

$$\frac{d}{dt}\int_{\Omega(t)} u(x(t),t)\,dx = -\int_{\partial\Omega(t)} J\cdot\nu\,dS + \int_{\Omega(t)} f(t,u)\,dx,$$

where $u(x(t),t)$ is the density of a species at position $x(t) \in \Omega(t)$ and time $t \ge 0$, J is the flux across the boundary $\partial\Omega(t)$, ν is the outward vector on $\partial\Omega(t)$ with $|\nu| = \rho(t)$, $f(t,u)$ is the reaction term within the domain. As in [17], we have

$$u_t + \nabla u\cdot a + u(\nabla\cdot a) = d\Delta u + f(t,u) \text{ in } \Omega(t), \tag{3}$$

where d is the diffusive coefficient with a random walk, $a = (x_1'(t), x_2'(t), \dots, x_n'(t))$ is flow velocity field generated by the variety of domain.

Adding the Robin boundary condition and initial density function, we get

$$\begin{cases} u_t + \nabla u\cdot a + u(\nabla\cdot a) = d\Delta u + f(t,u), & x\in\Omega(t),\ t>0,\\ \frac{\partial u(x,t)}{\partial\nu} + \beta u(x,t) = 0, & x\in\partial\Omega(t),\ t>0,\\ u(x,0) = u_0(x), & x\in\Omega(t). \end{cases}$$

Furthermore, we assume that domain varying is uniform and isotropic, that is, the varying of the domain takes place at the same proportion in all directions as time elapses. In mathematical terms, $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ can be described as

$$(x_1(t), x_2(t), \dots, x_n(t)) = \rho(t)\left(y_1(t), y_2(t), \dots, y_n(t)\right),\ y\in\Omega(0),$$

where $\rho(t)$ is the differential varying function with $\rho(0) = 1$ and $\lim_{t\to+\infty}\rho(t) = \rho_\infty \in (0,+\infty)$.

Under the above transformation and from [17], we have known that (3) becomes

$$v_t = \frac{d}{\rho^2(t)}\Delta v - \frac{n\rho'(t)}{\rho(t)}v + f(t,v),\ y\in\Omega(0),\ t>0.$$

For any $y\in\partial\Omega(0)$, we have $x = \rho(t)y \in \partial\Omega(t)$. Let γ denote the unit normal vector to the interface of $\Omega(0)$. Then we have

$$\frac{\partial v}{\partial\gamma} = \nabla v\cdot\gamma = \rho(t)\nabla u\cdot\gamma = \frac{\partial u}{\partial\nu}.$$

Taking $f(t,s) = s\left(a(t) - bs^q\right)$ and considering the Robin boundary condition, we then have the following insect dispersal problem on the growing domain $\Omega(t)$

$$\begin{cases} v_t = \frac{d}{\rho^2(t)}\Delta v - \frac{n\rho'(t)}{\rho(t)}v + v\left(a(t) - bv^q\right), & y\in\Omega(0),\ t>0,\\ \frac{\partial v(y,t)}{\partial\gamma} + \beta v(y,t) = 0, & y\in\partial\Omega(0),\ t>0,\\ v(y,0) = v_0(y), & y\in\Omega(0). \end{cases} \tag{4}$$

The extension of the model on a growing domain with the 0-Dirichlet boundary condition into the above model presents new opportunities and challenges for mathematicians. In the context of spatial ecology and physics, extensive research on stability, effects of boundary conditions, evolution of dispersal has been done

[2, 5, 8, 11, 16]. Note that problem (4) is non-autonomous which raises some essential difficulties to the study of this kind of problems. We refer to [7, 10, 14] and their references for this kind of problems. Despite the tremendous progress in mathematical analysis of reaction-diffusion equations in the last few decades, there remains a host of unsolved mathematical problems in stability of spatial competition models, in particular, non-autonomous parabolic equations [9].

Clearly, the steady state solution equation of (4) is

$$\begin{cases} -\frac{d}{\rho_\infty^2}\Delta u = u\left(a_\infty - bu^q\right), & y\in\Omega(0),\\ \frac{\partial u(y)}{\partial\gamma} + \beta u(y) = 0, & y\in\partial\Omega(0), \end{cases} \tag{5}$$

where $u(y) = \lim_{t\to+\infty} v(y,t)$. The results of Corollary 3.14 of [2] imply that there exists a unique positive solution $u^*(y)$ of (5) if $a_\infty > d\lambda_1/\rho_\infty^2$; and there is only trivial solution of (5) if $a_\infty \le d\lambda_1/\rho_\infty^2$. Next we give the following definition of upper and lower solutions of (4).

Definition 2.2. A function $v \in C^{2,1}(\Omega(0)\times(0,+\infty)) \cap C^{1,0}\left(\overline{\Omega}(0)\times[0,+\infty)\right)$ is called an upper solution of (4) if it satisfies

$$\begin{cases} v_t \ge \frac{d}{\rho^2(t)}\Delta v - \frac{n\rho'(t)}{\rho(t)}v + v\left(a(t) - bv^q\right), & y\in\Omega(0),\ t>0,\\ \frac{\partial v(y,t)}{\partial\gamma} + \beta v(y,t) \ge 0, & y\in\partial\Omega(0),\ t>0,\\ v(y,0) \ge v_0(y), & y\in\Omega(0). \end{cases} \tag{6}$$

Similarly, v is called a lower solution of (4) if it satisfies all the reversed inequalities in (6).

Then we have the following comparison principle.

Lemma 2.3. *Let $v(y,t)$ be a solution of (4), $\widetilde{v}(y,t)$ and $\widehat{v}(y,t)$ are upper and lower solutions of (4) respectively, then $\widehat{v}(y,t) \le v(y,t) \le \widetilde{v}(y,t)$ on $\overline{\Omega}(0)\times[0,+\infty)$.*

Proof. Let $w = \widetilde{v} - \widehat{v}$. Then by Definition 2.2 and the mean value theorem,

$$\begin{cases} w_t - \frac{d}{\rho^2(t)}\Delta w + \frac{n\rho'(t)}{\rho(t)}w \ge f_v\left(t,\widehat{\eta}\right)w, & y\in\Omega(0),\ t>0\\ \frac{\partial w(y,t)}{\partial\gamma} + \beta w(y,t) \ge 0, & y\in\partial\Omega(0),\ t>0,\\ w(y,0) \ge 0, & y\in\Omega(0), \end{cases}$$

where $f(t,s) = s\left(a(t) - bs^q\right)$, $\widehat{\eta}$ is an intermediate value between $\widetilde{v}$ and $\widehat{v}$. By Lemma 2.2.1 of [13], we have $\widetilde{v} \ge \widehat{v}$. Since v may be considered as a lower solution or an upper solution the relation $\widehat{v}(y,t) \le v(y,t) \le \widetilde{v}(y,t)$ follows immediately. □

Next we give a technical result.

Lemma 2.4. *Let $v(y,t)$ be a nonnegative nontrivial solution of*

$$\begin{cases} v_t = \frac{d}{\rho^2(t)}\Delta v - \frac{n\rho'(t)}{\rho(t)}v + v\left(a(t) - bv^q\right), & y\in\Omega(0),\ t>0,\\ \frac{\partial v(y,t)}{\partial\gamma} + \beta v(y,t) = 0, & y\in\partial\Omega(0),\ t>0,\\ v(y,0) = v_0(y), & y\in\Omega(0). \end{cases} \tag{7}$$

If $v_0(y) \in C^2\left(\overline{\Omega}(0)\right)$, $\frac{\partial v_0(y)}{\partial\gamma} + \beta v_0(y) = 0$ for $y\in\partial\Omega(0)$ and $\Delta v_0(y) \le 0$ in $\Omega(0)$, then $v(y,t) \in C^{2,1}\left(\overline{\Omega}(0)\times[0,+\infty)\right)$ and $\Delta v(y,t) \le 0$ for $y\in\Omega(0)$, $t>0$.

Proof. The standard parabolic regularity theory (Theorem 1.14 of [2]) shows that $v(y,t) \in C^{2,1}\left(\overline{\Omega}(0) \times [0,+\infty)\right)$. Let $w = \Delta v$, then we have

$$w_t \leq \frac{d}{\rho^2(t)}\Delta w - \frac{n\rho'(t)}{\rho}w + a(t)w - b(q+1)v^q w.$$

Clearly, we have $w(y,0) \leq 0$ for any $y \in \Omega(0)$. In addition, for any $y \in \partial\Omega(0)$ and $t > 0$, we have

$$\frac{\partial w}{\partial \gamma} + \beta w = \frac{\rho^2 bqv^q}{d}\frac{\partial v}{\partial \gamma} \leq 0.$$

Using Lemma 2.2.1 of [13], we get that $w(y,t) \leq 0$ for any $y \in \Omega(0)$, $t > 0$, which implies that $\Delta v(y,t) \leq 0$ for $y \in \Omega(0)$, $t > 0$. □

Finally, we prove two lemmata which will be used later.

Lemma 2.5. *For any $\epsilon > 0$, let $u_\epsilon(y)$ be any positive solution of*

$$\begin{cases} -\frac{d}{(\rho_\infty-\epsilon)^2}\Delta v = \left(a_\infty - (n+1)\epsilon\right)v - bv^{q+1}, & y \in \Omega(0), \\ \frac{\partial v(y)}{\partial \gamma} + \beta v(y) = 0, & y \in \partial\Omega(0). \end{cases} \tag{8}$$

Then $\limsup_{\epsilon\to 0^+} u_\epsilon(y) = u^*(y)$, where $u^*(y)$ is the unique positive of (8) with $\epsilon = 0$.

Proof. Clearly, $(a_\infty/b)^{1/q}$ is an upper solution of (8). So we have that $u_\epsilon(y) \leq (a_\infty/b)^{1/q}$ for any $y \in \overline{\Omega}(0)$. Let $w = \limsup_{\epsilon\to 0^+} u_\epsilon(y)$. From (8), we get

$$\begin{cases} -\frac{d}{\rho_\infty^2}\Delta w = aw - bw^{q+1}, & y \in \Omega(0), \\ \frac{\partial w(y)}{\partial \gamma} + \beta w(y) = 0, & y \in \partial\Omega(0). \end{cases}$$

It follows that $w \equiv u^*$. □

Similarly to the proof of Lemma 2.5, we can obtain the following result.

Lemma 2.6. *For any $\epsilon > 0$, let $u_\epsilon(y)$ be any positive solution of*

$$\begin{cases} -\frac{d}{(\rho_\infty+\epsilon)^2}\Delta v = \left(a_\infty + (n+1)\epsilon\right)v - bv^{q+1}, & y \in \Omega(0), \\ \frac{\partial v(y)}{\partial \gamma} + \beta v(y) = 0, & y \in \partial\Omega(0). \end{cases} \tag{9}$$

Then $\liminf_{\epsilon\to 0^+} u_\epsilon(y) = u^*(y)$, where $u^*(y)$ is the unique positive of (9) with $\epsilon = 0$.

3. **Stability results on varying domain.** In this section, we shall get some asymptotically stable results about the solutions of (4) on varying domain. From Theorem 2.5.1 of [13], we can know that (4) has a unique nonnegative solution $v^*(y,t)$. The following two theorems are our main results.

Theorem 3.1. *If $a_\infty < (d\lambda_1)/\rho_\infty^2$, then (4) has no positive equilibria and any solution decay to zero uniformly on $\overline{\Omega}(0)$ as $t \to +\infty$.*

Proof. Clearly, 0 is a lower solution of (4). Now, consider the following problem

$$\begin{cases} v_t = \frac{d}{\rho^2(t)}\Delta v - \frac{n\rho'(t)}{\rho(t)}v + v\left(a(t) - bv^q\right), & y \in \Omega(0),\ t > 0, \\ \frac{\partial v(y,t)}{\partial \gamma} + \beta v(y,t) = 0, & y \in \partial\Omega(0),\ t > 0, \\ v(y,0) = M\phi(y), & y \in \Omega(0), \end{cases} \tag{10}$$

where $\phi > 0$ is the eigenfunction corresponding to λ_1 and $M > 0$ is a constant. Choose M so large that $M\phi_1(y) \geq u_0(y)$ for any $y \in \Omega(0)$. For any solution $\widetilde{v}(t,y)$

of (10), we can see that $\widetilde{v}$ is an upper solution of (4). It follows from Lemma 2.3 that

$$0 \leq v^*(y,t) \leq \widetilde{v}(y,t) \text{ for any } y \in \overline{\Omega}(0) \text{ and } t \geq 0.$$

Since $\Delta\widetilde{v}(y,0) = M\Delta\phi(y) = -M\lambda_1\phi(y) \leq 0$, it follows from Lemma 2.4 that $\Delta\widetilde{v}(y,t) \leq 0$ for $y \in \Omega(0)$, $t > 0$.

Since $\lim_{t\to+\infty}\rho(t) = \rho_\infty$, for any $\epsilon > 0$, there exists a $T_0 > 0$, such that $\rho_\infty - \epsilon \leq \rho(t) \leq \rho_\infty + \epsilon$ for $t \geq T_1$. As $\lim_{t\to+\infty} a(t) = a_\infty$, for above $\epsilon > 0$, there exists a $\widetilde{T}_0 > 0$, such that $a_\infty - \epsilon \leq a(t) \leq a_\infty + \epsilon$ for $t \geq \widetilde{T}_0$. Since $\lim_{t\to+\infty}\rho'(t)/\rho(t) = 0$, for above ϵ, there exists a $\widehat{T}_0 > 0$, such that $-\epsilon \leq \rho'(t)/\rho(t) \leq \epsilon$ for $t \geq \widehat{T}_0$. Let $T_1 = \max\left\{T_0, \widetilde{T}_0, \widehat{T}_0\right\}$. Then $\widetilde{v}(y,t)$ satisfies that

$$\widetilde{v}_t \leq \frac{d}{(\rho_\infty + \epsilon)^2}\Delta\widetilde{v} + (a_\infty + (n+\epsilon)\epsilon)\,\widetilde{v} - b\widetilde{v}^{q+1}, \; y \in \Omega(0), \; t > T_1.$$

Now consider the following problem

$$\begin{cases} v_t = \frac{d}{(\rho_\infty+\epsilon)^2}\Delta v + v\left(a_\infty + (n+1)\epsilon - bv^q\right), & y \in \Omega(0), \; t > T_1, \\ \frac{\partial v(y,t)}{\partial\gamma} + \beta v(y,t) = 0, & y \in \partial\Omega(0), \; t > T_1, \\ v\left(y, T_1\right) = \widetilde{v}\left(y, T_1\right), & y \in \Omega(0). \end{cases} \tag{11}$$

Clearly, $\widetilde{v}$ is a lower solution of (11). For the unique positive solution $\overline{v}_\epsilon(y,t)$ of (11), by Lemma 2.3, we have that

$$\widetilde{v}(y,t) \leq \overline{v}_\epsilon(y,t) \text{ for any } y \in \overline{\Omega}(0) \text{ and } t \geq T_1.$$

Choose $\epsilon > 0$ sufficiently small such that $a_\infty + (n+1)\epsilon \leq d/\left(\rho_\infty + \epsilon\right)^2$. By Lemma 2.1, we have $\overline{v}_\epsilon(y,t) \to 0$ uniformly for $y \in \overline{\Omega}(0)$ as $t \to +\infty$. Therefore, $v^*(y,t) \to 0$ uniformly for $y \in \overline{\Omega}(0)$ as $t \to +\infty$. □

Remark 1. From the proof of Theorem 3.1, we can see that if $\rho'(t) \geq 0$ for any $t > 0$ and $a(t) \equiv a_\infty$, then the conclusion also holds for the case of $a_\infty = (d\lambda_1)/\rho_\infty^2$.

Theorem 3.2. *If $a_\infty > (d\lambda_1)/\rho_\infty^2$, then (4) has a unique positive steady state solution u^*, and all solutions of (4) satisfy $v^*(y,t) \to u^*(y)$ uniformly on $\overline{\Omega}(0)$ as $t \to +\infty$.*

Proof. In view of the proof of Theorem 3.1, $a_\infty + (n+1)\epsilon > d/\left(\rho_\infty + \epsilon\right)^2$ and Lemma 2.1 imply that $\overline{v}(y,t) \to u_\epsilon^*(y)$ uniformly for $y \in \overline{\Omega}(0)$ as $t \to +\infty$, where $u_\epsilon^*(y)$ is the unique positive solution of

$$\begin{cases} -\frac{d}{(\rho+\epsilon)_\infty^2}\Delta v = v\left(a_\infty + (n+1)\epsilon - bv^q\right), & y \in \Omega(0), \\ \frac{\partial v(y)}{\partial\gamma} + \beta v(y) = 0, & y \in \partial\Omega(0). \end{cases}$$

So we have

$$\limsup_{t\to+\infty} v^*(y,t) \leq u_\epsilon^*(y) \text{ for any } y \in \overline{\Omega}(0).$$

By Lemma 2.6, we have that

$$\limsup_{t\to+\infty} v^*(y,t) \leq u^*(y) \text{ for any } y \in \overline{\Omega}(0).$$

Now let $\widehat{v}(y,t)$ be the solution of

$$\begin{cases} v_t = \frac{d}{\rho^2(t)}\Delta v - \frac{n\rho'(t)}{\rho(t)}v + v\left(a(t) - bv^q\right), & y \in \Omega(0), \; t > T_1, \\ \frac{\partial v(y,t)}{\partial\gamma} + \beta v(y,t) = 0, & y \in \partial\Omega(0), \; t > T_1, \\ v\left(y, T_1\right) = \delta\phi(y), & y \in \Omega(0), \end{cases}$$

where δ is a positive constant such that $\delta\phi(y) \leq v^*(y, T_1)$ for any $y \in \Omega(0)$. Then we can easily show that $\widehat{v}(y,t)$ is a lower solution of (4) in $\overline{\Omega}(0) \times [T_1, +\infty)$.

Since $\Delta\widehat{v}(y, T_1) = -\delta\lambda_1\phi(y) \leq 0$, it follows from Lemma 2.4 that $\Delta\widehat{v}(y,t) \leq 0$ for $y \in \Omega(0) \times (T_1, +\infty)$. Clearly, we have that

$$\widehat{v}_t \geq \frac{d}{(\rho_\infty - \epsilon)^2}\Delta\widehat{v} + ((a_\infty - (n+1)\epsilon)\,\widehat{v} - b\widehat{v}^{q+1},\ y \in \Omega(0),\ t > T_1.$$

Now consider the problem

$$\begin{cases} v_t = \frac{d}{(\rho_\infty-\epsilon)^2}\Delta v + (a_\infty - (n+1)\epsilon)\, v - bv^{q+1}, & y \in \Omega(0),\ t > T_1, \\ \frac{\partial v(y,t)}{\partial \gamma} + \beta v(y,t) = 0, & y \in \partial\Omega(0),\ t > T_1, \\ v\,(y, T_1) = \delta\phi(y), & y \in \Omega(0). \end{cases} \tag{12}$$

For the unique positive solution $\underline{v}_\epsilon(y,t)$ of (12), by the comparison principle, we have that

$$\widehat{v}(y,t) \geq \underline{v}_\epsilon(y,t) \ \text{ for any } \ y \in \overline{\Omega}(0) \ \text{ and } \ t \geq T_1.$$

Choose $\epsilon > 0$ sufficiently small such that $a_\infty > d/\,(\rho_\infty - \epsilon)^2 + (n+1)\epsilon$. By Lemma 2.1, we have $\underline{v}_\epsilon(y,t) \to u_\epsilon(y)$ uniformly for $y \in \overline{\Omega}(0)$ as $t \to +\infty$, where $u_\epsilon(y)$ is the unique positive solution of

$$\begin{cases} -\frac{d}{(\rho_\infty-\epsilon)^2}\Delta v = (a_\infty - (n+1)\epsilon)\, v - bv^{q+1}, & y \in \Omega(0), \\ \frac{\partial v(y)}{\partial \gamma} + \beta v(y) = 0, & y \in \partial\Omega(0). \end{cases}$$

Therefore, we have that

$$\liminf_{t\to+\infty} v^*(y,t) \geq u_\epsilon(y) \text{ for } \ y \in \overline{\Omega}(0).$$

By Lemma 2.5, we get that

$$\liminf_{t\to+\infty} v^*(y,t) \geq u^*(y) \ \text{ for any } \ y \in \overline{\Omega}(0).$$

Therefore, we have $v^*(y,t) \to u^*(y)$ uniformly on $\overline{\Omega}(0)$ as $t \to +\infty$. □

If $\rho(t) \equiv 1$, then problem (4) can degenerate to (1). So we have the following two corollary.

Corollary 1. *If $a_\infty < d\lambda_1$, then (1) has no positive equilibria and any solution decay to zero uniformly on $\overline{\Omega}$ as $t \to +\infty$.*

Corollary 2. *If $a_\infty > d\lambda_1$, then (1) has a unique positive steady state solution u^*, and all solutions to (4) satisfies $u(x,t) \to u^*(x)$ uniformly on $\overline{\Omega}$ as $t \to +\infty$.*

Remark 2. It is well known that $\lambda_1(\beta)$ is increasing with respect to β. So our results show that the Robin boundary condition takes a positive effect on the asymptotic stability of positive steady state solution of problem while it takes a negative effect on the asymptotic stability of the trivial solution.

Remark 3. The results of Theorem 3.1 and 3.2 show that the fact of $\rho_\infty > 1$ takes a positive effect on the asymptotic stability of positive steady state solution of problem while it takes a negative effect on the asymptotic stability of the trivial solution. On the other hand, the fact of $\rho_\infty < 1$ takes a negative effect on the asymptotic stability of positive steady state solution of problem while it takes a positive effect on the asymptotic stability of the trivial solution. These facts show that the varying (growing or shrinking) of domain in finite time or $\rho_\infty = 1$ has no effect on the asymptotic stability of positive steady state solution of problem.

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