

ON THE EXISTENCE OF OPTIMAL CONTROL FOR
SEMILINEAR ELLIPTIC EQUATIONS WITH NONLINEAR
NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. An optimal control problem governed by a class of semilinear elliptic equations with nonlinear Neumann boundary conditions is studied in this paper. It is pointed out that the cost functional considered may not be convex. Using a relaxation method, some existence results of an optimal control are obtained.

1. Introduction. Existence of optimal controls is an important topic in the theory of optimal controls, and it has some actual sense in the practice. It is well known to researchers working in the field of optimal control theory that to guarantee the existence of optimal controls we need an assumption of convexity on a control domain and cost functional. Along this line, many results are available. We refer the readers to the books by Cesari [5], Li and Yong [10] and the papers [11], [15] for further details.

In the absence of convexity, the problem will be more difficult. To the best of our knowledge, the first result on the existence of optimal controls in the absence of convexity was established by Neustadt [19] for the finite-dimensional linear systems. Later, more general cases were studied in the literatures (see [2], [3], [4], [21] and the references cited therein, for example). However, for the infinite-dimensional case, relevant results are not rich enough. The readers can refer to [7], [8], [9], [20]. When a convex condition is not assumed, relaxed controls have been proved to be an important tool for studying existence of optimal controls (see [12], [13], [14], [16], [17], for example). The concept of relaxed controls has evolved from generalized curves introduced by Young [24] and McShane [18] in the late 1930s. The largest advantage of relaxation is that the space of admissible controls can be extended to a larger space, and both the control system and the cost functional are convexified.

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With the aid of relaxed controls, Lou [12], [13] discussed the existence and nonexistence of optimal controls for linear elliptic equations. Later, motivated by the idea presented in [12], [13], the author [16] considered the case of semilinear elliptic controlled system.

This present paper can be considered as a sequent work of [16] and there are three main differences between them. (i) In this paper, we will consider a state equation with a nonlinear Neumann boundary condition not a Dirichlet boundary condition. (ii) The control domain $[0,1]$ in [16] is generalized to $[a,b]$ here. (iii) The present existence results are more general. The technique used in this paper is similar to that in [12] or [16], however, the presence of the nonlinear Neumann boundary condition will bring out some troubles when analyzing the boundary-value of some functions. For this reason, we cannot obtain the expected nonexistence results and only get the existences. To prove the existence theorems, we first consider the optimal relaxed control problem corresponding to the initial (classical) control problem, and then give the existence and the maximum principle for optimal relaxed controls. This process is standard since the optimal relaxed control problem has been convexified. The next step is devoted to analyzing the maximum principle for an optimal relaxed control and a solution of the adjoint system detailedly. Finally, to obtain the existence of optimal controls it is sufficient to prove that the support of an optimal relaxed control is a singleton almost everywhere. Here we use the fact that when an optimal relaxed control is a Dirac measure almost everywhere, then it essentially becomes an optimal control.

The rest of this paper is organized as follows. In Section 2, we give the formulation of the optimal control problem and the main results. Section 3 is devoted to presenting the existence and the maximum principle for an optimal relaxed control. The proofs of the main results are presented in Section 4. Finally, Section 5 is a appendix in which two basic results are proved.

2. Formulation of the optimal control problem and the main results. Let Ω be a bounded domain of \mathbb{R}^n with smooth boundary Γ . Consider the following elliptic controlled system:

$$\begin{cases} -\Delta y = f(y) + u & \text{in } \Omega, \\ \partial_\nu y = g(y) & \text{on } \Gamma, \end{cases} \quad (1)$$

where ∂_ν is the derivative in the direction of the outward normal ν on Γ , the function u is a control, and y is a state.

Let the control domain $U = [a, b]$, and the set of admissible controls \mathcal{U} be defined by

$$\mathcal{U} \equiv \{u : \Omega \rightarrow U \mid u \text{ is measurable}\}.$$

We impose the following assumptions on f and g :

$$(S) \quad f \in C^1(\mathbb{R}), f(0) > 0, \quad f'(y) \leq -r_1, \quad \forall y \in \mathbb{R}, \quad (2)$$

for some constant $r_1 > 0$, and there exists a constant $C_1 > 0$ such that

$$|f(y)| \leq C_1(1 + |y|^q) \quad (3)$$

for $q > 0$, and, moreover, for any $K > 0$, there exists a constant $M_K > 0$ such that

$$|f'(y)| \leq M_K, \quad \forall |y| \leq K. \quad (4)$$

The function g has the same properties as f with (2) replaced by

$$g'(y) \leq -r_2, \forall y \in \mathbb{R},$$

for some constant $r_2 \geq 0$.

Now we introduce the following cost functional:

$$J(u) = \int_{\Omega} (Ay^2 + Byu + Cu^2)dx, \tag{5}$$

where (y, u) is a state-control pair, and A, B, C are given constants.

Our optimal control problem can be stated as follows:

(P) Find a control $\bar{u} \in \mathcal{U}$ such that

$$J(\bar{u}) = \inf_{u \in \mathcal{U}} J(u). \tag{6}$$

Any \bar{u} satisfying (6) is said of an optimal control.

The purpose of this paper is to prove the existence of an optimal control \bar{u} under the different conditions. Before stating the main results, we first introduce the following two systems:

$$\begin{cases} -\Delta \xi = f(\xi) + a & \text{in } \Omega, \\ \partial_{\nu} \xi = g(\xi) & \text{on } \Gamma, \end{cases} \tag{7}$$

and

$$\begin{cases} -\Delta \eta = f(\eta) + b & \text{in } \Omega, \\ \partial_{\nu} \eta = g(\eta) & \text{on } \Gamma. \end{cases} \tag{8}$$

For the simplification of form, our main results are stated separately according to the signs of A, B being same or not. It is mentioned that when discussing the existence only the case of $AB \geq 0, A + B \neq 0, C < 0$ was studied in [16].

Theorem 2.1. *Suppose that (S) holds and $C < 0$. Let ξ, η satisfy (7), (8), respectively, $l_1 = \inf_{x \in \Omega} \xi(x)$, and $l_2 = \sup_{x \in \Omega} \eta(x)$. Then any one of the following conditions is satisfied, Problem (P) admits at least one optimal control.*

(i) $b > a \geq 0, A < 0, B > 0$, and

$$\frac{-Bf(l_2) - 2Ba}{2l_2} \leq A \leq \frac{-Bb}{2l_1}. \tag{9}$$

(ii) $b > a \geq 0, A > 0, B < 0$, and

$$A \geq \frac{-Bb}{l_1}. \tag{10}$$

(iii) $a < b \leq 0, A < 0, B > 0$, and

$$\frac{-Bb}{2l_1} \leq A \leq \frac{-Ba}{l_2}. \tag{11}$$

(iv) $a < b \leq 0, A > 0, B < 0$, and

$$\frac{-Ba}{2l_2} \leq A \leq \frac{-Bf(l_1) - 2Bb}{2l_1}. \tag{12}$$

Theorem 2.2. *Suppose that (S) holds and $C < 0$. Let ξ, η, l_1 and l_2 be defined as above. Then any one of the following conditions is satisfied, Problem (P) admits at*

least one optimal control.

(i) $b > a \geq 0$, $A, B > 0$, and

$$A \leq \frac{-Bf(l_1) - 2Bb}{2l_2}. \tag{13}$$

(ii) $b > a \geq 0$, $A, B < 0$, and (13) holds for “ \leq ” being replaced by “ \geq ”.

(iii) $a < b \leq 0$, $A, B > 0$, and

$$A \leq \frac{-Bf(l_2) - 2Ba}{2l_1}. \tag{14}$$

(iv) $a < b \leq 0$, $A, B < 0$, and (14) holds for “ \leq ” being replaced by “ \geq ”.

Remark 1. If $C \geq 0$, then Cesari’s condition holds and Problem **(P)** admits at least one optimal control.

3. Existence and maximum principle for optimal relaxed controls. This section is mainly devoted to considering the optimal relaxed control problem. Firstly, it is necessary to recall the concept of relaxed controls and the relations between classical controls and relaxed controls.

Let $C(U)$ be the space of continuous functions endowed with the maximum norm, and $\mathcal{M}(U) = C(U)^*$ is the space of Radon measures in U . The subset of $\mathcal{M}(U)$ formed by the probability measures in U is denoted by $\mathcal{M}_+^1(U)$. In addition, we denote by \mathcal{R} the subset of the Banach space $L^\infty(\Omega; \mathcal{M}(U)) = L^1(\Omega; C(U))^*$ formed by all $\mathcal{M}_+^1(U)$ -valued $C(U)$ -weakly measurable functions in Ω . That is to say that $\sigma \in \mathcal{R}$ if and only if

$$\sigma(x) \in \mathcal{M}_+^1(U) \text{ a.e. } x \in \Omega,$$

and

$$x \mapsto \int_U h(v)\sigma(x)(dv) \text{ is measurable for any } h \in C(U).$$

Each element of \mathcal{R} will be called a relaxed control. Respectively, an element of \mathcal{U} is called a classical control.

It is well known that the set of relaxed controls \mathcal{R} is convex and sequentially compact; moreover, \mathcal{U} is dense in \mathcal{R} with the weak star topology of $L^\infty(\Omega; \mathcal{M}(U))$ (see [22], Theorem IV.2.1, p. 272, and Theorem IV.2.6, p. 275). Furthermore, we mention that $\sigma_k \rightarrow \sigma$ in \mathcal{R} means that

$$\int_\Omega dx \int_U h(x, v)\sigma_k(x)(dv) \rightarrow \int_\Omega dx \int_U h(x, v)\sigma(x)(dv), \quad \forall h \in L^1(\Omega; C(U)).$$

We now state the optimal relaxed control problem corresponding to Problem **(P)** in the following way.

(RP). Find a relaxed control $\bar{\sigma} \in \mathcal{R}$ such that

$$J(\bar{\sigma}) = \inf_{\sigma \in \mathcal{R}} J(\sigma),$$

where

$$J(\sigma) = \int_\Omega dx \int_U (Ay^2 + Byv + Cv^2)\sigma(x)(dv), \tag{15}$$

and y satisfies the following relaxed controlled system:

$$\begin{cases} -\Delta y = f(y) + \int_U v\sigma(x)(dv) & \text{in } \Omega, \\ \partial_\nu y = g(y) & \text{on } \Gamma. \end{cases} \tag{16}$$

It is mentioned that \mathcal{U} can be embedded in \mathcal{R} by identifying each $u \in \mathcal{U}$ with the Dirac measure-valued function $\delta_u \in \mathcal{R}$. Moreover $J(\delta_u)$ defined by (15) coincides with $J(u)$ defined by (5).

The following proposition is concerned with the existence and uniqueness of a solution of (16). This result is basic and the proof is similar to that of Theorem 6.11 in [10] (p.78). For the readers' convenience, we will present the proof in the last section of this paper.

Proposition 1. *Let (S) hold. Then for any $1 \leq p < \infty$ and $\sigma \in \mathcal{R}$, (16) admits a unique solution $y \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover, there exists a constant K_p independent of σ , such that*

$$\|y\|_{W^{1,p}(\Omega) \cap L^\infty(\Omega)} \leq K_p. \tag{17}$$

The existence theorem and maximum principle for optimal relaxed controls can be established in essentially the same way as those for classical control problems.

Lemma 3.1. *Suppose that (S) holds. Then Problem (RP) admits at least one solution.*

Proof. By (17), we have that

$$\bar{J} \equiv \inf_{\sigma \in \mathcal{R}} J(\sigma) > -\infty.$$

Let $\sigma_k \in \mathcal{R}$ satisfy

$$\lim_{k \rightarrow +\infty} J(\sigma_k) = \bar{J}.$$

Since \mathcal{R} is convex and sequentially compact, we can suppose that σ_k converges to some $\bar{\sigma}$ in \mathcal{R} . By (17) and the Sobolev embedding theorem, we can suppose that

$$y_k \rightarrow \bar{y} \text{ in } C(\bar{\Omega}) \tag{18}$$

for some $\bar{y} \in W^{1,p}(\Omega)$ with $p > n$, where y_k is the solution of (16) corresponding to σ_k . Finally, by (S) and (18), we can get that \bar{y} is the solution of (16) corresponding to $\bar{\sigma}$, and $\bar{\sigma}$ is an optimal relaxed control. \square

Lemma 3.2. *Suppose that (S) holds. Let $\bar{\sigma}$ be an optimal relaxed control of Problem (RP), and \bar{y} be the optimal state corresponding to $\bar{\sigma}$, i.e.,*

$$\begin{cases} -\Delta \bar{y} = f(\bar{y}) + \int_U v \bar{\sigma}(x)(dv) & \text{in } \Omega, \\ \partial_\nu \bar{y} = g(\bar{y}) & \text{on } \Gamma. \end{cases} \tag{19}$$

Then there exists a $\bar{\psi} \in W^{1,p}(\Omega)$ ($p \geq 1$), such that

$$\begin{cases} -\Delta \bar{\psi} = f'(\bar{y})\bar{\psi} - \left[2A\bar{y} + B \int_U v \bar{\sigma}(x)(dv) \right] & \text{in } \Omega, \\ \partial_\nu \bar{\psi} = g'(\bar{y})\bar{\psi} & \text{on } \Gamma, \end{cases} \tag{20}$$

and

$$\text{supp } \bar{\sigma}(x) \subseteq \left\{ w \in U \mid H(\bar{y}(x), w, \bar{\psi}(x)) = \max_{v \in U} H(\bar{y}(x), v, \bar{\psi}(x)) \right\} \text{ for a.e. } x \in \Omega, \tag{21}$$

where

$$H(y, w, \psi) = w\psi - (Byw + Cw^2), \quad \forall (y, w, \psi) \in \mathbb{R} \times U \times \mathbb{R}. \tag{22}$$

Sketch of the proof. Let $\bar{\sigma}$ be an optimal relaxed control, then for any $\sigma \in \mathcal{R}$, and $\alpha \in (0, 1)$, we have $\sigma^\alpha = \bar{\sigma} + \alpha(\sigma - \bar{\sigma}) \in \mathcal{R}$ since \mathcal{R} is convex. Let y^α be the state corresponding to σ^α , then we can prove that

$$y^\alpha = \bar{y} + \alpha Y + o(\alpha),$$

where

$$\begin{cases} -\Delta Y = f'(\bar{y})Y + \int_U v(\sigma(x) - \bar{\sigma}(x))(dv) & \text{in } \Omega, \\ \partial_\nu Y = g'(\bar{y})Y & \text{on } \Gamma. \end{cases}$$

On the other hand, by optimality,

$$0 \leq \frac{J(\sigma^\alpha) - J(\bar{\sigma})}{\alpha}.$$

Passing to the limit, we get

$$0 \leq \int_\Omega dx \int_U [(2A\bar{y} + Bv\bar{\sigma}(x))Y + (B\bar{y}v + Cv^2)(\sigma(x) - \bar{\sigma}(x))] (dv).$$

Let $\bar{\psi}$ be the solution of the adjoint equation of (20), then we have that

$$\int_\Omega dx \int_U [(\bar{\psi}v - B\bar{y}v - Cv^2)(\sigma(x) - \bar{\sigma}(x))] (dv) \leq 0.$$

Thus, (21) follows from the above inequality immediately. □

Before finishing this section, we introduce another useful lemma, which can be found in [12] (see Lemma 2.3).

Lemma 3.3. *Let M be a constant. If $\phi \in W^{m,1}(\Omega)$, $m \geq 1$, then*

$$\partial^\rho \phi(x) = 0 \text{ a.e. on } \{x \in \Omega \mid \phi(x) = M\} \quad \forall 1 \leq |\rho| \leq m.$$

Here and later, for simplicity, the set $\{x \in \Omega \mid \phi(x) = M\}$ will be denoted by $\{\phi = M\}$. Especially, by Lemma 3.3, if $\phi \in W^{2,1}(\Omega)$, then we have that

$$-\Delta \phi(x) = 0 \text{ a.e. on } \{\phi = M\} \tag{23}$$

for any constant M .

4. The proofs of the main results. Before proving the main results, it is necessary to present the following comparison principle which will be proved in the next section.

Proposition 2. *Suppose that (S) holds. Let $y_1, y_2 \in W^{2,p}(\Omega)$ satisfy the inequalities*

$$\begin{cases} -\Delta y_1 - f(y_1) \geq -\Delta y_2 - f(y_2) & \text{in } \Omega, \\ \partial_\nu y_1 - g(y_1) \geq \partial_\nu y_2 - g(y_2) & \text{on } \Gamma. \end{cases}$$

Then $y_1 \geq y_2$. Moreover, if $y_1 \neq y_2$ then $y_1(x) > y_2(x)$ for every $x \in \Omega$.

Now, we begin to prove Theorem 2.1.

Proof of Theorem 2.1. Without loss of generality, we only prove (i), and the other cases are similar. Let us begin to prove (i). It follows from (3), (4), (17) and L^p estimate of elliptic equations that $\bar{y} \in W^{2,p}(\Omega)$ for any $p \geq 1$. Thus, $\bar{y} \in C^{1,\alpha}(\bar{\Omega})$ ($\alpha \in (0, 1 - \frac{n}{p})$) for $p > n$ by the Sobolev embedding theorem. Hence,

$$\left| 2A\bar{y} + B \int_U v\bar{\sigma}(x)(dv) \right| \leq L$$

for some constant $L > 0$. Similar to the above arguments, we have that

$$\bar{\psi} \in W^{2,p}(\Omega) \cap C^{1,\alpha}(\bar{\Omega}) \text{ for } p > n \text{ and } \alpha \in (0, 1 - \frac{n}{p}).$$

That is, we have that

$$\bar{y}, \bar{\psi} \in W^{2,p}(\Omega) \cap C^{1,\alpha}(\bar{\Omega}) \text{ for } p > n \text{ and } \alpha \in (0, 1 - \frac{n}{p}). \tag{24}$$

Now, let us make an observation on (21). Let

$$\bar{\varphi}(x) = \bar{\psi}(x) - B\bar{y}(x) \text{ for } x \in \Omega, \tag{25}$$

then

$$\bar{\varphi} \in W^{2,p}(\Omega) \cap C^{1,\alpha}(\bar{\Omega}) \text{ for } p > n \text{ and } \alpha \in (0, 1 - \frac{n}{p}) \tag{26}$$

by (24).

Now, let us observe (21). Since $C < 0$,

$$\frac{\partial^2}{\partial w^2} H(y, w, \psi) = -2C > 0 \quad \forall (y, w, \psi) \in \mathbb{R} \times U \times \mathbb{R}.$$

Thus, it follows from (21) that

$$\text{supp } \bar{\sigma}(x) \subseteq \begin{cases} \{b\}, & \text{if } \bar{\varphi}(x) > C(a+b), \\ \{a\}, & \text{if } \bar{\varphi}(x) < C(a+b) \\ \{a, b\}, & \text{if } \bar{\varphi}(x) = C(a+b), \end{cases} \text{ for a.e. } x \in \Omega. \tag{27}$$

To prove that Problem **(P)** admits one solution, we only need to prove that for almost all $x \in \Omega$, $\text{supp } \bar{\sigma}(x)$ is a singleton. By (27), it is sufficient to prove that for almost all $x \in \{\bar{\varphi} = C(a+b)\}$, $\text{supp } \bar{\sigma}(x)$ is a singleton. We claim that for almost all $x \in \{\bar{\varphi} = C(a+b)\}$,

$$\text{supp } \bar{\sigma}(x) = \{b\}.$$

Otherwise, if we define the set $E \equiv \{\text{supp } \bar{\sigma}(x) \neq \{b\}\} \cap \{\bar{\varphi} = C(a+b)\}$, then

$$|E| > 0, \tag{28}$$

where $|E|$ denotes the Lebesgue measure of E . Consequently, by (25), (26), (23), (19) and (20), for almost all $x \in E$, it holds that

$$\begin{aligned} 0 &= -\Delta \bar{\varphi} = -\Delta \bar{\psi} + B\Delta \bar{y} \\ &= f'(\bar{y})\bar{\psi} - 2A\bar{y} - Bf(\bar{y}) - 2B \int_U v\bar{\sigma}(x)(dv). \end{aligned} \tag{29}$$

Let ξ, η be defined by (7), (8), respectively. Then Proposition 2 shows that

$$0 < \xi \leq \bar{y} \leq \eta \text{ in } \Omega \tag{30}$$

since $b > a \geq 0, f(0), g(0) > 0$, and (30) implies that $l_1 \neq 0$. Furthermore, noting that $A < 0$, and $B > 0$, then by (9) and (30), we have that

$$- \left[2A\bar{y} + B \int_U v\bar{\sigma}(x)(dv) \right] \geq -(2Al_1 + Bb) \geq 0 \text{ in } \Omega,$$

which means that $\bar{\psi} \geq 0$ in Ω . Then the first term on the right side of (29)

$$f'(\bar{y})\bar{\psi} \leq 0 \text{ in } \Omega \tag{31}$$

since $f'(\bar{y}) < 0$ by (2). On the other hand, it follows from (28) and (30) that

$$\bar{y} < \eta \text{ in } E. \tag{32}$$

Therefore, by the monotonicity of f , (32) and (9), we have that

$$-2A\bar{y} - Bf(\bar{y}) - 2B \int_U v\bar{\sigma}(x)(dv) < -2Al_2 - Bf(l_2) - 2Ba \leq 0 \text{ in } E. \quad (33)$$

Combing (29), (31) and (33), for almost all $x \in E$, we have that

$$0 = -\Delta\bar{\varphi} = f'(\bar{y})\bar{\psi} - 2A\bar{y} - Bf(\bar{y}) - 2B \int_U v\bar{\sigma}(x)(dv) < 0.$$

This is a contradiction. Hence, the measure of E is zero, and we conclude the proof of (i). \square

Proof of Theorem 2.2. Theorem 2.2 can be proved using the similar arguments in the proof of Theorem 2.1 (i), and therefore we omit it. \square

Appendix. In this section, we will prove Proposition 1 and Proposition 2.

The proof of Proposition 1. Step 1. Let $1 \leq p < \infty$ and $m > 0$. We define

$$f_m(y) = \begin{cases} f(-m), & y < -m, \\ f(y), & |y| \leq m, \\ f(m), & y > m. \end{cases}$$

In the same way, we define another function g_m . Consider the following truncated problem:

$$\begin{cases} -\Delta y_m = f_m(y_m) + \int_U v\sigma(x)(dv) & \text{in } \Omega, \\ \partial_\nu y_m = g_m(y_m) & \text{on } \Gamma. \end{cases} \quad (34)$$

Now, for any $z \in L^p(\Omega)$, by (3) and the definitions of f_m and g_m , we see that

$$|f_m(z(x))|, |g_m(z(x))| \leq M_m$$

for some constant $M_m > 0$. Then by L^p theory for linear elliptic equations, there exists a unique solution $z_m \in W^{1,p}(\Omega)$ of the following equation (with fixed z):

$$\begin{cases} -\Delta z_m = f_m(z) + \int_U v\sigma(x)(dv) & \text{in } \Omega, \\ \partial_\nu z_m = g_m(z) & \text{on } \Gamma. \end{cases} \quad (35)$$

Moreover,

$$\|z_m\|_{W^{1,p}(\Omega)} \leq L \left(\|f_m(z(\cdot))\|_{L^p(\Omega)} + \left\| \int_U v\sigma(\cdot)(dv) \right\|_{L^p(\Omega)} + \|g_m(z(\cdot))\|_{L^p(\Gamma)} \right) \leq L_p,$$

where $L_p > 0$ is a constant, which, in particular, is independent of z . Thus, we see that the map $z \mapsto z_m$ defined through (35) is continuous and compact from some fixed ball in $L^p(\Omega)$ into itself. Hence by the Schauder fixed-point theorem, there exists a fixed point y_m of this map. Clearly, $y_m \in W^{1,p}(\Omega)$ is a solution of (35).

Step 2. For any $m > 0$, we will prove $\|y_m\|_{L^\infty(\Omega)} \leq K$, where the constant $K > 0$ is independent of m . It is mentioned that this uniform estimate can be deduced from [23] (Theorem 4.1) and which is mainly based on the Moser iteration technique. For the readers' convenience, next we only deal with the special cases of $q = 1$ in (3) and $n \geq 2$, however, it is sufficient for the readers to understand the main idea.

First we are going to show that $y_m^+ = \max\{y_m, 0\}$ belongs to $L^\infty(\Omega)$. For $M > 0$ we define $v_{m,M}(x) = \min\{y_m^+(x), M\}$. Letting $K(t) = t$ if $t \leq M$ and $K(t) = M$

if $t > M$, then $K \circ y_m^+ = v_{m,M} \in W^{1,2}(\Omega)$ and hence $v_{m,M} \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$. For real $k \geq 0$ we choose $\varphi = v_{m,M}^{2k+1}$, then $\nabla \varphi = (2k+1)v_{m,M}^{2k} \nabla v_{m,M}$ and $\varphi \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$. Notice that $y_m(x) \leq 0$ implies that directly $v_{m,M}(x) = 0$. Therefore, multiplying (34) by $\varphi = v_{m,M}^{2k+1}$ and then integrating in Ω by parts, we have that

$$(2k+1) \int_{\Omega} \nabla y_m^+ \cdot \nabla v_{m,M} v_{m,M}^{2k} dx = \int_{\Omega} \left[f_m(y_m^+) + \int_U v \sigma(x)(dv) \right] v_{m,M}^{2k+1} dx + \int_{\Gamma} g_m(y_m^+) v_{m,M}^{2k+1} d\sigma. \quad (36)$$

Defining the three terms in (36) to be I_1, I_2 and I_3 , respectively, then one gets

$$\begin{aligned} I_1 &= (2k+1) \int_{\Omega} \nabla y_m^+ \cdot \nabla v_{m,M} v_{m,M}^{2k} dx \\ &= (2k+1) \int_{\Omega} |\nabla v_{m,M}|^2 v_{m,M}^{2k} dx \\ &= \frac{2k+1}{(k+1)^2} \int_{\Omega} |\nabla v_{m,M}^{k+1}|^2 dx. \end{aligned} \quad (37)$$

The assumption (3) along with the Hölder inequality implies that

$$\begin{aligned} |I_2| &= \left| \int_{\Omega} \left[f_m(y_m^+) + \int_U v \sigma(x)(dv) \right] v_{m,M}^{2k+1} dx \right| \\ &\leq \int_{\Omega} (1 + C_1 + C_1 y_m^+) v_{m,M}^{2k+1} dx \\ &\leq \int_{\Omega} (1 + C_1 + C_1 y_m^+) (y_m^+)^{2k+1} dx \\ &= (1 + C_1) \int_{\Omega} (y_m^+)^{2k+1} dx + C_1 \int_{\Omega} (y_m^+)^{2(k+1)} dx \\ &\leq e_1 \left(\int_{\Omega} (y_m^+)^{2(k+1)} dx \right)^{\frac{2k+1}{2(k+1)}} + C_1 \int_{\Omega} (y_m^+)^{2(k+1)} dx, \end{aligned} \quad (38)$$

where $e_1 = (1 + C_1)|\Omega|^{\frac{1}{2(k+1)}}$. By the same arguments for the boundary integral, we have that

$$\begin{aligned} |I_3| &= \left| \int_{\Gamma} g_m(y_m^+) v_{m,M}^{2k+1} d\sigma \right| \\ &\leq e_2 \left(\int_{\Gamma} (y_m^+)^{2(k+1)} d\sigma \right)^{\frac{2k+1}{2(k+1)}} + C_1 \int_{\Gamma} (y_m^+)^{2(k+1)} d\sigma. \end{aligned} \quad (39)$$

Applying the estimate (37)-(39) to (36), one gets

$$\begin{aligned} &\frac{2k+1}{(k+1)^2} \int_{\Omega} |\nabla v_{m,M}^{k+1}|^2 dx \\ &\leq e_1 \left(\int_{\Omega} (y_m^+)^{2(k+1)} dx \right)^{\frac{2k+1}{2(k+1)}} + C_1 \int_{\Omega} (y_m^+)^{2(k+1)} dx \\ &\quad + e_2 \left(\int_{\Gamma} (y_m^+)^{2(k+1)} d\sigma \right)^{\frac{2k+1}{2(k+1)}} + C_1 \int_{\Gamma} (y_m^+)^{2(k+1)} d\sigma. \end{aligned} \quad (40)$$

We have $\lim_{M \rightarrow \infty} v_{m,M}(x) = y_m^+(x)$ for a.e. $x \in \Omega$ and can apply Fatou's Lemma which results in

$$\begin{aligned} & \frac{2k+1}{(k+1)^2} \int_{\Omega} |\nabla(y_m^+)^{k+1}|^2 dx \\ & \leq e_1 \left(\int_{\Omega} (y_m^+)^{2(k+1)} dx \right)^{\frac{2k+1}{2(k+1)}} + C_1 \int_{\Omega} (y_m^+)^{2(k+1)} dx \\ & \quad + e_2 \left(\int_{\Gamma} (y_m^+)^{2(k+1)} d\sigma \right)^{\frac{2k+1}{2(k+1)}} + C_1 \int_{\Gamma} (y_m^+)^{2(k+1)} d\sigma. \end{aligned} \quad (41)$$

We have either

$$\left(\int_{\Omega} (y_m^+)^{2(k+1)} dx \right)^{\frac{2k+1}{2(k+1)}} \leq 1 \text{ or } \left(\int_{\Omega} (y_m^+)^{2(k+1)} dx \right)^{\frac{2k+1}{2(k+1)}} \leq \int_{\Omega} (y_m^+)^{2(k+1)} dx,$$

respectively, either

$$\left(\int_{\Gamma} (y_m^+)^{2(k+1)} d\sigma \right)^{\frac{2k+1}{2(k+1)}} \leq 1 \text{ or } \left(\int_{\Gamma} (y_m^+)^{2(k+1)} d\sigma \right)^{\frac{2k+1}{2(k+1)}} \leq \int_{\Gamma} (y_m^+)^{2(k+1)} d\sigma.$$

Using the calculation above to (41), we obtain

$$\begin{aligned} & \frac{2k+1}{(k+1)^2} \int_{\Omega} |\nabla(y_m^+)^{k+1}|^2 dx \\ & \leq e_3 \int_{\Omega} (y_m^+)^{2(k+1)} dx + e_4 \int_{\Gamma} (y_m^+)^{2(k+1)} d\sigma + e_5. \end{aligned} \quad (42)$$

Adding on both sides of (42) the positive integral $\frac{2k+1}{(k+1)^2} \int_{\Omega} (y_m^+)^{2(k+1)} dx$ yields

$$\begin{aligned} & \frac{2k+1}{(k+1)^2} \left[\int_{\Omega} |\nabla(y_m^+)^{k+1}|^2 dx + \int_{\Omega} (y_m^+)^{2(k+1)} dx \right] \\ & \leq e_6(2k+1) \int_{\Omega} (y_m^+)^{2(k+1)} dx + e_4 \int_{\Gamma} (y_m^+)^{2(k+1)} d\sigma + e_5 \end{aligned} \quad (43)$$

due to the fact $\frac{2k+1}{(k+1)^2} < 2k+1$ for all $k \geq 0$. Next we want to estimate the boundary integral by an integral in the domain Ω . Using the embedding theorem and Young's inequality, we have that

$$\begin{aligned} \int_{\Gamma} (y_m^+)^{2(k+1)} d\sigma & \leq C_2 \left(\delta \|(y_m^+)^{k+1}\|_{W^{1,2}(\Omega)}^2 \right. \\ & \quad \left. + C(\delta) \|(y_m^+)^{k+1}\|_{L^2(\Omega)}^2 \right) \end{aligned} \quad (44)$$

holds for a free parameter specified later. Applying (44) to (43) shows that

$$\begin{aligned} & \frac{2k+1}{(k+1)^2} \left[\int_{\Omega} |\nabla(y_m^+)^{k+1}|^2 dx + \int_{\Omega} (y_m^+)^{2(k+1)} dx \right] \\ & \leq e_6(2k+1) \int_{\Omega} (y_m^+)^{2(k+1)} dx + e_7 \delta \|(y_m^+)^{k+1}\|_{W^{1,2}(\Omega)}^2 \\ & \quad + e_7 C(\delta) \|(y_m^+)^{k+1}\|_{L^2(\Omega)}^2 + e_5. \end{aligned}$$

We take $\delta = \frac{2k+1}{2e_7(k+1)^2}$ to get

$$\begin{aligned} & \frac{2k+1}{2(k+1)^2} \left[\int_{\Omega} |\nabla(y_m^+)^{k+1}|^2 dx + \int_{\Omega} (y_m^+)^{2(k+1)} dx \right] \\ & \leq e_6(2k+1) \int_{\Omega} (y_m^+)^{2(k+1)} dx + e_7 C(\delta) \|(y_m^+)^{k+1}\|_{L^2(\Omega)}^2 + e_5 \\ & \leq e_8(2k+1 + C(\delta)) \int_{\Omega} (y_m^+)^{2(k+1)} dx + e_5, \end{aligned} \quad (45)$$

where it holds

$$2k+1 + C(\delta) = 2k+1 + \frac{1}{4\delta} = 2k+1 + \frac{e_7(k+1)^2}{2(2k+1)} \leq e_9(k+1)^2.$$

Applying the calculation above to (45) provides

$$\begin{aligned} & \frac{2k+1}{2(k+1)^2} \left[\int_{\Omega} |\nabla(y_m^+)^{k+1}|^2 dx + \int_{\Omega} (y_m^+)^{2(k+1)} dx \right] \\ & \leq e_{10}(2k+1)^2 \left[\int_{\Omega} (y_m^+)^{2(k+1)} dx + 1 \right], \end{aligned}$$

equivalently,

$$\|(y_m^+)^{k+1}\|_{W^{1,2}(\Omega)}^2 \leq (2k+1)(k+1)^2 e_{11} \left[\int_{\Omega} (y_m^+)^{2(k+1)} dx + 1 \right].$$

By Sobolev's embedding theorem a positive constant e_{12} exists such that

$$\|(y_m^+)^{k+1}\|_{L^{p^*}(\Omega)} \leq e_{12} \|(y_m^+)^{k+1}\|_{W^{1,2}(\Omega)}, \quad (46)$$

where $p^* = \frac{2n}{n-2}$ if $n > 2$, and $p^* = 4$ if $n = 2$. We get

$$\begin{aligned} \|y_m^+\|_{L^{(k+1)p^*}(\Omega)} &= \|(y_m^+)^{k+1}\|_{L^{p^*}(\Omega)}^{\frac{1}{k+1}} \\ &\leq e_{12}^{\frac{1}{k+1}} \|(y_m^+)^{k+1}\|_{W^{1,2}(\Omega)}^{\frac{1}{k+1}} \\ &\leq e_{12}^{\frac{1}{k+1}} \left[(2k+1)^{\frac{1}{2}} (k+1) \right]^{\frac{1}{k+1}} e_{11}^{\frac{1}{2(k+1)}} \left[\int_{\Omega} (y_m^+)^{2(k+1)} dx + 1 \right]^{\frac{1}{2(k+1)}}. \end{aligned}$$

Since $[(2k+1)^{\frac{1}{2}}(k+1)]^{\frac{1}{\sqrt{k+1}}} \geq 1$ and $\lim_{k \rightarrow \infty} [(2k+1)^{\frac{1}{2}}(k+1)]^{\frac{1}{\sqrt{k+1}}} = 1$, there exists a constant $e_{13} > 1$ such that $[(2k+1)^{\frac{1}{2}}(k+1)]^{\frac{1}{k+1}} \leq e_{13}^{\frac{1}{\sqrt{k+1}}}$. We obtain

$$\|y_m^+\|_{L^{(k+1)p^*}(\Omega)} \leq e_{12}^{\frac{1}{k+1}} e_{13}^{\frac{1}{\sqrt{k+1}}} e_{11}^{\frac{1}{2(k+1)}} \left[\int_{\Omega} (y_m^+)^{2(k+1)} dx + 1 \right]^{\frac{1}{2(k+1)}}. \quad (47)$$

Now, we will use the bootstrap arguments similarly as in the proof of Lemma 3.2 in [6] starting with $2(k_1+1) = p^*$ to get

$$\|y_m^+\|_{L^{(k+1)p^*}(\Omega)} \leq C(k) \quad (48)$$

for any finite $k > 0$ which shows that $y_m^+ \in L^r(\Omega)$ for any $r \in (1, +\infty)$. To prove the uniform estimate with respect to k we argue as follows. If there is a sequence $k_n \rightarrow \infty$ such that

$$\int_{\Omega} (y_m^+)^{2(k_n+1)} dx \leq 1,$$

we have immediately

$$\|y_m^+\|_{L^\infty(\Omega)} \leq 1,$$

(cf. the proof of Lemma 3.2 in [6]). In the opposite case there exists $k_0 > 0$ such that

$$\int_{\Omega} (y_m^+)^{2(k+1)} dx > 1$$

for any $k \geq k_0$. Then we conclude from (47)

$$\|y_m^+\|_{L^{(k+1)p^*}(\Omega)} \leq e_{12}^{\frac{1}{k+1}} e_{13}^{\frac{1}{\sqrt{k+1}}} e_{14}^{\frac{1}{2(k+1)}} \|y_m^+\|_{L^{2(k+1)}(\Omega)} \quad (49)$$

for any $k \geq k_0$ and where $e_{14} = 2e_{11}$. Choosing $k := k_1$ such that $2(k_1 + 1) = (k_0 + 1)p^*$ yields

$$\|y_m^+\|_{L^{(k_1+1)p^*}(\Omega)} \leq e_{12}^{\frac{1}{k_1+1}} e_{13}^{\frac{1}{\sqrt{k_1+1}}} e_{14}^{\frac{1}{2(k_1+1)}} \|y_m^+\|_{L^{2(k_1+1)}(\Omega)}. \quad (50)$$

Next, we choose k_2 in (49) such that $2(k_2 + 1) = (k_1 + 1)p^*$ to get

$$\begin{aligned} \|y_m^+\|_{L^{(k_2+1)p^*}(\Omega)} &\leq e_{12}^{\frac{1}{k_2+1}} e_{13}^{\frac{1}{\sqrt{k_2+1}}} e_{14}^{\frac{1}{2(k_2+1)}} \|y_m^+\|_{L^{2(k_2+1)}(\Omega)} \\ &= e_{12}^{\frac{1}{k_2+1}} e_{13}^{\frac{1}{\sqrt{k_2+1}}} e_{14}^{\frac{1}{2(k_2+1)}} \|y_m^+\|_{L^{(k_1+1)p^*}(\Omega)}. \end{aligned} \quad (51)$$

By induction we obtain

$$\|y_m^+\|_{L^{(k_n+1)p^*}(\Omega)} \leq e_{12}^{\frac{1}{k_n+1}} e_{13}^{\frac{1}{\sqrt{k_n+1}}} e_{14}^{\frac{1}{2(k_n+1)}} \|y_m^+\|_{L^{(k_{n-1}+1)p^*}(\Omega)}, \quad (52)$$

where the sequence $\{k_n\}$ is chosen such that $2(k_n + 1) = (k_{n-1} + 1)p^*$ with $k_0 > 0$. One easily verifies that $k_n + 1 = (\frac{p^*}{2})^n (k_0 + 1)$. Thus

$$\|y_m^+\|_{L^{(k_n+1)p^*}(\Omega)} \leq e_{12}^{\sum_{i=1}^n \frac{1}{k_i+1}} e_{13}^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}} e_{14}^{\sum_{i=1}^n \frac{1}{2(k_i+1)}} \|y_m^+\|_{L^{(k_0+1)p^*}(\Omega)},$$

with $r_n = (k_n + 1)p^* \rightarrow \infty$ as $n \rightarrow \infty$. Since $\frac{1}{k_i+1} = (\frac{2}{p^*})^i \frac{1}{k_0+1}$ and $p^* > 2$ there is a constant $e_{15} > 0$ such that

$$\|y_m^+\|_{L^{(k_n+1)p^*}(\Omega)} \leq e_{15} \|y_m^+\|_{L^{(k_0+1)p^*}(\Omega)} < \infty. \quad (53)$$

Let us assume that $y_m^+ \notin L^\infty(\Omega)$. Then there exist $\eta > 0$ and a set A of positive measure in Ω such that $y_m^+(x) \geq e_{15} \|y_m^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta$ for $x \in A$. It follows that

$$\begin{aligned} \|y_m^+\|_{L^{(k_n+1)p^*}(\Omega)} &\geq \left(\int_A |y_m^+(x)|^{(k_n+1)p^*} dx \right)^{\frac{1}{(k_n+1)p^*}} \\ &\geq \left(e_{15} \|y_m^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta \right) |A|^{\frac{1}{(k_n+1)p^*}}. \end{aligned}$$

Passing to the limits inferior in the inequality above yields

$$\liminf_{n \rightarrow \infty} \|y_m^+\|_{L^{(k_n+1)p^*}(\Omega)} \geq e_{15} \|y_m^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta,$$

which is a contradiction to (53) and hence, $y_m^+ \in L^\infty(\Omega)$. Moreover, by (48) and (53), it holds that

$$\|y_m^+\|_{L^{(k_n+1)p^*}(\Omega)} \leq e_{15} \|y_m^+\|_{L^{(k_0+1)p^*}(\Omega)} \leq C < \infty$$

for some constant $C > 0$ independent of m . Therefore, we have that

$$\|y_m^+\|_{L^\infty(\Omega)} = \lim_{n \rightarrow \infty} \|y_m^+\|_{L^{(k_n+1)p^*}(\Omega)} \leq C.$$

In a similar way one shows that $y_m^- = \max\{-y_m, 0\}$ belongs to $L^\infty(\Omega)$. This proves that $y_m = y_m^+ - y_m^- \in L^\infty(\Omega)$ and $\|y_m\|_{L^\infty(\Omega)} \leq K$ for some constant $K > 0$ independent of m .

Step 3. Then, if we take $m > K$, $y_m = y$ is a solution of (35) and satisfies (17). Finally, using the monotonicity assumptions on f , g in (S), we obtain the uniqueness of y , and the proof of Proposition 1 is completed. \square

Proof of Proposition 2. Let $w = y_1 - y_2$, then w satisfies the following inequalities

$$\begin{cases} -\Delta w + a_0 w \geq 0 & \text{in } \Omega, \\ \partial_\nu w + b_0 w \geq 0 & \text{on } \Gamma, \end{cases}$$

where $a_0 = -\int_0^1 f'(y_2 + \tau(y_1 - y_2))d\tau$ and $b_0 = -\int_0^1 g'(y_2 + \tau(y_1 - y_2))d\tau$. By (4) and (17), we have $a_0 \in L^\infty(\Omega)$, and (2) implies $a_0 > 0$. Moreover, it follows from the condition $g'(y) \leq -r_2$ ($r_2 \geq 0$) that $b_0 \geq 0$. Then by the maximum principle ([1], Lemma 1), Proposition 2 is proved. \square

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