EVOLUTION EQUATIONS AND
CONTROL THEORY
Volume **6**, Number **3**, September **2017**

doi:10.3934/eect.2017022

pp. **427–436**

SENSITIVITY ANALYSIS IN SET-VALUED OPTIMIZATION UNDER STRICTLY MINIMAL EFFICIENCY

Zhenhua Peng* and Zhongping Wan

School of Mathematics and Statistics
Wuhan University
Wuhan 430072, China

Weizhi Xiong

College of Mathematics and Science
Tongren University
Tongren 554300, China

(Communicated by Hedy Attouch)

Abstract. In this paper, the behavior of the perturbation map is analyzed quantitatively by virtue of contingent derivatives and generalized contingent epiderivatives for the set-valued maps under strictly minimal efficiency. The purpose of this paper is to provide some well-known results concerning sensitivity analysis by applying a separation theorem for convex sets. When the results regress to multiobjective optimization, some related conclusions are obtained in a multiobjective programming problem.

1. **Introduction.** It is well known that stability and sensitivity analysis is not only theoretically interesting but also practically important in approximation, optimization theory and variational problems. A number of interesting results have been provided in usual scalar optimization [5, 13]. Tanino [18, 19] proposed the concept of perturbation maps for set-valued maps and obtained some interesting results concerning sensitivity analysis in vector optimization by virtue of the contingent derivatives. The concept of the TP−derivative proposed in [16] was used to weaken some conditions in [18]. Shi [17] investigated the behavior of perturbation maps in the parametrized convex vector optimization problem. Along with this thought, many developments concerning sensitivity analysis [11, 15] are obtained.

Since the scope of the weak efficient point and efficient one is too large, the concept of proper efficient point has been developing. Fu and Chen [6], Fu and Cheng [7], Cheng and Fu [4] introduced the concept of strict efficiency and provided a perfect property: every strictly efficient point can be depicted by a strictly positive function. On the other hand, it maintains the main features of a super efficient point. Furthermore, the existence conditions of a strictly efficient point is much

2000 *Mathematics Subject Classification.* Primary: 90C29, 46G05; Secondary: 49Q12.

Key words and phrases. Sensitivity analysis, perturbation maps, contingent derivatives, generalized contingent epiderivatives, strictly minimal efficiency.

The authors are supported by the Natural Science Foundation of China No. 71471140, Natural science program of Guizhou Provincial Department of Education[2015]456 and Collaborative Fund of the Science and Teachnology Department of Guizhou Province[2014]7490.

* Corresponding author: pzhjearya@gmail.com.

weaker than that of a super efficient point. Therefore, the theoretical research of strict efficiency deserves to be developed.

In this paper, we discuss the sensitivity for the strict efficient element of a perturbation map. Let G be a set-valued map, and assume that $W(u)$ represents strictly efficient solution map of $G(u)$. The purpose of this paper is to obtain the relationship between the contingent derivative of W with respect to u and the set of strictly efficient points of the contingent derivative $DG(\hat{u},\hat{y})(u)$ and the relationship between the generalized contingent epiderivative of W with respect to u and the set of strictly efficient points of the generalized contingent epiderivative $D_gG(\hat{u},\hat{y})(u)$, respectively.

2. Contingent derivative and strictly minimal efficient elements of set-valued map.

Throughout this paper, if not otherwise stated, let U and Y be two Banach spaces, $F : U \to 2^Y$ be a set-valued map, $C \subset Y$ be a pointed closed convex cone with interior $\text{int}C \neq \emptyset$. Denote by 0_U and 0_Y the origin points of U and Y, respectively. U^* and Y^* are the topological dual spaces of U and Y, respectively. The dual cone of C is defined by $C^* = \{f \in Y^* : f(c) \leq 0, \forall c \in C\}$.

Let M be a nonempty subset of Y, denote by $\text{int}M$, $\text{cl}M$ and $\text{cone}M$ the interior, closure and cone hull of M, respectively. A nonempty convex subset B of C is called a base of C if $0 \notin \text{cl}B$ and $C = \text{cone}B = \bigcup_{\lambda \geq 0} \{\lambda b : b \in B\}$. Let $\delta := \inf\{||b|| : b \in B\}$. It follows from $0 \notin \text{cl}B$ that $\delta > 0$. Suppose that V_0 is a closed unit ball in Y, i.e., $V_0 = \{y \in Y : ||y|| \leq 1\}$. For any $\varepsilon \in (0,\delta)$, let

$$S_\varepsilon(B) := \text{cone}(\varepsilon V_0 + B),$$

$$C_\varepsilon(B) := \text{clcone}(\varepsilon V_0 + B).$$

For a set-valued map $F : U \to 2^Y$, the domain, graph and epigraph of F are defined respectively by

$$\text{dom}F := \{u \in U : F(u) \neq \emptyset\},$$

$$\text{graph}F := \{(u,y) \in U \times Y : y \in F(u)\},$$

$$\text{epi}F := \{(u,y) \in U \times Y : y \in F(u) + C\}.$$

Lemma 2.1. [12] *If* $0 \leq \varepsilon_1 < \varepsilon_2 < \delta$, *then* $C_{\varepsilon_1}(B) \setminus \{0\} \subset \text{int}C_{\varepsilon_2}(B)$.

Lemma 2.2. [2] *If* $0 < \varepsilon < \delta$, *then* $C_\varepsilon(B)$ *is a pointed closed convex cone.*

Definition 2.3. [10] Let $F : U \to 2^Y$ be a set-valued map and S be a nonempty convex subset of U. Then F is said to be $C-$convex on S if and only if for all $x_1, x_2 \in S$ and $\lambda \in [0,1]$, we have

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + C.$$

Definition 2.4. [1] Let Q be a nonempty subset of U, $\hat{u} \in \text{cl}Q$. The contingent cone of Q at $\hat{u}$, denoted by $T(Q,\hat{u})$, is given by

$$T(Q,\hat{u}) := \{u \in U : \exists t_n \to 0^+, u_n \to u, \text{such that } \hat{u} + t_n u_n \in Q, \forall n \in N\}, \quad (1)$$

where N denotes the set of positive integers.

Remark 1. [1] (1) is equivalent to

$$T(Q,\hat{u}) = \{u \in U : \exists \lambda_n \to +\infty, u_n \in Q, \text{such that } u_n \to \hat{u} \text{ and } \lambda_n(u_n - \hat{u}) \to u\}.$$

Definition 2.5. [1] Let $F : U \to 2^Y$ be a set-valued map and $(\hat{u}, \hat{y}) \in \text{graph}F$. The contingent derivative of F at $(\hat{u}, \hat{y})$ is the set-valued map $DF(\hat{u}, \hat{y}) : U \to 2^Y$ defined by

$$\text{graph}DF(\hat{u}, \hat{y}) = T(\text{epi}F, (\hat{u}, \hat{y})).$$

Definition 2.6. [3] Let $F : U \to 2^Y$ be a set-valued map and $(\hat{u}, \hat{y}) \in \text{graph}F$. The generalized contingent epiderivative of F at $(\hat{u}, \hat{y})$ is defined by

$$D_g F(\hat{u}, \hat{y})(u) := \text{Min}\{y \in Y : (u, y) \in T(\text{epi}F, (\hat{u}, \hat{y}))\}, \forall u \in U.$$

Definition 2.7. [4] Let A be a nonempty subset of Y and B be a base of C. $\bar{y} \in A$ is called a strictly efficient point of A with respect to B, written as $\bar{y} \in FE(A, B)$, if there is a neighborhood V of zero element such that

$$\text{cl}[\text{cone}(A - \bar{y})] \cap (V - B) = \emptyset.$$

Remark 2. [4] $\bar{y} \in FE(A, B)$ if and only if there is a neighborhood V of zero element such that

$$\text{cone}(A - \bar{y}) \cap (V - B) = \emptyset.$$

Definition 2.8. [1] Let $S \subset U$. A set-valued map $F : U \to 2^Y$ is called lower semicontinuous at $\hat{x} \in S$ if and only if for any $x_n \in S$ with $x_n \to \hat{x}$ and any $\hat{y} \in F(\hat{x})$, there exists a sequence $y_n \in F(x_n)$ such that $y_n \to \hat{y}$.

Definition 2.9. [14] Let A be a nonempty subset of Y. $\bar{y} \in A$ is called a weak minimal point of A, written as $\bar{y} \in \text{Wmin}(A, C)$, iff

$$(A - \bar{y})] \cap (-\text{int}C) = \emptyset.$$

Definition 2.10. [19] Let Q be a nonempty subset of U. The normal cone $N_Q(\hat{u})$ to Q at $\hat{u}$ is the negative polar cone of the tangent cone $T(Q, \hat{u})$, i.e.,

$$N_Q(\hat{u}) = [T(Q, \hat{u})]^0 = \{\varphi \in U^* : \varphi(u) \leq 0, \forall u \in T(Q, \hat{u})\}.$$

When Q is convex and $\hat{u} \in Q$, we have $N_Q(\hat{u}) = \{\varphi \in U^* : \varphi(\hat{u}) \geq \varphi(u), \forall u \in Q\}$.

Similar to Definition 5.4 in [19], we introduce the following concept.

Definition 2.11. [19] Let $A \subset Y$ and $A + C$ be a nonempty subset of Y. If $\hat{y} \in FE(A, B)$ and there exists $\varepsilon_0 \in (0, \delta)$ such that

$$N_{A+C}(\hat{y}) \subset C^*_{\varepsilon_0}(B),$$

where $C^*_{\varepsilon_0}(B) = \{\varphi \in Y^* : \varphi(y) \leq 0, \forall y \in C_{\varepsilon_0}(B)\}$, then $\hat{y}$ is called the normally strictly efficient point of A.

3. Contingent derivative of the perturbation map.

Let $G(u)$ be a set-valued map from U to Y, where U is the Banach space of a perturbation parameter vector and Y is the objective space. We define another set-valued map W from U to Y by

$$W(u) = FE(G(u), B), \tag{2}$$

for every $u \in U$, and call it the perturbation map, since it is a generalization of the perturbation map in scalar optimization, vector optimization and set-valued optimization. The purpose of this section is to investigate relationship between the contingent derivatives of W and that of G.

Definition 3.1. Let $C \subset Y$. G is said to be $C-$minicomplete by W near $\hat{u}$, iff there exists a neighborhood $V(\hat{u})$ of $\hat{u}$ such that

$$G(u) \subset W(u) + C, \forall u \in V(\hat{u}).$$

Lemma 3.2. [9] *For a cone* $K \subset Y$ *and its dual cone* $K^* = \{\varphi \in Y^* : \varphi(k) \leq 0, k \in K\}$*, we have* $\varphi(k) < 0$ *for* $\varphi \in K^* \setminus \{0_{Y^*}\}$*,* $k \in \text{int}K$*, and* $\varphi \in \text{int}K^*$*,* $k \in K \setminus \{0_Y\}$*.*

Lemma 3.3. [19] *If* $G : U \to 2^Y$ *is a* $C-$*convex set-valued map and* $\hat{u} \in \text{int}U$*, then* $G + C$ *is lower semicontinuous at* $\hat{u}$*.*

Theorem 3.4. *Suppose that* G *is* $C-$*minicomplete by* W *near* $\hat{u}$*, then for any* $u \in U$*,*
(i) $DG(\hat{u}, \hat{y})(u) \subset DW(\hat{u}, \hat{y})(u)$.
(ii) $D_gG(\hat{u}, \hat{y})(u) \subset D_gW(\hat{u}, \hat{y})(u)$.

Proof. We only prove that (i) hold since the proof of (ii) is similar to that of (i). Let $y \in DG(\hat{u}, \hat{y})(u)$, then there exist $t_n \to 0^+$, $u_n \to u$, $y_n \to y$ such that

$$\hat{y} + t_n y_n \in G\left(\hat{u} + t_n u_n\right) + C. \tag{3}$$

Consequently, for any neighborhood $V(\hat{u})$ of $\hat{u}$ there exists $N_1 \in N$ such that

$$\hat{u} + t_n u_n \in V(\hat{u}), n > N_1.$$

Since G is $C-$minicomplete by W near $\hat{u}$, one obtains

$$G\left(\hat{u} + t_n u_n\right) \subset W\left(\hat{u} + t_n u_n\right) + C,$$

which together with (3) gives

$$\hat{y} + t_n y_n \in W\left(\hat{u} + t_n u_n\right) + C + C \subset W\left(\hat{u} + t_n u_n\right) + C.$$

By the definition of the contingent derivative, one obtains

$$y \in DW(\hat{u}, \hat{y})(u).$$

□

The following example illustrates that the $C-$minicompleteness of G is essential in Theorem 3.4.

Example 1. Let $U = R$, $Y = R^2$, $C = \{(t_1, t_2) \in R^2 : t_1 \geq 0, t_2 \geq 0\}$ and $B = \{(t_1, t_2) \in R^2 : t_1 + t_2 = 1, t_1 \geq 0, t_2 \geq 0\}$. The set-valued map $G : U \to 2^Y$ is defined by

$$G(u) = \{y \in R^2 : y_1 \geq 0, y_2 > -y_1\} \cup \{(0, 0)\}, \forall u \in U.$$

Let $(\hat{u}, \hat{y}) = \{(0, (0, 0))\}$, a direct calculation gives

$$W(u) = \{(0, 0)\}, \forall u \in U,$$
$$DG(\hat{u}, \hat{y})(u) = \{y \in R^2 : y_1 \geq 0, y_2 \geq -y_1\}, \forall u \in U,$$
$$DW(\hat{u}, \hat{y})(u) = \{y \in R^2 : y_1 \geq 0, y_2 \geq 0\}, \forall u \in U,$$
$$D_gG(\hat{u}, \hat{y})(u) = \{y \in R^2 : y_1 \geq 0, y_2 = -y_1\}, \forall u \in U,$$
$$D_gW(\hat{u}, \hat{y})(u) = \{(0, 0)\}, \forall u \in U.$$

Naturally, G is not $C-$mincomplete by W near $\hat{u}$, and for any $u \in U$, we have

$$DG(\hat{u}, \hat{y})(u) \not\subset DW(\hat{u}, \hat{y})(u),$$
$$D_gG(\hat{u}, \hat{y})(u) \not\subset D_gW(\hat{u}, \hat{y})(u).$$

Theorem 3.5. *Suppose that* G *is* $C-$*minicomplete by* W *near* $\hat{u}$*, then for any* $u \in U$*,*
(i) $FE(DG(\hat{u}, \hat{y})(u), B) + C \subset DW(\hat{u}, \hat{y})(u)$.
(ii) $FE(D_gG(\hat{u}, \hat{y})(u), B) \subset D_gW(\hat{u}, \hat{y})(u)$.

Proof. We only prove that (i) hold since the proof of (ii) is similar to that of (i). Let $y \in FE(DG(\hat{u},\hat{y})(u),B)+C$, Then there exist $\bar{y} \in FE(DG(\hat{u},\hat{y})(u),B)$ and $c \in C$ such that $y = \bar{y} + c$. It follows from Theorem 3.4 that

$$\bar{y} \in DG(\hat{u},\hat{y})(u) \subset DW(\hat{u},\hat{y})(u).$$

Consequently, there exist $t_n \to 0^+$, $u_n \to u$, $y_n \to \bar{y}$ such that

$$\hat{y} + t_n y_n \in W(\hat{u} + t_n u_n) + C.$$

From $c \in C$ and C is a convex cone, it follows that

$$\begin{aligned} \hat{y} + t_n(y_n + c) \quad &= \hat{y} + t_n y_n + t_n c \\ &\in W(\hat{u} + t_n u_n) + C + C \\ &\subset W(\hat{u} + t_n u_n) + C. \end{aligned}$$

Since $y_n + c \to \bar{y} + c$ and $u_n \to u$, one obtains

$$y + c \in T(\mathrm{epi}W, (\hat{u},\hat{y})).$$

Therefore

$$y = \bar{y} + c \in DW(\hat{u},\hat{y})(u).$$

□

Example 2. Let $U = Y = \mathbb{R}$ and $C = \mathbb{R}_+$. Consider the following set-valued map:

$$G(u) = \begin{cases} \{-1\}, & \text{if } u < 0, \\ \{\sqrt{u}, u^2\}, & \text{otherwise.} \end{cases}$$

Let $(\hat{u},\hat{y}) = \{(0,0)\}$ and $B = \{1\}$, a direct calculation gives

$$W(u) = \begin{cases} \{-1\}, & \text{if } u < 0, \\ \{u^2\}, & \text{if } 0 \le u \le 1, \\ \{\sqrt{u}\}, & \text{otherwise.} \end{cases}$$

$$DG(0,0)(u) = DW(0,0)(u) = \begin{cases} \mathbb{R}, & \text{if } u \le 0, \\ \{y : y \ge 0\}, & \text{otherwise.} \end{cases}$$

$$D_g G(0,0)(u) = D_g W(0,0)(u) = \begin{cases} 0, & \text{if } u > 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Consequently, the conclusions of Theorem 3.5 hold.

Theorem 3.6. *Suppose that $G(u)$ is a $C-$convex set-valued map, $\hat{u} \in \mathrm{int}U$, $\hat{y} \in \mathrm{Wmin}(DG(\hat{u},\hat{y})(\hat{u}),C)$, C has a compact base B, U^* and Y^* are * weak compact, and there exists a $\varepsilon_0 \in (0,\delta)$ such that*

$$N_{G(\hat{u})+C}(\hat{y}) \subset C^*_{\varepsilon_0}(B).$$

Then
(i) $DW(\hat{u},\hat{y})(u) \subset FE(DG(\hat{u},\hat{y})(u),B)+C, \forall u \in U.$
(ii) $D_g W(\hat{u},\hat{y})(u) \subset FE(D_g G(\hat{u},\hat{y})(u),B), \forall u \in U.$

Proof. We only prove that (i) hold since the proof of (ii) is similar to that of (i). Let $y \in DW(\hat{u},\hat{y})(u)$, from Definition 2.7 and (2), it follows that $W(u) \subset G(u)$, hence $DW(\hat{u},\hat{y})(u) \subset DG(\hat{u},\hat{y})(u)$, then $y \in DG(\hat{u},\hat{y})(u)$. If $y \notin FE(DG(\hat{u},\hat{y})(u),B)+C$, then for any $c \in C$ one has $y - c \notin FE(DG(\hat{u},\hat{y})(u),B)$. Thus for any $V \in N(0)$, one has

$$\mathrm{cone}(DG(\hat{u},\hat{y})(u) - y + c) \cap (V - B) \neq \emptyset. \tag{4}$$

From $y \in DW(\hat{u}, \hat{y})(u)$, it follows that there exist $t_k \to 0^+$, $(u_k, y_k) \to (u, y)$, such that

$$\hat{y} + t_k y_k \in W(\hat{u} + t_k u_k) + C.$$

Consequently, there exists $c_k \in C$ such that

$$\hat{y} + t_k y_k - c_k \in W(\hat{u} + t_k u_k).$$

Hence there exist $\alpha_k \geq 0$ and $b_k \in B$ such that

$$c_k = \alpha_k b_k$$

and

$$\hat{y} + t_k y_k - \alpha_k b_k \in W(\hat{u} + t_k u_k),$$

which implies that $(\hat{u} + t_k u_k, \hat{y} + t_k y_k - \alpha_k b_k)$ is boundary points of the convex set epi$G$. By a separation theorem for convex sets, there exists $(\varphi_k, \phi_k) \in U^* \times Y^* \setminus \{0_{U^*}, 0_{Y^*}\}$ such that

$$\varphi_k(u') + \phi_k(y') \leq \varphi_k(\hat{u} + t_k u_k) + \phi_k(\hat{y} + t_k y_k - \alpha_k b_k), \forall (u', y') \in \text{epi}G. \tag{5}$$

By the assumption that U^* and Y^* are * weak compact, we may assume without loss of generality that $(\varphi_k, \phi_k) \to^{*w} (\varphi, \phi) \neq \{0_{U^*}, 0_{Y^*}\}$. In (5), let $k \to +\infty$, one obtains

$$\varphi(u') + \phi(y') \leq \varphi(\hat{u}) + \phi(\hat{y}) - \phi(\alpha b), \forall (u', y') \in \text{epi}G,$$

where $\alpha_k \to \alpha$ and $b_k \to b$. If $\phi = 0_{Y^*}$, then $\varphi \neq 0_{U^*}$ and $\varphi(u') \leq \varphi(\hat{u})$. Since $\hat{u} \in \text{int}U$, then $\varphi = 0_{U^*}$. Therefore $\phi \neq 0_{Y^*}$.

From Lemma 3.3 it follows that $G + C$ is lower semicontinuous at $\hat{u}$. For any $\tilde{y} \in G(\hat{u}) + C$ there exists a sequence $\tilde{y}_k \in G(\hat{u} + t_k u_k) + C$ such that $\tilde{y}_k \to \tilde{y}$. It follows from (5) that

$$\varphi_k(\hat{u} + t_k u_k) + \phi_k(\tilde{y}_k) \leq \varphi_k(\hat{u} + t_k u_k) + \phi_k(\hat{y} + t_k y_k - \alpha_k b_k).$$

Let $k \to +\infty$, one obtains

$$\phi(\tilde{y}) \leq \phi(\hat{y}) - \phi(\alpha b).$$

Then

$$\phi \in N_{G(\hat{u})+C+\alpha b}(\hat{y}) \subset N_{G(\hat{u})+C}(\hat{y}),$$

which together with $N_{G(\hat{u})+C}(\hat{y}) \subset C^*_{\varepsilon_0}(B)$ gives

$$\phi \in C^*_{\varepsilon_0}(B) \setminus \{0_{Y^*}\}. \tag{6}$$

By (4), $0 < \varepsilon' < \varepsilon_0 < \delta$, and $-\varepsilon' V_0 \in N(0)$, one has

$$\text{cone}(DG(\hat{u}, \hat{y})(u) - y + c) \cap (-\varepsilon' V_0 - B) \neq \emptyset.$$

From $0_Y \notin -\varepsilon' V_0 - B$, it follows that there exist $t_0 > 0$ and $y_0 \in DG(\hat{u}, \hat{y})(u)$ such that

$$t_0(y_0 - y + c) \in -\varepsilon' V_0 - B.$$

Then

$$t_0(y_0 - y + c) \in C_{\varepsilon'}(B) \setminus \{0_Y\}.$$

From Lemma 2.1, it follows that

$$t_0(y_0 - y + c) \in -\text{int}C_{\varepsilon_0}(B).$$

Since $-\text{int}C_{\varepsilon_0}(B)$ is a cone, one obtains

$$y_0 - y + c \in -\text{int}C_{\varepsilon_0}(B).$$

It follows from (6) and Lemma 3.2 that

$$\phi(y_0) - \phi(y) + \phi(c) > 0. \tag{7}$$

Since $c \in C \subset C_{\varepsilon_0}(B)$ and (6), one obtains

$$\phi(c) < 0.$$

Which together with (7) give

$$\phi(y_0) - \phi(y) > 0. \tag{8}$$

Since $y_0 \in DG(\hat{u}, \hat{y})(u)$, there exist $\bar{t}_k \to 0^+$, $(\bar{u}_k, \bar{y}_k) \to (u, y_0)$ such that

$$\hat{y} + \bar{t}_k \bar{y}_k \in G(\hat{u} + \bar{t}_k \bar{u}_k) + C. \tag{9}$$

It follows from $\hat{y} \in \mathrm{Wmin}(DG(\hat{u}, \hat{y})(\hat{u}), B)$ that $\hat{y} \in DG(\hat{u}, \hat{y})(\hat{u})$. Then there exist $\hat{t}_k \to 0^+$, $(\hat{u}_k, \hat{y}_k) \to (\hat{u}, \hat{y})$ such that

$$\hat{y} + \hat{t}_k \hat{y}_k \in G(\hat{u} + \hat{t}_k \hat{u}_k) + C. \tag{10}$$

Since $t_k \to 0^+$, we may assume $t_k \leq \bar{t}_k$ and $t_k \leq \hat{t}_k$ by taking a subsequence if necessary. By (9), $\hat{y} \in G(\hat{u})$ and $t_k \leq \bar{t}_k$, one obtains

$$(\hat{u} + t_k \bar{u}_k, \hat{y} + t_k \bar{y}_k) = \frac{t_k}{\bar{t}_k}(\hat{u} + \bar{t}_k \bar{u}_k, \hat{y} + \bar{t}_k \bar{y}_k) + (1 - \frac{t_k}{\bar{t}_k})(\hat{u}, \hat{y}) \in \mathrm{epi}G.$$

From (5) it follows that

$$\varphi_k(\hat{u} + t_k \bar{u}_k) + \phi_k(\hat{y} + t_k \bar{y}_k) \leq \varphi_k(\hat{u} + t_k u_k) + \phi_k(\hat{y} + t_k y_k - \alpha_k b_k).$$

Then

$$\varphi_k(\bar{u}_k) + \phi_k(\bar{y}_k) \leq \varphi_k(u_k) + \phi_k(y_k - \frac{\alpha_k}{t_k} b_k). \tag{11}$$

We shall prove that $\frac{\alpha_k}{t_k} \to 0$. Suppose to the contrary, then for some $\varepsilon_0' > 0$, we may assume without loss of generality that $\frac{\alpha_k}{t_k} \geq \varepsilon_0'$, by taking a subsequence if necessary. By (10), $\hat{y} \in G(\hat{u})$ and $t_k \leq \hat{t}_k$, one obtains

$$(\hat{u} + t_k \hat{u}_k, \hat{y} + t_k \hat{y}_k) = \frac{t_k}{\hat{t}_k}(\hat{u} + \hat{t}_k \hat{u}_k, \hat{y} + \hat{t}_k \hat{y}_k) + (1 - \frac{t_k}{\hat{t}_k})(\hat{u}, \hat{y}) \in \mathrm{epi}G.$$

Consequently, there exist $\hat{\alpha}_k \geq 0$ and $\hat{b}_k \in B$ such that

$$\hat{y} + t_k \hat{y}_k - \hat{\alpha}_k \hat{b}_k \in G(\hat{u} + t_k \hat{u}_k),$$

where $\hat{\alpha}_k \to \hat{\alpha}$ and $\hat{b}_k \to \hat{b}$. Then

$$\begin{aligned} \hat{y} + t_k \left(\hat{y}_k - \varepsilon_0' \hat{b}_k\right) &= \hat{y} + t_k \hat{y}_k - \hat{\alpha}_k \hat{b}_k + \hat{\alpha}_k \hat{b}_k - \varepsilon_0' t_k \hat{b}_k \\ &\in G(\hat{u} + t_k \hat{u}_k) + C. \end{aligned}$$

Since $\hat{y}_k - \varepsilon_1 \hat{b}_k \to \hat{y} - \varepsilon_0' \hat{b}$, one obtains

$$\hat{y} - \varepsilon_0' \hat{b} \in DG(\hat{u}, \hat{y})(\hat{u}).$$

However, this contradicts the assumption

$$\hat{y} \in \mathrm{Wmin}(DG(\hat{u}, \hat{y})(\hat{u}), C).$$

By taking limits of (11) with $k \to +\infty$, we have

$$\phi(y_0) \leq \phi(y),$$

which is a contradiction to (8).Therefore

$$y \in FE(DG(\hat{u}, \hat{y})(u), B) + C.$$

□

Similar to the proof of Theorem 3.6, the following Theorem is obtained.

Theorem 3.7. *Suppose that $G(u)$ is a $C-$convex set-valued map, $\hat{u} \in \mathrm{int}U$, $\hat{y}$ is a normally strictly efficient point of $DG(\hat{u},\hat{y})(\hat{u})$, C have a compact base B, U^* and Y^* are * weak compact. Then*
(i) $DW(\hat{u},\hat{y})(u) \subset FE(DG(\hat{u},\hat{y})(u),B)+C, \forall u \in U.$
(ii) $D_gW(\hat{u},\hat{y})(u) \subset FE(D_gG(\hat{u},\hat{y})(u),B), \forall u \in U.$

We provide the following example to explain Theorems 3.6 and 3.7.

Example 3. Let $U = Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. Consider the following set-valued map:

$$G(u) = \begin{cases} \{(-1,-1)\}, & \text{if } u_1 < 0, u_2 < 0, \\ \{(-1,\sqrt{u_2}),(-1,u_2)\}, & \text{if } u_1 < 0, u_2 \ge 0, \\ \{(\sqrt{u_1},-1),(u_1,-1)\}, & \text{if } u_1 \ge 0, u_2 < 0, \\ \{(\sqrt{u_1},u_2^2),(u_1,u_2^2)\}, & \text{if } u_1 \ge 0, u_2 \ge 0. \end{cases}$$

Let $(\hat{u},\hat{y}) = ((0,0),(0,0))$ and $B = \{(y_1,y_2) \in R^2 : y_1+y_2 = 1\}$, a direct calculation gives

$$W(u) = \begin{cases} \{(-1,-1)\}, & \text{if } u_1 < 0, u_2 < 0, \\ \{(-1,u_2)\}, & \text{if } u_1 < 0, 0 \le u_2 \le 1, \\ \{(-1,\sqrt{u_2})\}, & \text{if } u_1 < 0, u_2 > 1, \\ \{(u_1,-1)\}, & \text{if } 0 \le u_1 \le 1, u_2 < 0, \\ \{(\sqrt{u_1},-1)\}, & \text{if } u_1 > 1, u_2 < 0, \\ \{(u_1,u_2^2)\}, & \text{if } 0 \le u_1 \le 1, u_2 \ge 0, \\ \{(\sqrt{u_1},u_2^2)\}, & \text{if } u_1 > 1, u_2 \ge 0. \end{cases}$$

$$\begin{aligned} & DG((0,0),(0,0))(u) = DW((0,0),(0,0))(u) \\ = & \begin{cases} \mathbb{R}^2, & \text{if } u_1 \le 0, u_2 \le 0, \\ \{(y_1,y_2) : y_1 \in \mathbb{R}, y_2 \ge u_2\}, & \text{if } u_1 \le 0, u_2 > 0, \\ \{(y_1,y_2) : y_1 \ge u_1, y_2 \in \mathbb{R}\}, & \text{if } u_1 > 0, u_2 \le 0, \\ \{(y_1,y_2) : y_1 \ge u_1, y_2 \ge 0\}, & \text{if } u_1 > 0, u_2 > 0. \end{cases} \end{aligned}$$

$$\begin{aligned} & D_gG((0,0),(0,0))(u) = D_gW((0,0),(0,0))(u) \\ = & \begin{cases} \{(y_1,y_2) : y_1 \ge u_1, y_2 \ge 0\}, & \text{if } u_1 > 0, u_2 > 0, \\ \emptyset, & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, it is easy to verify that the conclusions of Theorems 3.6 and 3.7 hold.

Theorem 3.8. *Suppose that $G(u)$ is a $C-$convex set-valued map, G is $C-$mini-complete by W near $\hat{u}$, $\hat{u} \in \mathrm{int}U$, $\hat{y} \in \mathrm{Wmin}(DG(\hat{u},\hat{y})(\hat{u}),C)$, C has a compact base B, U^* and Y^* are * weak compact, and there exists $\varepsilon_0 \in (0,\delta)$ such that*

$$N_{G(\hat{u})+C}(\hat{y}) \subset C^*_{\varepsilon_0}(B).$$

Then
(i) $DW(\hat{u},\hat{y})(u) = DG(\hat{u},\hat{y})(u) = FE(DW(\hat{u},\hat{y})(u),B)+C$
$\qquad = FE(DG(\hat{u},\hat{y})(u),B)+C, \forall u \in U.$
(ii) $D_gW(\hat{u},\hat{y})(u) = D_gG(\hat{u},\hat{y})(u) = FE(D_gW(\hat{u},\hat{y})(u),B)+C$
$\qquad = FE(D_gG(\hat{u},\hat{y})(u),B)+C, \forall u \in U.$

Proof. We only prove that (i) hold since the proof of (ii) is similar to that of (i). From Theorem 3.4 and its proof, it follows that

$$FE(DG(\hat{u},\hat{y})(u),B)+C \subset DG(\hat{u},\hat{y})(u) \subset DW(\hat{u},\hat{y})(u).$$

Which together with the conclusions of Theorem 3.6 gives

$$FE(DG(\hat{u},\hat{y})(u),B)+C = DG(\hat{u},\hat{y})(u) = DW(\hat{u},\hat{y})(u).$$

Then

$$\begin{aligned} FE(DW(\hat{u},\hat{y})(u),B)+C &= FE(DG(\hat{u},\hat{y})(u),B)+C \\ &= DG(\hat{u},\hat{y})(u) = DW(\hat{u},\hat{y})(u). \end{aligned}$$

□

Similar to the proof of Theorem 3.8, the following Theorem is obtained.

Theorem 3.9. *Suppose that $G(u)$ is a $C-$convex set-valued map, G is $C-$minicomplete by W near $\hat{u}$, $\hat{u} \in \mathrm{int}U$, $\hat{y}$ is a normally strictly efficient point of $DG(\hat{u},\hat{y})(\hat{u})$, C have a compact base B, U^* and Y^* are * weak compact. Then*
(i) $DW(\hat{u},\hat{y})(u) = DG(\hat{u},\hat{y})(u) = FE(DW(\hat{u},\hat{y})(u),B)+C$
$= FE(DG(\hat{u},\hat{y})(u),B)+C, \forall u \in U.$
(ii) $D_gW(\hat{u},\hat{y})(u) = D_gG(\hat{u},\hat{y})(u) = FE(D_gW(\hat{u},\hat{y})(u),B)+C$
$= FE(D_gG(\hat{u},\hat{y})(u),B)+C, \forall u \in U.$

4. **Sensitivity analysis in multiobjective programming.** In this section, we apply the preceding results to the following multiobjective programming problem:

$$\begin{aligned} \min \quad & f(x) = (f_1(x), f_2(x), \ldots, f_m(x)) \\ \text{s.t.} \quad & g(x) = (g_1(x), g_2(x), \ldots, g_p(x)) \le 0, \\ & h(x) = (h_1(x), h_2(x), \ldots, h_q(x)) = 0, x \in \mathbb{R}^n. \end{aligned}$$

Let X be the set-valued map from $\mathbb{R}^{p+q}$ to $\mathbb{R}^n$ defined by

$$X(u,v) = \{x \in \mathbb{R}^n : g(x) \le u, h(x) = v\}, \text{ for } u \in \mathbb{R}^p \text{ and } v \in \mathbb{R}^q.$$

Consequently, the feasible set map G from $\mathbb{R}^{p+q}$ to $\mathbb{R}^m$ defined by

$$G(u,v) = f(X(u,v)) = \{y \in \mathbb{R}^m : y = f(x), g(x) \le u, h(x) = v\}.$$

Obviously, the nominal values of the parameter vector u and v are 0. Taking $\hat{x} \in X(0,0)$, and denoting by $J(\hat{x}) := \{j : g_j(\hat{x}) = 0\}$ the index set of the active constraints at $\hat{x}$. Assuming that $f_i, i = 1,2,\ldots,m$, $g_j, j = 1,2,\ldots,p$ and $h_k, k = 1,2,\ldots,k$ are continuously differentiable, we can apply Theorems 3.5, 3.6 and 3.7 to obtain the well-known results in the multiobjective programming problem.

Theorem 4.1. *Suppose that G is $C-$minicomplete by W near $\hat{u}$, then for any $u \in U$,*
(i) $FE(\nabla f(\hat{x})DX(0,\hat{x})(u),B)+C \subset DW(0,\hat{y})(u)$.
(ii) $FE(\nabla f(\hat{x})D_gX(0,\hat{x})(u),B) \subset D_gW(0,\hat{y})(u)$,
where $DX(0,\hat{x})(u) = \{x : \langle \nabla g_j(\hat{x}), x\rangle \le u_j, \langle \nabla h(\hat{x}), x\rangle = v, \text{ for all } j \in J(\hat{x})\}$.

Theorem 4.2. *Suppose that $G(u)$ is a $C-$convex set-valued map, $\hat{u} \in \mathrm{int}U$, $\hat{y}$ is a normally strictly efficient point of $DG(\hat{u},\hat{y})(\hat{u})$, C have a compact base B, U^* and Y^* are * weak compact. Then*
(i) $DW(0,\hat{y})(u) \subset FE(\nabla f(\hat{x})DX(0,\hat{x})(u),B)+C, \forall u \in U.$
(ii) $D_gW(0,\hat{y})(u) \subset FE(\nabla f(\hat{x})D_gX(0,\hat{x})(u),B), \forall u \in U,$
where $DX(0,\hat{x})(u) = \{x : \langle \nabla g_j(\hat{x}), x\rangle \le u_j, \langle \nabla h(\hat{x}), x\rangle = v, \text{ for all } j \in J(\hat{x})\}$.

REFERENCES

[1] J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.
[2] J. M. Borwein and D. Zhuang, Super efficiency in vector optimization, *Transactions of the American Mathematical Society*, **338** (1993), 105–122.
[3] G. Y. Chen and J. Jahn, Optimality conditions for set-valued optimization problems, *Mathematical Methods of Operations Research*, **48** (1998), 187–200.
[4] Y. H. Cheng and W. T. Fu, Strong efficiency in a locally convex space, *Mathematical Methods of Operations Research*, **50** (1999), 373–384.

[5] A. V. Fiacco, *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming*, Academic Press, New York, 1983.

[6] W. T. Fu and X. Q. Chen, On approximation families of cones and strictly efficient points, *Acta Mathematica Sinica*, **40** (1997), 933–938.

[7] W. T. Fu and Y. H. Cheng, On the strict efficiency in a locally convex space, *Journal of Systems Science and Mathematical Sciences*, **12** (1999), 40–44.

[8] X. H. Gong, H. B. Dong and S. Y. Wang, Optimality conditions for proper efficient solutions of vector set-valued optimization, *Journal of Mathematical Analysis and Applications*, **284** (2003), 332–350.

[9] Y. D. Hu and C. Ling, Connectedness of cone superefficient point sets in locally convex topological vector spaces, *Journal of Optimization Theory and Applications*, **107** (2000), 433–446.

[10] J. Jahn, *Vector Optimization: Theory, Applications, and Extensions*, Springer, Berlin Heidelberg, 2004.

[11] S. J. Li, Sensitivity and stability for contingent derivative in multiobjective optimization, *Mathematica Applicata*, **11** (1998), 49–53.

[12] Z. F. Li and S. Y. Wang, Connectedness of super efficient sets in vector optimization of set-valued maps, *Mathematical Methods of Operations Research*, **48** (1998), 207–217.

[13] R. T. Rockafellar, Lagrange multipliers and subderivatives of optimal value functions in nonlinear programming, *Mathematical Programming Study*, **17** (1982), 28–66.

[14] W. D. Rong and Y. N. Wu, $\varepsilon-$weak minimal solution of vector optimization problems with set-valued maps, *Journal of Optimization Theory and Applications*, **106** (2000), 569–579.

[15] B. H. Sheng and S. Y. Liu, Sensitivity analysis in vector optimization under Benson proper efficiency, *Journal of Mathematical Research & Exposition*, **22** (2002), 407–412.

[16] D. S. Shi, Contingent derivative of the perturbation map in multiobjective optimization, *Journal of Optimization Theory and Applications*, **70** (1991), 385–396.

[17] D. S. Shi, Sensitivity analysis in convex vector optimization, *Journal of Optimization Theory and Applications*, **77** (1993), 145–159.

[18] T. Tanino, Sensitivity analysis in multiobjective optimization, *Journal of Optimization Theory and Applications*, **56** (1988), 479–499.

[19] T. Tanino, Stability and sensitivity analysis in convex vector optimization, *SIAM Journal on Control and Optimization*, **26** (1988), 521–536.

Received December 2015; revised May 2017.

E-mail address: pzhjearya@gmail.com
E-mail address: mathwanzhp@whu.edu.cn
E-mail address: xiongweizhi@hotmail.com