

## STABILITY AND INSTABILITY OF SOLUTIONS TO THE DRIFT-DIFFUSION SYSTEM

TAKAYOSHI OGAWA

Tohoku University, Mathematical Institute  
Sendai 980-8578, Japan

HIROSHI WAKUI

Mathematical Institute, Tohoku University  
Sendai 980-8578, Japan

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**ABSTRACT.** We consider the large time behavior of a solution to a drift-diffusion equation for degenerate and non-degenerate type. We show an instability and uniform unbounded estimate for the semi-linear case and uniform bound and convergence to the stationary solution for the case of mass critical degenerate case for higher space of dimension bigger than two.

**1. Introduction.** We consider the Cauchy problem of the drift-diffusion system in  $n$ -dimensional Euclidian space  $\mathbb{R}^n$ :

$$\begin{cases} \partial_t \rho - \Delta \rho^\alpha + \nabla \cdot (\rho \nabla \psi) = 0, & t > 0, x \in \mathbb{R}^n, \\ -\Delta \psi = \rho, & t > 0, x \in \mathbb{R}^n, \\ \rho(0, x) = \rho_0(x) \geq 0, & x \in \mathbb{R}^n, \end{cases} \quad (1)$$

where  $\alpha \geq 1$  is the adiabatic constant,  $\rho_0 \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$  and  $\rho_0 \geq 0$  for the initial data. The equation can be derived from the damped compressible Navier-Stokes-Poisson equations ([6], [8]) or from the damped compressible Euler-Poisson system.

The system (1) has a scaling invariant property. For a positive parameter  $\lambda > 0$ , we define a scaled function  $(\rho_\lambda, \psi_\lambda)$  by

$$\begin{cases} \rho_\lambda(t, x) \equiv \lambda^{\frac{2}{2-\alpha}} \rho(\lambda^{\frac{2}{2-\alpha}} t, \lambda x), \\ \psi_\lambda(t, x) \equiv \lambda^{\frac{2}{2-\alpha}-2} \psi(\lambda^{\frac{2}{2-\alpha}} t, \lambda x), \end{cases} \quad t > 0, \quad x \in \mathbb{R}^n. \quad (2)$$

Then  $(\rho_\lambda, \psi_\lambda)$  also solves the system (1) except for the initial condition. The Bochner space  $L^\theta(\mathbb{R}_+; L^p(\mathbb{R}^n))$  is invariant under the scaling (2) subjected to

$$\frac{2}{2-\alpha} = \frac{2}{\theta(2-\alpha)} + \frac{n}{p}.$$

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In particular, if  $\theta = \infty$ , then  $L^p(\mathbb{R}^n)$  with

$$p = p_c \equiv \frac{n}{2}(2 - \alpha)$$

is the invariant Lebesgue space under the scaling (2). According to the Fujita-Kato classical result, it is interesting and important to construct a solution in the scaling invariant space  $L^{p_c}(\mathbb{R}^n)$ . When  $\alpha = 2 - \frac{2}{n}$ , we should emphasize that the invariant scaling (2) preserves the  $L^1$ -norm of solutions and the  $L^1$ -norm of solutions is conserved.

**1.1. Semi-linear case.** The case of  $\alpha = 1$ , the equation (1) is a semi-linear problem and if  $n = 2$ , then we have the  $L^1$ -invariant case. If  $n \geq 3$ , then it is super critical case. In any case, the solution can be formulated by the Duhamel principle via the semi-group.

**Definition 1.1.** Let  $1 < p < \infty$ ,  $\alpha = 1$  and  $n \geq 3$ . For any  $\rho_0 \in L^p(\mathbb{R}^n)$  we call  $\rho \in C([0, T]; L^p(\mathbb{R}^n))$  as a mild solution to (1) if

$$\rho(t) = e^{t\Delta}\rho_0 + \int_0^t e^{(t-s)\Delta}\nabla \cdot (\rho(s)\psi(s))ds,$$

where  $\{e^{t\Delta}\}_{t \geq 0}$  denotes the heat evolution semi-group and  $\psi(s) = (-\Delta)^{-1}\rho(s)$ .

It is known that the local mild solution in  $L^p(\mathbb{R}^n)$  for  $\frac{n}{2} \leq p < n$  exists and has regularity sufficient to obtain the strong solution for all  $n \geq 2$ . Besides if the data is non-negative, then the corresponding solution is strictly positive and satisfies the two conservation laws. Then the global behavior of solution is well understood. If the initial data is small as  $\|\rho_0\|_1 \leq 8\pi$ , then the solution exists global in time [11], [12], [14]. If the data is large  $\|\rho_0\|_1 > 8\pi$ , then the solution blows up in a finite time [2], [9], [12], [13]. For the higher dimensional cases, the global behavior of solutions is not well understood than two dimensional case. This is partially because the problem is super critical and nonlinear effect is much stronger than the dissipative effect. Nevertheless, there is a global existence result as follows:

**Proposition 1** ([4], [5], [17]). *Assume that*

$$\|\rho_0\|_{\frac{n}{2}} < B_n \equiv \frac{8}{nS_b^2}, \tag{3}$$

where

$$S_b^2 = \frac{1}{\pi n(n-2)} \left( \frac{\Gamma(n)}{\Gamma(\frac{n}{2})} \right)^{\frac{2}{n}}$$

is the best possible constant of the Sobolev inequality

$$\|f\|_{\frac{2n}{n-2}} \leq S_b \|\nabla f\|_2, \quad f \in \dot{H}^1(\mathbb{R}^n).$$

Then the positive solution  $\rho(t)$  in  $L^{\frac{n}{2}}(\mathbb{R}^n)$  satisfies

$$\sup_{t>0} \|\rho(t)\|_{\frac{n}{2}} \leq B_n.$$

Furthermore the solution exists globally in time and

$$\sup_{t>0} t^{\frac{n}{2}(1-\frac{1}{p})} \|\rho(t)\|_p \leq C(B_n, \|\rho_0\|_1)$$

for all  $1 < p \leq \infty$ .

For  $1 \leq p \leq \infty$  and  $s > 0$ , we denote the weighted Lebesgue space  $L^p_s(\mathbb{R}^n)$  as

$$L^p_s(\mathbb{R}^n) \equiv \{f \in L^p(\mathbb{R}^n) \mid |x|^s f \in L^p(\mathbb{R}^n)\}$$

through this paper.

**1.2. Instability of semi-linear case.** There are few results for the finite time blowing up of solutions. Corrias-Perthame-Zaag [5] obtained the finite time blowing up for the higher dimensional solutions, where they did not use the entropy functional. Namely there exists a large constant  $M_n = M(n) > 0$  such that if the data satisfies

$$M_n \leq \frac{\|\rho_0\|_1^{\frac{n}{n-2}}}{\int_{\mathbb{R}^n} |x - x_0|^2 \rho_0(x) dx}, \tag{4}$$

then the solution blows up in a finite time. We recall the entropy functional and the uniform boundedness for the semi-linear case:

$$H[\rho(t)] \equiv \int_{\mathbb{R}^n} \rho(t) \log \rho(t) dx - \frac{1}{2} \int_{\mathbb{R}^n} \rho(t) \psi(t) dx, \tag{5}$$

$$H[\rho(t)] + \int_0^t \int_{\mathbb{R}^n} \rho(s) \left| \nabla (\log \rho(s) - \psi(s)) \right|^2 dx ds \leq H[\rho_0]. \tag{6}$$

We show the instability result for the solution to the Cauchy problem (1) as follows:

**Theorem 1.2** ([17]). *Let  $n \geq 3$ ,  $b > 0$ . Assume that the initial data  $\rho_0$  is non-negative in  $L^{\frac{n}{2}}(\mathbb{R}^n) \cap L^1_b(\mathbb{R}^n)$  and there exists a constant  $C_n = C(n) > 0$  such that*

$$H[\rho_0] < \frac{n}{b} \|\rho_0\|_1 \log \left( \frac{\|\rho_0\|_1^{1+\frac{b}{n}}}{C_n \int_{\mathbb{R}^n} |x - \bar{x}|^b \rho_0(x) dx} \right), \tag{7}$$

where  $\bar{x}$  is the center of  $b$ th-order moment defined by

$$\int_{\mathbb{R}^n} |\bar{x} - y|^b \rho_0(y) dy = \inf_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |x - y|^b \rho_0(y) dy,$$

and  $H[\rho_0]$  is defined in (5).

- (1) Then the corresponding local solution is not uniformly bounded in  $L^{\frac{n}{2}}(\mathbb{R}^n)$ .
- (2) If  $b \geq 2$ , then the solution blows up in a finite time. Namely

$$\limsup_{t \rightarrow T_*} \|\rho(t)\|_p = \infty \tag{8}$$

for all  $\frac{n}{2} \leq p \leq \infty$ .

- (3) If  $n \geq 3$  and initial data is radially symmetric, then the solution blows up in a finite time in the sense (8).

The analogous result can be found for the bounded domain with a Neumann boundary condition in Biler [1]. For the case  $b = 2$ , Calvez-Corrias-Ebde [4] showed a similar result. Such a result can be also true for a system of the equation (see, Kurokiba-Ogawa [10]). The above theorem shows that the blow up phenomena can be possibly shown in the critical Lebesgue space  $L^{\frac{n}{2}}(\mathbb{R}^n)$  provided the solution of the Cauchy problem has a non-finite second moment. We should like to note that the conditions (4) and (7) treat different classes of solutions: In some cases, the former condition covers a wider class of data while in the following, the latter is

wider: Under the assumption (3) for the global decaying solution, there exists a constant  $\tilde{C}_n < C_n$  such that the data satisfies

$$H[\rho_0] > \frac{n}{2} \|\rho_0\|_1 \log \left( \frac{\|\rho_0\|_1^{1+\frac{2}{n}}}{\tilde{C}_n \int_{\mathbb{R}^n} |x - \bar{x}|^2 \rho_0(x) dx} \right).$$

The proof of Theorem 1.2 relies on the improved version of the generalized Shannon inequality: For any non-negative function  $f \in L^1_b(\mathbb{R}^n)$ ,

$$\int_{\mathbb{R}^n} f(x) \log (f(x))^{-1} dx \leq \frac{n}{b} \|f\|_1 \log \left( \frac{b e c_{n,b} \int_{\mathbb{R}^n} |x - \bar{x}|^b f(x) dx}{n \|f\|_1^{1+\frac{b}{n}}} \right),$$

where  $\bar{x} \in \mathbb{R}^n$  is a center of  $b$ th-moment given by

$$\int_{\mathbb{R}^n} |\bar{x} - y|^b f(y) dy = \inf_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |x - y|^b f(y) dy$$

and  $c_{n,b} = \left( \frac{2\pi^{\frac{n}{2}}}{b} \Gamma\left(\frac{n}{b}\right) \Gamma\left(\frac{n}{2}\right)^{-1} \right)^{\frac{b}{n}}$ . See, for the proof, [17].

**2. Degenerate drift-diffusion equation.** When  $\alpha > 1$  in (1), then the system degenerates and we need to introduce the notion of a weak solution since the regularity of solution generally fails.

**Definition 2.1.** Let  $\alpha \geq 2 - \frac{4}{n+2}$  and  $\rho_0 \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$  be non-negative. We call  $(\rho(t), \psi(t))$  is a weak solution to (1) if

- (1)  $\rho(t, x) \geq 0$  for almost all  $(t, x) \in [0, T) \times \mathbb{R}^n$ .
- (2)  $\rho \in L^\infty(0, T; L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n))$ ,  $\nabla \rho^{\alpha-\frac{1}{2}} \in L^2(0, T; L^2(\mathbb{R}^n))$  and
- (3) for any test  $\phi \in C^\infty([0, T) \times \mathbb{R}^n)$ , with  $\text{supp } \phi(t) \subset\subset \mathbb{R}^n$  ( $0 \leq t < T$ )

$$\begin{aligned} & \int_{\mathbb{R}^n} \rho(t)\phi(t) dx - \int_{\mathbb{R}^n} \rho_0(x)\phi(0) dx \\ &= \int_0^t \int_{\mathbb{R}^n} (\nabla \rho(s)^\alpha \cdot \nabla \phi(s) + \rho(s) \nabla \psi(s) \cdot \nabla \phi(s) + \rho(s) \partial_s \phi(s)) dx ds, \end{aligned}$$

where  $\psi(t) = (-\Delta)^{-1} \rho(t) := \frac{1}{n(n-2)\omega_n} |x|^{-(n-2)} * \rho(t)$ .

If the adiabatic constant is  $\alpha = 2 - \frac{2}{n}$ , then it is known that the weak solution does not exist globally in time (see, Suzuki-Takahashi [18], [19]). In the degenerate case, the entropy bound (6) also holds with exchanging the entropy

$$H[\rho(t)] = \frac{1}{\alpha - 1} \|\rho(t)\|_\alpha^\alpha - \frac{1}{2} \int_{\mathbb{R}^n} \rho(t)\psi(t) dx \tag{9}$$

instead of (5).

Blanchet-Carrillo-Laurençot [3] showed the following: We recall the Hardy-Littlewood-Sobolev inequality:

$$\int_{\mathbb{R}^n} f(-\Delta)^{-1} f dx \leq C_{HLS,\alpha} \|f\|_1^{1-\sigma} \|f\|_\alpha^{1+\sigma}, \tag{10}$$

where  $C_{HLS,\alpha} > 0$  is the best possible constant and  $\sigma = \frac{\alpha}{\alpha-1} \cdot \frac{n-2}{n} - 1$ . There exists an extremal function that attains the best constant of (10) (see for instance, [3], [7]).

**Proposition 2** ([3], [15]). *Let  $n \geq 3$  and  $\alpha = 2 - \frac{2}{n}$ . Let  $C_{HLS,\alpha}$  be the best possible constant for the Hardy-Littlewood-Sobolev inequality (10) and set  $M_* \equiv \left(\frac{2n}{n-2} C_{HLS,\alpha}^{-1}\right)^{\frac{n}{2}}$ . Then for  $\rho_0 \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$  with  $\rho_0 \geq 0$ , the weak solution  $\rho(t)$  for (1) satisfies the followings:*

- (1) *If  $\|\rho_0\|_1 < M_*$ , then the weak solution exists globally in time.*
- (2) *For any  $M > M_*$ , there exists an initial data  $\rho_0$  such that  $M = \|\rho_0\|_1$ ,  $|x|^2 \rho_0 \in L^1(\mathbb{R}^n)$ ,  $H[\rho_0]$  defined by (9) satisfies  $H[\rho_0] < 0$  and the corresponding solution  $\rho(t)$  to (1) blows up in a finite time.*
- (3) *For  $|x|^2 \rho_0 \in L^1(\mathbb{R}^n)$ , if  $\|\rho_0\|_1 = M_*$ , then the weak solution exists globally in time.*

From Proposition 2, the solution can be classified into either the global existence or the finite time blow up. Proposition 2 shows further that if the initial data of (1) has the threshold mass and finite second moment, then the corresponding weak solution exists globally in time. When the initial data has a smaller mass than  $M_*$ , the weak solution decays and its asymptotic profile is shown as a solution of the Barenblatt type (cf. [16]). Here we claim the large time behavior of the weak solution for the mass critical case  $\alpha = 2 - \frac{2}{n}$ . The following theorem is one of our main results.

**Theorem 2.2.** *Let  $n \geq 3$  and  $\alpha = 2 - \frac{2}{n}$ . Assume that  $\rho_0 \in L^1_2(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$ ,  $\|\rho_0\|_1 = M_*$ . Assume that for a weak solution  $(\rho, \psi)$  to (1) obtained in Proposition 2 satisfies*

$$\int_0^\infty \left( \int_{\mathbb{R}^n} \rho(t) \left| \nabla \left( \frac{\alpha}{\alpha-1} \rho(t)^{\alpha-1} - \psi(t) \right) \right|^2 dx \right)^{\frac{1}{2}} dt < \infty \tag{11}$$

and either of the followings:

- (1) *The solution  $\rho(t)$  is radially symmetric, or*
- (2) *the entropy functional  $H[\rho]$  given by (9) satisfies <sup>1</sup>*

$$\int_0^\infty H[\rho(s)] ds < \infty. \tag{12}$$

Then for any  $p \in [1, \infty]$ , there exists a constant  $C = C(p, n, \rho_0) > 0$  such that for all  $t > 0$ ,

$$\|\rho(t)\|_p \leq C. \tag{13}$$

Namely the solution is uniformly bounded.

To state the stability, we introduce the solution of the nonlinear elliptic equation related to the problem (1). Let  $x_0 \in \mathbb{R}^n$  and  $R > 0$ . Consider the stationary problem

$$\begin{cases} -\frac{\alpha}{\alpha-1} \Delta V^{\alpha-1} = V, & x \in B_R(x_0), \\ V > 0, & x \in B_R(x_0), \\ V = 0, & x \in B_R(x_0)^c. \end{cases} \tag{14}$$

The other our main result is the following:

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<sup>1</sup>If the solution  $\rho(t)$  is radially symmetric, then (12) follows automatically.

**Theorem 2.3.** *Let  $n \geq 3$ ,  $\alpha = 2 - \frac{2}{n}$ , the initial data satisfy  $\rho_0 \in L^1_2(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$  and  $\|\rho_0\|_1 = M_*$ . Assume that the assumption (1) or (2) in Theorem 2.2 holds. Then there exists a sequence  $\{t_j\}_{j=1}^\infty \subset (0, \infty)$ ,  $t_j \rightarrow \infty$  ( $j \rightarrow \infty$ ) such that the solution  $\rho(t_j)$  converges the minimal mass solution  $V$  for (14) with  $\|V\|_1 = M_*$  up to dilation and spatial translation. Namely, for some  $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^n$  and  $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{R}_+$ ,*

$$\left\| \rho(t_j) - \lambda_j^n V(\lambda_j(\cdot - x_j)) \right\|_p \rightarrow 0$$

as  $j \rightarrow \infty$ .

**Remark 1.** We should remark that the solution to the stationary problem (14) satisfies conditions (11) and (12) in Theorem 2.2. We hence show in Theorem 2.3 that the conversed statement is asymptotically valid. Indeed, conditions (11) and (12) guarantee the stability of solutions to (1) when  $\|\rho_0\|_1 = M_*$ . In other words, if the solution  $\rho$  corresponding to the initial data  $\rho_0$  with  $\|\rho_0\|_1 = M_*$  stays near the solution to (14) in the sense of (11) and (12), then the stationary solution  $V$  is stable.

*Proof of Theorem 2.2.* Suppose that the condition (12) holds. Then the sequence  $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^n$  of Theorem 2.3 converges to  $\bar{x} := (\int_{\mathbb{R}^n} x \rho_0 dx) / M_*$  as  $j \rightarrow \infty$ , see [3]. We assume on the contrary that there exists  $\{t_j\}_{j=1}^\infty \subset \mathbb{R}_+$  such that  $\|\rho(t_j)\|_\alpha \rightarrow \infty$  as  $j \rightarrow \infty$ . When  $\|\rho(t_j)\|_\alpha \rightarrow \infty$  as  $j \rightarrow \infty$ , Blanchet-Carrillo-Laurençot [3] proved that there exist subsequence of  $\{t_j\}_{j=1}^\infty$  and sequence  $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^n$  such that

$$\left\| \rho(t_j) - \lambda_j^n V(\lambda_j(\cdot - x_j)) \right\|_1 \rightarrow 0 \quad (j \rightarrow \infty), \tag{15}$$

where  $\lambda_j := \|\rho(t_j)\|_\alpha^{\frac{\alpha}{n(\alpha-1)}}$ . From Proposition 2, we see that  $H[\rho(s)] \geq 0$  for  $s > 0$  and

$$\begin{aligned} \int_{\mathbb{R}^n} |x - \bar{x}|^2 \rho(t) dx &= \int_{\mathbb{R}^n} |x - \bar{x}|^2 \rho_0(x) dx + 2(n-2) \int_0^t H[\rho(s)] ds \\ &\geq \int_{\mathbb{R}^n} |x - \bar{x}|^2 \rho_0(x) dx \\ &> 0 \end{aligned} \tag{16}$$

for  $t > 0$ . We claim that  $\limsup_{j \rightarrow \infty} \int_{\mathbb{R}^n} |x - \bar{x}|^2 \rho(t_j) dx = 0$ . To prove this, we divide the integral domain into two part, that is, for fixed  $R > |\bar{x}|$ ,

$$\begin{aligned} \int_{\mathbb{R}^n} |x - \bar{x}|^2 \rho(t_j) dx &= \left( \int_{B_{2R}(\bar{x})} + \int_{B_{2R}(\bar{x})^c} \right) |x - \bar{x}|^2 \rho(t_j) dx \\ &\equiv I_{j,R}^1 + I_{j,R}^2. \end{aligned} \tag{17}$$

The property (14) of  $V$  and (15) guarantee that

$$\lim_{j \rightarrow \infty} I_{j,R}^1 = 0, \quad R > 0. \tag{18}$$

Indeed, we observe

$$\begin{aligned} \int_{B_{2R}(\bar{x})} |x - \bar{x}|^2 \rho(t_j) dx &\leq \int_{B_{2R}(\bar{x})} |x - \bar{x}|^2 |\rho(t_j) - \lambda_j^n V(\lambda_j(x - x_j))| dx \\ &\quad + \int_{B_{2R}(\bar{x})} |x - \bar{x}|^2 \lambda_j^n V(\lambda_j(x - x_j)) dx \\ &\leq 4R^2 \left\| \rho(t_j) - \lambda_j^n V(\lambda_j(\cdot - x_j)) \right\|_1 \end{aligned}$$

$$\begin{aligned}
 & + \int_{B_{2R\lambda_j}(\bar{x}-x_j)} |\lambda_j^{-1}y + x_j - \bar{x}|^2 V(y) dy \\
 \leq & 4R^2 \|\rho(t_j) - \lambda_j^\alpha V(\lambda_j(\cdot - x_j))\|_1 \\
 & + 2\lambda_j^{-2} \int_{\text{supp } V} |y|^2 V(y) dy \\
 & + 2|x_j - \bar{x}|^2 \int_{\text{supp } V} V(y) dy.
 \end{aligned} \tag{19}$$

The first term of right hand side in (19) tends to 0 as  $j \rightarrow \infty$  from (15). Since  $\lambda_j \rightarrow \infty$  and  $|x_j - \bar{x}| \rightarrow 0$  as  $j \rightarrow \infty$ , the second and third term converges to 0. Therefore we obtain (18). By the choice of the parameter  $R$ , it holds that  $B_{2R}(\bar{x})^c \subset B_R(0)^c$ . Thus we have

$$\begin{aligned}
 I_{j,R}^2 & \leq \int_{B_{2R}(\bar{x})^c} 2(|x|^2 + |\bar{x}|^2)\rho(t_j) dx \\
 & \leq 4 \int_{B_R(0)^c} |x|^2 \rho(t_j) dx \\
 & \leq 4 \sup_{t>0} \int_{B_R(0)^c} |x|^2 \rho(t) dx.
 \end{aligned} \tag{20}$$

If the solution satisfies (11), then the right hand side of (20) converges to 0 as  $R \rightarrow \infty$ . For the proof of this fact, see [3] and [20]. Therefore we observe from (17) that

$$\limsup_{R \rightarrow \infty} \limsup_{j \rightarrow \infty} I_{j,R}^2 = 0. \tag{21}$$

Combining (18) with (21), we obtain

$$\limsup_{j \rightarrow \infty} \int_{\mathbb{R}^n} |x - \bar{x}|^2 \rho(t_j) dx = 0.$$

This contradicts to (16). Hence we conclude (13) in the case  $p = \alpha$ . If the solution  $\rho$  is radially symmetric, the same argument carries over as  $\bar{x} = 0$ . By the Nash-Moser iteration argument, we obtain the uniform bound of  $\|\rho(t)\|_p$  for any  $p \in [1, \infty]$ , see [19].  $\square$

*Proof of Theorem 2.3.* We note that the entropy  $H$  satisfies

$$H[\rho(t)] + \int_0^t \int_{\mathbb{R}^n} \rho(s) \left| \nabla \left( \frac{\alpha}{\alpha-1} \rho(s)^{\alpha-1} - \psi(s) \right) \right|^2 dx ds \leq H[\rho_0]$$

Since  $\|\rho(t)\|_1 = \|\rho_0\|_1 = M_*$ , it holds that  $H[\rho(t)] \geq 0$  for  $t > 0$ . Then we observe

$$\int_0^\infty \int_{\mathbb{R}^n} \rho(t) \left| \nabla \left( \frac{\alpha}{\alpha-1} \rho(t)^{\alpha-1} - \psi(t) \right) \right|^2 dx dt \leq H[\rho_0] < \infty. \tag{22}$$

From (22), there exists  $\{t_j\}_{j=1}^\infty \subset \mathbb{R}_+$  such that  $t_j \rightarrow \infty$  as  $j \rightarrow \infty$  and

$$I_j := \int_{\mathbb{R}^n} \rho(t_j) \left| \nabla \left( \frac{\alpha}{\alpha-1} \rho(t_j)^{\alpha-1} - \psi(t_j) \right) \right|^2 dx \rightarrow 0 \tag{23}$$

as  $j \rightarrow \infty$ . We note that the sequence  $\{I_j\}$  is bounded and

$$I_j = \left( \frac{2\alpha}{2\alpha-1} \right)^2 \int_{\mathbb{R}^n} |\nabla \rho(t_j)^{\alpha-\frac{1}{2}}|^2 dx$$

$$- 2 \int_{\mathbb{R}^n} \rho(t_j)^{\alpha+1} dx + \int_{\mathbb{R}^n} \rho(t_j) |\nabla(-\Delta)^{-1} \rho(t_j)|^2 dx.$$

By (13) and the boundedness of  $\{I_j\}_{j=1}^\infty$ , there exist  $C, C' > 0$  independent of  $j$  such that

$$\begin{aligned} & \left(\frac{2\alpha}{2\alpha-1}\right)^2 \int_{\mathbb{R}^n} |\nabla \rho(t_j)^{\alpha-\frac{1}{2}}|^2 dx + \int_{\mathbb{R}^n} \rho(t_j) |\nabla(-\Delta)^{-1} \rho(t_j)|^2 dx \\ & \leq 2 \|\rho(t_j)\|_{\alpha+1}^{\alpha+1} + C \\ & \leq C'. \end{aligned} \tag{24}$$

Inequalities (24) and (13) show that  $\{\rho(t_j)^{\alpha-\frac{1}{2}}\}_{j=1}^\infty$  is bounded in  $H^1(\mathbb{R}^n)$ . When we assume (1) or (2) of Theorem 2.2, it holds that

$$\sup_{t>0} \int_{\mathbb{R}^n} |x|^2 \rho(t) dx < +\infty. \tag{25}$$

Then for some  $\varepsilon \in (0, 1]$  and  $0 < p, q < 1$ , we have

$$\int_{\mathbb{R}^n} |x|^{1+\varepsilon} \rho(t_j)^{\alpha-\frac{1}{2}} dx \leq \left(\int_{\mathbb{R}^n} |x|^2 \rho(t_j) dx\right)^p \left(\int_{\mathbb{R}^n} \rho(t_j)^q dx\right)^r. \tag{26}$$

According to (13), (25) and (26), we observe that  $\{\rho(t_j)^{\alpha-\frac{1}{2}}\}_{j=1}^\infty$  is bounded in  $H^1(\mathbb{R}^n) \cap L^1_{1+\varepsilon}(\mathbb{R}^n)$  which is compactly embedded in  $L^1(\mathbb{R}^n)$ . Thus there exist a subsequence of  $\{t_j\}_{j=1}^\infty$  and  $W \in L^1(\mathbb{R}^n)$  such that

$$\rho(t_j)^{\alpha-\frac{1}{2}} \rightarrow W \quad \text{in } L^1(\mathbb{R}^n) \quad (j \rightarrow \infty). \tag{27}$$

As  $\rho(t_j)^{\alpha-\frac{1}{2}} \rightarrow W$  in  $L^1(\mathbb{R}^n)$  we may choose a subsequence of  $\{t_j\}_{j=1}^\infty$  (we still denote it as  $\{t_j\}_{j=1}^\infty$ ) so that  $\rho(t_j) \rightarrow W^{\frac{1}{\alpha-\frac{1}{2}}}$  ( $j \rightarrow \infty$ ) almost everywhere in  $\mathbb{R}^n$ . Thus

$$\rho(t_j, x) \rightarrow U(x) := W^{\frac{1}{\alpha-\frac{1}{2}}}(x) \quad \text{a.e. in } \mathbb{R}^n \quad (j \rightarrow \infty). \tag{28}$$

On the other hand, we may also choose a subsequence of  $\{t_j\}$  such that  $\rho(t_j)^{\alpha-\frac{1}{2}}$  and  $\nabla \rho(t_j)^{\alpha-\frac{1}{2}}$  converge weakly in  $L^2(\mathbb{R}^n)$ . We note that such weak limits of those sequence coincides with  $U^{\alpha-\frac{1}{2}}$  and  $\nabla U^{\alpha-\frac{1}{2}}$ , respectively by (28) and Lebesgue's dominated convergence theorem. Besides from Theorem 2.2, we see that the weak limit  $U$  of  $\rho(t_j)$  also in  $L^p(\mathbb{R}^n)$  for any  $1 \leq p \leq \infty$  by taking a further subsequence. Then the Banach-Steinhaus theorem shows by passing subsequence that

$$\begin{aligned} \|U^{\alpha-\frac{1}{2}}\|_2^2 & \leq \liminf_{j \rightarrow \infty} \|\rho(t_j)^{\alpha-\frac{1}{2}}\|_2^2, \\ \|\nabla U^{\alpha-\frac{1}{2}}\|_2^2 & \leq \liminf_{j \rightarrow \infty} \|\nabla \rho(t_j)^{\alpha-\frac{1}{2}}\|_2^2, \\ \|U\|_p & \leq \liminf_{j \rightarrow \infty} \|\rho(t_j)\|_p \end{aligned} \tag{29}$$

and  $U^{\alpha-\frac{1}{2}} \in H^1(\mathbb{R}^n)$ . On account of (13), it follows that  $\{(-\Delta)^{-1} \rho(t_j)\}_{j=1}^\infty$  is a bounded sequence in  $\dot{W}^{2,\alpha}(\mathbb{R}^n)$ . Therefore we observe that  $(-\Delta)^{-1} \rho(t_j)$  converge to a function in  $L^p_{loc}(\mathbb{R}^n)$  for some  $p \geq 1$  by Rellich-Kondrachov's theorem. In addition, (25), (27) and Vitali's convergence theorem imply that  $\rho(t_j) \rightarrow U$  in  $L^1(\mathbb{R}^n)$  as  $j \rightarrow \infty$  and we have

$$\|U\|_1 = \lim_{j \rightarrow \infty} \|\rho(t_j)\|_1 = M_*.$$



By the interpolation inequality, (28) and (13), we obtain

$$\begin{aligned} \|\rho(t_j) - U\|_q &\leq \|\rho(t_j) - U\|_1^{1-\theta} \|\rho(t_j) - U\|_\infty^\theta \\ &\leq C \|\rho(t_j) - U\|_1^{1-\theta} \rightarrow 0 \quad (j \rightarrow \infty), \end{aligned} \tag{30}$$

where  $\frac{1}{q} = \frac{1-\theta}{1}$  and  $C > 0$  is independent of  $j$ . The non-negativity of  $I_j$  and (23) lead

$$\rho(t_j) \left| \nabla \left( \frac{\alpha}{\alpha-1} \rho(t_j)^{\alpha-\frac{1}{2}} - \psi(t_j) \right) \right|^2 \rightarrow 0 \quad \text{a.e. in } \mathbb{R}^n.$$

From Hölder’s inequality, Hardy-Littlewood-Sobolev’s inequality and (13),

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \rho(t_j) |\nabla(-\Delta)^{-1} \rho(t_j)|^2 dx - \int_{\mathbb{R}^n} U |\nabla(-\Delta)^{-1} U|^2 dx \right| \\ &\leq \int_{\mathbb{R}^n} |(\rho(t_j) - U) |\nabla(-\Delta)^{-1} \rho(t_j)|^2| dx \\ &\quad + \int_{\mathbb{R}^n} \left| U \nabla(-\Delta)^{-1} (\rho(t_j) - U) \cdot \nabla(-\Delta)^{-1} \rho(t_j) \right| dx \\ &\quad + \int_{\mathbb{R}^n} \left| U \nabla(-\Delta)^{-1} U \cdot \nabla(-\Delta)^{-1} (\rho(t_j) - U) \right| dx \\ &\leq \|\rho(t_j) - U\|_{p_1} \|(-\Delta)^{-\frac{1}{2}} \rho(t_j)\|_{2q_1}^2 \\ &\quad + \|U\|_{2\alpha-1} \|(-\Delta)^{-\frac{1}{2}} (\rho(t_j) - U)\|_{p_2} \|(-\Delta)^{-\frac{1}{2}} \rho(t_j)\|_{q_2} \\ &\quad + \|U\|_{2\alpha-1} \|(-\Delta)^{-\frac{1}{2}} U\|_{p_3} \|(-\Delta)^{-\frac{1}{2}} (\rho(t_j) - U)\|_{q_3} \\ &\leq \|\rho(t_j) - U\|_{p_1} \|\rho(t_j)\|_{\frac{2nq_1}{n+2q_1}}^2 \\ &\quad + \|U\|_{2\alpha-1} \|(\rho(t_j) - U)\|_{\frac{np_2}{n+p_2}} \|\rho(t_j)\|_{\frac{nq_2}{n+q_2}} \\ &\quad + \|U\|_{2\alpha-1} \|U\|_{\frac{np_3}{n+p_3}} \|\rho(t_j) - U\|_{\frac{nq_3}{n+q_3}}, \end{aligned} \tag{31}$$

where  $\frac{1}{p_1} + \frac{1}{q_1} = 1$ ,  $\frac{1}{2\alpha-1} + \frac{1}{p_2} + \frac{1}{q_2} = 1$ ,  $\frac{1}{2\alpha-1} + \frac{1}{p_3} + \frac{1}{q_3} = 1$ . The right hand side of last terms in (31) tend to 0 as  $j \rightarrow \infty$  by (30). Thus we observe

$$\int_{\mathbb{R}^n} \rho(t_j) |\nabla(-\Delta)^{-1} \rho(t_j)|^2 dx \rightarrow \int_{\mathbb{R}^n} U |\nabla(-\Delta)^{-1} U|^2 dx. \tag{32}$$

Combining (29), (30) and (32), we have

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^n} U \left| \nabla \left( \frac{\alpha}{\alpha-1} U^{\alpha-1} - (-\Delta)^{-1} U \right) \right|^2 dx \\ &= \int_{\mathbb{R}^n} |\nabla U^{\alpha-\frac{1}{2}}|^2 dx - 2 \int_{\mathbb{R}^n} U^{\alpha+1} dx + \int_{\mathbb{R}^n} U |\nabla(-\Delta)^{-1} U|^2 dx \\ &\leq \liminf_{j \rightarrow \infty} \left[ \int_{\mathbb{R}^n} |\nabla \rho(t_j)^{\alpha-\frac{1}{2}}|^2 dx - 2 \int_{\mathbb{R}^n} \rho(t_j)^{\alpha+1} dx \right. \\ &\quad \left. + \int_{\mathbb{R}^n} \rho(t_j) |\nabla(-\Delta)^{-1} \rho(t_j)|^2 dx \right] \\ &= \liminf_{j \rightarrow \infty} I_j \\ &= 0. \end{aligned}$$

Therefore we obtain

$$\int_{\mathbb{R}^n} U \left| \nabla \left( \frac{\alpha}{\alpha-1} U^{\alpha-1} - (-\Delta)^{-1} U \right) \right|^2 dx = 0.$$

Since  $M_* = \|U\|_1 > 0$ , there exists  $B_R(x_0) \subset \mathbb{R}^n$  such that  $\text{supp } U \subset B_R(x_0)$ . After a proper shifting, there exists  $A \in \mathbb{R}$  such that

$$\frac{\alpha}{\alpha-1} U^{\alpha-1} - (-\Delta)^{-1} U = A, \quad x \in \text{supp } U$$

or equivalently

$$-\frac{\alpha}{\alpha-1} \Delta U^{\alpha-1} = U, \quad x \in \text{supp } U. \quad (33)$$

Since  $U$  satisfies (33) and  $\|U\|_1 = M_*$ ,  $U$  coincides with the least mass non-trivial solutions  $V$  to (14).  $\square$

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*E-mail address:* [ogawa@math.tohoku.ac.jp](mailto:ogawa@math.tohoku.ac.jp)

*E-mail address:* [hiroshi\\_wakui@m.tohoku.ac.jp](mailto:hiroshi_wakui@m.tohoku.ac.jp)