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EXISTENCE AND NONEXISTENCE OF NONTRIVIAL SOLUTIONS OF SOME NONLINEAR FOURTH ORDER ELLIPTIC EQUATIONS

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Abstract. In this paper, we are concerned with the existence and nonexistence of nontrivial solutions for nonlinear elliptic equations involving a biharmonic operator. Concerning the second order equations, a complementary result was obtained for the problem of interior, exterior and whole space. The main purpose of this paper is to discuss whether the complementary result mentioned above is still valid for the nonlinear fourth order equations. We introduce "Kelvin type transformation" for a biharmonic operator to convert an exterior problem to an interior problem. The existence results in case of super-critical exterior problem are shown by introducing a weighted version of Sobolev-Poincaré type inequality, and the nonexistence results are shown by giving a Pohozaev-type identity for fourth order equations.

1. **Introduction.** Let us consider the following nonlinear fourth order equations:

$$\text{(E)}\begin{cases} \Delta^2 u &= |u|^{q-2}u \quad \text{in } \Omega, \\ u = &\dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where, $1 < q < \infty$ and $\Omega \subset \mathbb{R}^N$ $(N > 4)$ is a domain with smooth boundary. The operator $\Delta^2(\cdot) := \Delta(\Delta(\cdot))$ is called a biharmonic operator, which appears in the equations of a vibrating plate. The solution of the equation (E) is related to the element $u \in H_0^2(\Omega)$ which attains the best constant of the Sobolev inequality

$$\|u\|_{L^q} \le C\|\Delta u\|_{L^2}, \ \forall u \in H_0^2(\Omega), \ q \le 2^*(2), \tag{S}$$

Here the exponent $2^*(k) := \dfrac{2N}{N-2k}$ (if $2k < N$), $:= \infty$ (if $2k \ge N$).

In general, the existence and nonexistence of nontrivial solutions of nonlinear elliptic equations depends on the exponent q and the shape of domain Ω. As for the case of the following second order equations,

$$\begin{cases} -\Delta u &= |u|^{q-2}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

three types of problem (E) with $\Omega = \mathbb{R}^N$, $\Omega = \Omega_0$ star-shaped bounded (interior problem) and $\Omega = \mathbb{R}^N \setminus \overline{\Omega_0}$ (exterior problem) are complementary to each others, as is shown in the following table (cf. [8], [4], [3]).

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	$1 < q < 2^*(1)$	$q = 2^*(1)$	$q > 2^*(1)$
interior	$\exists$ pos. sol.	no pos. sol.	no sol.
$\mathbb{R}^N$	no sol.	$\exists$ pos. sol.	no sol.
exterior	no sol.	no pos. sol.	$\exists$ pos. sol (in a radial case)

The main purpose of this paper is to discuss whether the complementary result mentioned above is valid for the nonlinear fourth order equations.

It seems difficult to show the existence for the case where q is super critical and Ω is an exterior domain due to the absence of embedding theorems. In the previous work by S-Hashimoto and Ôtani [3], they overcame these difficulties by means of the transformation which reduces exterior problem to the problem in annuli. However, although this type of transformation is effective for the Laplacian, not for the biharmonic operator. Therefore, we here make use of another "Kelvin type transformation" for the biharmonic operator. It is introduced by Lin [5] to prove radial symmetry of positive solutions for (E) with $\Omega = \mathbb{R}^N$, $q = 2^*(2)$.

Lemma 1. *For the function u, we define $v(y) = |x|^{N-4}u(x)$ with $y = \dfrac{x}{|x|^2}$. $\Omega \ni x \mapsto y \in \tilde{\Omega}$. The equation* (E) *is transformed into the following equation* (E)$'$.

$$(\mathrm{E})' \begin{cases} \Delta^2 v(y) &= |y|^{(N-4)q-2N}|v|^{q-2}v(y) \quad y \in \tilde{\Omega}, \\ v(y) &= \dfrac{\partial v}{\partial n}(y) = 0 \qquad\qquad\quad y \in \partial\tilde{\Omega}. \end{cases}$$

Proof. First of all, let $v(y) = |x|^s u(x)$ for $s > 0$. Then,

$$\frac{\partial}{\partial y_i}v(y) = -s|x|^s x_i u + |x|^{s+2}\frac{\partial}{\partial x_i}u - 2|x|^s x \cdot \nabla u x_i \tag{1}$$

$$\begin{aligned} \Delta_y v(y) &= \Delta_y(|x|^s u(x)) \\ &= |x|^{s+4}\Delta_x u - (N-s-2)|x|^{s+2}(su + 2x \cdot \nabla u). \end{aligned} \tag{2}$$

We here take $s = N - 4$,

$$\Delta_y v = |x|^N \Delta u - 2|x|^{N-2}\{(N-4)u + 2x \cdot \nabla u\},$$

$$\Delta_y(\Delta_y v) = \Delta_y(|x|^N \Delta u) - 2\Delta_y[|x|^{N-2}\{(N-4)u + 2x \cdot \nabla u\}].$$

To calculate the first term, we take $s = N$ in (2). Then,

$$\Delta_y(|x|^N \Delta u) = |x|^{N+4}\Delta(\Delta u) + 2|x|^{N+2}\{N\Delta u + 2x \cdot \nabla(\Delta u)\}.$$

Similarly, taking $s = N - 2$,

$$-2\Delta_y(|x|^{N-2}(N-4)u) = -2(N-4)|x|^{N+2}\Delta u.$$

From the fact that $\Delta(x \cdot \nabla u) = 2\Delta u + x \cdot \nabla(\Delta u)$,

$$\begin{aligned} -2\Delta_y(|x|^{N-2}2x \cdot \nabla u) &= -4|x|^{N+2}\Delta(x \cdot \nabla u) \\ &= -8|x|^{N+2}\Delta u - 4|x|^{x+2}x \cdot \nabla(\Delta u). \end{aligned}$$

We finally deduce,

$$\Delta_y^2 v(y) = |x|^{N+4}\Delta_x^2 u(x).$$

We note that the boundary condition yields from $\nabla u = 0$ on $\partial\Omega$ and (1). □

In view of the transformation mentioned above, We consider the following equation $(\mathrm{E})_\alpha$ with real parameter α in bounded domains.

$$(\mathrm{E})_\alpha \begin{cases} \Delta^2 u &= |x|^\alpha |u|^{q-2}u \quad \text{in } \Omega, \\ u &= \dfrac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

The main theorems are stated in section 2. section 3 contains the proof of the nonexistence theorem and we prove the existence theorem in section 4.

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2. **Main results.** Our main results about the equation $(\mathrm{E})_\alpha$ are stated as follows.

Theorem 1 (Nonexistence)**.** *Let Ω be a bounded star-shaped domain and let $\alpha > -N$.*

(i) *If $q > \dfrac{2(\alpha+N)}{N-4}$, then $(E)_\alpha$ has no nontrivial classical solution.*

(ii) *If $q = \dfrac{2(\alpha+N)}{N-4}$ and if we assume that Ω is strictly star-shaped, then $(E)_\alpha$ has no positive classical solution.*

Theorem 2 (Existence)**.** *Let Ω be a bounded domain.*

(i) *If $q < \min\{\dfrac{2(\alpha+N)}{N-4}, 2^*(2)\}$ and $q \neq 2$, then $(E)_\alpha$ has a nontrivial weak solution in $H_0^2(\Omega)$.*

(ii) *If $\Omega = B_R(0)$, $q < \dfrac{2(\alpha+N)}{N-4}$ and $q \neq 2$, then $(E)_\alpha$ has a radial weak solution in $H_0^2(\Omega)$.*

Remark 1. When $\alpha = 0$, we have the following regularity result of the solutions of $(\mathrm{E})_0$.

Proposition 1. *Let $q < 2^*(2)$. Then every weak solution of $(\mathrm{E})_0$ belonging to $H_0^2(\Omega)$ is in $C^4(\overline{\Omega})$. Moreover, if q is even integer, u is in $C^\infty(\overline{\Omega})$.*

Proof. At first, we can show the following lemma by the similar way to Lemma 3.1 of Drábek-Ôtani [2].

Lemma 2. *Let $u \in W^{2,r_k}(\Omega)$ be a solution of $(\mathrm{E})_0$ and let $0 < r_k < \dfrac{q-1}{q}\dfrac{N}{2}$. Then $u \in W^{2,r_{k+1}}(\Omega)$ with $\dfrac{1}{r_{k+1}} = \dfrac{q-1}{r_k} - \dfrac{2q}{N}$.*

Proof. Assume $u \in W^{2,r_k}(\Omega)$ $(2r_k < N)$, then $u \in L^{s_k}(\Omega)$ with $\dfrac{1}{s_k} = \dfrac{1}{r_k} - \dfrac{2}{N}$ by Sobolev's embedding theorem. Since $|u|^{q-2}u \in L^{t_k}(\Omega)$ with $t_k = \dfrac{s_k}{q-1}$, by elliptic estimate we have $\Delta u \in W^{2,t_k}(\Omega)$. Since $2qr_k < (q-1)N$, $2t_k < N$ and $W^{2,t_k}(\Omega) \subset L^{Nt_k/(N-2t_k)}(\Omega)$. Therefore $\Delta u \in L^{r_{k+1}}(\Omega)$, that is $u \in W^{2,r_{k+1}}(\Omega)$. □

From this lemma, we can show that u belongs to $L^\alpha(\Omega)$ for all $\alpha > 1$. Let $(q-1)N \leq 4q$ in the first place. Then if we set $r_1 = 2$, $\Delta u \in W^{2,t_1}(\Omega)$ with $\dfrac{1}{t_1} - \dfrac{2}{N} = (q-1)(\dfrac{1}{2} - \dfrac{2}{N}) - \dfrac{2}{N} \leq 0$. Therefore $u \in W^{2,\beta}(\Omega)$ for any $\beta > 1$. Moreover $u \in L^\alpha(\Omega)$ for any $\alpha > 1$.

Secondly, let $(q-1)N > 4q$. We also set $r_1 = 2$ and $\dfrac{1}{r_1} - \dfrac{2q}{N(q-2)} = -\rho < 0$. Let us suppose for all $k \in \mathbb{N}$, $0 < r_k < \dfrac{q-1}{q}\dfrac{N}{2}$. We are going to prove two cases.

(i) The case $q \leq 2$: Since $\dfrac{1}{r_{k+1}} - \dfrac{1}{r_k} = \dfrac{q-2}{r_k} - \dfrac{2q}{N}$, $\dfrac{1}{r_{k+1}} \leq \dfrac{1}{r_k} - \dfrac{2q}{N} \leq \dfrac{1}{r_1} - \dfrac{2qk}{N}$,

The right side is negative when k goes to infinity. This is a contradiction.
(ii) The case $q > 2$: From the recurrence formula of Lemma 2,

$$\frac{1}{r_k} = \frac{2q}{N(q-2)} - \rho\,(q-1)^{k-1}$$

for all k satisfying $0 < r_k < \frac{q-1}{q}\frac{N}{2}$. However, $\lim_{k\to\infty} (q-1)^{k-1} = +\infty$ and this leads to a contradiction.

Therefore there exists $j \in \mathbb{N}$ such that $0 < r_{j-1} < \frac{q-1}{q}\frac{N}{2}$ and $r_j \notin (0, \frac{q-1}{q}\frac{N}{2})$. Since $\frac{1}{r_{k+1}} = (q-1)\left\{\frac{1}{r_k} - \frac{2q}{N(q-1)}\right\}$, $\frac{1}{r_j} > 0$ if and only if $\frac{1}{r_{j-1}} > \frac{2q}{N(q-1)}$. Hence $r_j \geq \frac{q-1}{q}\frac{N}{2}$. By the Sobolev embedding theorem, we can deduce $u \in L^\alpha(\Omega)$ for any $\alpha > 1$.

Using elliptic estimate again, we can show that $u \in W^{2,\beta}(\Omega)$ for some $\beta > \frac{N}{2}$. By virtue of the embedding $W^{2,\beta}(\Omega) \subset C^\theta(\overline{\Omega})$ for some $\theta \in (0,1)$, $u \in C^\theta(\overline{\Omega})$. Since $\Delta(\Delta u) = |u|^{q-2}u \in C^{\theta'}(\overline{\Omega})$ for some $\theta' \in (0,1)$, $\Delta u \in C^{2,\theta'}(\overline{\Omega})$ by Schauder estimate. Therefore $u \in C^4(\overline{\Omega})$. Moreover if q is even integer, since the mapping $t \mapsto |t|^{q-2}t$ is C^∞, the argument above implies that $u \in C^\infty(\overline{\Omega})$. □

By applying Kelvin-type transformation to above theorem, we obtain the following complementary result between interior and exterior problems.

Corollary 1. *As for the equation (E), the following existence and nonexistence results hold.*

- (i) *If $1 < q < 2^*(2)$, there exists a nontrivial classical solution for the interior problem, while there exists no nontrivial classical solution for the exterior problem when Ω^c is star-shaped.*
- (ii) *If $q > 2^*(2)$, there exists no nontrivial classical solution for the interior problem when Ω is star-shaped, while there exists a radial classical solution for the problem when Ω^c is a ball.*

Proof. Noting that by inversion stated above, a domain which is star-shaped with respect to the origin is transformed to an exterior of a star-shaped domain, and vice versa. If u is a solution of (E) in an exterior domain, it is transformed into a solution of (E)$'$. Therefore, since $\alpha = (N-4)q - 2N$ in $(\mathrm{E})_\alpha$, there is no solution for the exterior problem if $q < 2^*(2)$.

On the other hand, if $q > 2^*(2)$, since the exponent of a coefficient of the transformed problem $\alpha = (N-4)q - 2N > \frac{N-4}{2}q - N$, there exists a nontrivial solution for the problem in exterior of a ball. □

3. **Pohozaev-type identity for solutions of fourth order equations.** In this section, we introduce the Pohozaev-type identity for fourth order equations and prove theorems. First of all, we show the identity for the solution of $(\mathrm{E})_\alpha$.

Lemma 3. *Let $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ be a solution of $(E)_\alpha$ with $\alpha > -N$. Then the following identity holds.*

$$\begin{aligned} &\frac{1}{2}\int_{\partial\Omega} |\Delta u|^2 (x\cdot\vec{n})dS \\ &\quad = \frac{\alpha+N}{q}\int_\Omega |x|^\alpha |u|^q dx + \frac{4-N}{2}\int_\Omega |\Delta u|^2 dx. \end{aligned} \tag{3}$$

Proof. We calculate $\displaystyle\int_\Omega (E)_\alpha (x\cdot\nabla u)dx$.

$$\int_\Omega |x|^\alpha |u|^{q-2} u (x\cdot\nabla u)dx = -\frac{\alpha+N}{q}\int_\Omega |x|^\alpha |u|^q dx.$$

$$\begin{aligned} &\int_\Omega \Delta(\Delta u)(x\cdot\nabla u)dx \\ &= \int_{\partial\Omega} \nabla(\Delta u)\vec{n}(x\cdot\nabla u)dS - \int_\Omega \nabla(\Delta u)\cdot\nabla(x\cdot\nabla u)dx \\ &= -\int_{\partial\Omega} \Delta u(\nabla(x\cdot\nabla u)\cdot\vec{n})dS \\ &\quad + \int_\Omega \Delta u \Delta(x\cdot\nabla u)dx =: -I_{\partial\Omega} + I_\Omega. \end{aligned}$$

As for the integral on the boundary, since $\nabla u = 0$ on $\partial\Omega$,

$$\nabla(x\cdot\nabla u)\cdot\vec{n} = \nabla u\cdot\vec{n} + \sum_{i,j} x_j \frac{\partial^2 u}{\partial x_i \partial x_j} n_i.$$

Moreover, since $u = \nabla u = 0$,

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial n^2} n_i n_j.$$

Therefore,

$$\begin{aligned} I_{\partial\Omega} &= \int_{\partial\Omega} \Delta u(\nabla u\cdot\vec{n})dS + \sum_{i,j=1}^N \int_{\partial\Omega} \Delta u x_j \frac{\partial}{\partial x_i}\frac{\partial u}{\partial x_j} n_i dS \\ &= \int_{\partial\Omega} |\Delta u|^2 (x\cdot\vec{n})dS. \end{aligned}$$

On the other hand,

$$\begin{aligned} I_\Omega &= \int_\Omega \Delta u(2\Delta u + x\cdot\nabla(\Delta u))dx \\ &= 2\int_\Omega |\Delta u|^2 dx + \int_\Omega x\cdot\nabla(\frac{1}{2}|\Delta u|^2)dx \\ &= 2\int_\Omega |\Delta u|^2 dx + \frac{1}{2}\int_{\partial\Omega} |\Delta u|^2(x\cdot\vec{n})dS - \frac{N}{2}\int_\Omega |\Delta u|^2 dx \\ &= \frac{1}{2}\int_{\partial\Omega} |\Delta u|^2 (x\cdot\vec{n})dS + \frac{4-N}{2}\int_\Omega |\Delta u|^2 dx \end{aligned}$$

Therefore (3) holds. □

Proof of Theorem 1. (i) From Lemma 3, the star-shapedness assumption on Ω and the relation

$$\int_\Omega |\Delta u|^2 dx = \int_\Omega |x|^\alpha |u|^q dx,$$

we deduce

$$\left(\frac{\alpha+N}{q} + \frac{4-N}{2}\right)\int_\Omega |x|^\alpha |u|^q dx \geq 0.$$

Since $\dfrac{4-N}{2} + \dfrac{\alpha+N}{q} < 0 \iff q > \dfrac{2(\alpha+N)}{N-4}$, $(\mathrm{E})_\alpha$ has no nontrivial solution.
(ii) For the critical case, (3) implies

$$\int_{\partial\Omega} |\Delta u|^2 (x\cdot \vec{n}(x)) dS = 0.$$

If Ω is strictly star-shaped, i.e., there exists $\delta > 0$ such that $(x \cdot \vec{n}(x)) \geq \delta$, we deduce

$$\int_{\partial\Omega} |\Delta u|^2 dS = 0.$$

Since $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$, $\Delta u = 0$ on $\partial\Omega$. Therefore $-\Delta u$ satisfies

$$-\Delta(-\Delta u) = |u|^{q-2}u > 0 \text{ in } \Omega, \quad -\Delta u = 0 \text{ on } \partial\Omega.$$

By the maximum principle, $-\Delta u \geq 0$ in Ω. Therefore, by the Hopf-type maximum principle, we deduce that there exists $\rho > 0$ such that

$$\frac{\partial u}{\partial n} \leq -\rho < 0.$$

This contradicts the boundary condition $\dfrac{\partial u}{\partial n} = 0$. □

Remark 2. (1) As for the case $\Omega = \mathbb{R}^N$, we can repeat the same argument as Lemma 3 with $\partial\Omega = \emptyset$ (i.e., all integrals of $\partial\Omega$ disappear) and derive the following result.

Corollary 2. *Let $\Omega = \mathbb{R}^N$, then every solution of* $(\mathrm{E})_0$ *belonging to $C^4(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ satisfies the following Pohozaev identity.*

$$\left(\frac{N}{q} + \frac{4-N}{2}\right)\int_{\mathbb{R}^N} |u|^q dx = 0. \tag{4}$$

(2) We can deduce the following nonexistence result for the case $\Omega = \mathbb{R}^N$ from the corollary above.

Proposition 2. *When $\Omega = \mathbb{R}^N$, (E) has no nontrivial solution if $q \neq 2^*(2)$.*

Remark 3. We note that there exist positive solutions for $(\mathrm{E})_\alpha$ with $\alpha = 0$, $q = 2^*(2)$ and $\Omega = \mathbb{R}^N$. The existence of positive solutions was shown by Lions in [6]. Moreover, in the paper of Lin ([5]), every smooth positive solution u is radially symmetric about some point.

Proposition 3. *When $\Omega = \mathbb{R}^N$, (E) has a smooth positive solution u if $q = 2^*(2)$ and u has the following form:*

$$u(x) = C_N \frac{\varepsilon^{(N-4)/2}}{(|x-x_0|^2 + \varepsilon^2)^{(N-4)/2}}$$

where $\varepsilon > 0$, $x_0 \in \mathbb{R}^N$ and

$$C_N = [(N-4)(N-2)N(N+2)]^{\frac{N-4}{8}}.$$

From Propositions 2, 3 and Corollary 1, as for the existence and nonexistence of positive solutions, a complementary result is obtained for the problem of interior, exterior and whole space.

	$1 < q < 2^*(2)$	$q = 2^*(2)$	$q > 2^*(2)$
interior	$\exists$ pos. sol.	no pos. sol.	no sol.
$\mathbb{R}^N$	no sol.	$\exists$ pos. sol.	no sol.
exterior	no sol.	no pos. sol.	$\exists$ pos. sol (in a radial case)

4. **Weighted Sobolev-Poincaré type inequality.** To prove the existence theorem for fourth order equations $(\mathrm{E})_\alpha$, we need a weighted Sobolev-Poincaré type inequality. At first, we prepare an inequality for sub critical case.

Proposition 4. *Let* $1 < q < \min\left\{\dfrac{2(\alpha+N)}{N-4}, 2^*(2)\right\}$. *Then there exists* $C > 0$, *for all* $u \in H_0^2(\Omega)$,

$$\left(\int_\Omega |x|^\alpha |u|^q dx\right)^{1/q} \leq C\|\Delta u\|_{L^2}. \tag{5}$$

Proof. For the function $u \in H_0^2(\Omega)$, the Sobolev inequality

$$\|u\|_{L^{2^*(2)}} \leq K\|\Delta u\|_{L^2}$$

holds. Then by the Hölder inequality,

$$\begin{aligned}\int_\Omega |x|^\alpha |u|^q dx &\leq \left(\int_\Omega |x|^{\frac{2N\alpha}{2N-(N-4)q}}\right)^{\frac{2N-(N-4)q}{2N}} \left(\int_\Omega |u|^{2^*(2)}\right)^{q/2^*} \\ &\leq C\|\Delta u\|_{L^2}^q.\end{aligned}$$

Here $\dfrac{2N\alpha}{2N-(N-4)q} > -N$ implies the integral of the power of $|x|$ is summable. □

As for the case $q \geq 2^*(2)$ (critical or super critical), we can show the embedding results in the case of radial symmetry (cf. Ni [7], Dalmasso [1]).

Proposition 5. *Let* $\Omega = B_R(0)$ *and let* $u \in H^2_{0,rad}(B_R)$ *(i.e.* $u \in H_0^2(B_R)$ *and radially symmetric). If* $1 < q < \dfrac{2(\alpha+N)}{N-4}$, *then the inequality (5) holds.*

Proof. From the boundary condition $u(R) = u_r(R) = 0$,

$$\begin{aligned}u(R) - u(x) = -u(x) &= \int_{|x|}^R u_r(t)dt \\ &= [tu_r(t)]_r^R - \int_r^R tu_{rr}(t)dt \\ &= -ru_r(r) - \int_r^R tu_{rr}dt \\ &= \int_r^R (r-t)u_{rr}dt.\end{aligned}$$

Therefore

$$\begin{aligned}|u(x)| &\le \int_{|x|}^{R} t|u_{rr}(t)|dt \\ &\le \left(\int_{|x|}^{R} |u_{rr}(t)|^2 t^{N-1} dt\right)^{1/2} \left(\int_{|x|}^{R} t^{3-N} dt\right)^{1/2}. \qquad (6)\end{aligned}$$

Here,

$$\int_{|x|}^{R} t^{3-N} dt = \frac{1}{4-N}\left(R^{4-N} - |x|^{4-N}\right) \le \frac{1}{N-4}|x|^{4-N}. \qquad (7)$$

To deal with the first integral, we note

$$\Delta u = u_{rr} + \frac{N-1}{r} u_r$$

and integrate

$$\begin{aligned}\int_{B_R} |\Delta u|^2 dx &= \omega_N \int_0^R \left(u_{rr} + \frac{N-1}{r} u_r\right)^2 r^{N-1} dr \\ &= \omega_N \int_0^R u_{rr}^2 r^{N-1} dr + 2\omega_N (N-1) \int_0^R u_r u_{rr} r^{N-2} dr \\ &\quad + \omega_N (N-1)^2 \int_0^R u_r^2 r^{N-3} dr. \qquad (8)\end{aligned}$$

Using integration by parts, we get

$$\int_0^R u_r u_{rr} r^{N-2} dr = -\frac{N-2}{2} \int_0^R u_r^2 r^{N-3} dr. \qquad (9)$$

Then (8) and (9) imply

$$\int_{B_R} (\Delta u)^2 dx \ge \omega_N \int_0^R u_{rr}^2 r^{N-1} dr. \qquad (10)$$

Therefore,

$$\begin{aligned}\int_{|x|}^{R} |u_{rr}(t)|^2 t^{N-1} dt &\le \int_0^R |u_{rr}(t)|^2 t^{N-1} dt \\ &\le \omega_N^{-1} \int_{B_R} (\Delta u)^2 dx \qquad (11)\end{aligned}$$

holds. Then, from (6), (7) and (11), there exists $K > 0$ such that

$$|u(x)| \le K \frac{\|\Delta u\|_{L^2(B_R)}}{|x|^{(N-4)/2}}. \qquad (12)$$

Multiplying (12) by $|x|^\alpha$ and integrating over Ω yields

$$\int_{B_R} |x|^\alpha |u(x)|^q dx \le \frac{\|\Delta u\|_{L^2}^q}{C} \int_{B_R} |x|^{\alpha - \frac{N-4}{2} q} dx.$$

Therefore, if $\alpha > \dfrac{N-4}{2} q - N$, then the integral of right hand side is summable and (5) holds. □

Since the equation has a power nonlinearity, the following proposition is useful for proving this theorem.

Proposition 6 (Ôtani, [8]). *Let X be a real Banach space and we define*

$$\begin{aligned} \Phi(X) &:= \{\varphi : X \to (-\infty, +\infty], \textit{lower semi continuous convex}, \ \varphi \not\equiv +\infty\}, \\ D(\varphi) &:= \{x \in X; \varphi(x) < +\infty\}, \\ \partial\varphi(x) &:= \{x^* \in X^*; \varphi(w) - \varphi(x) \geq {}_{X^*}\langle x^*, w - x\rangle_X \ \forall w \in D(\varphi)\}, \\ D(\partial\varphi) &:= \{x \in X; \partial\varphi(x) \neq \emptyset\}. \end{aligned}$$

Let ϕ^i $(i = 1, 2)$ be nonnegative functions ϕ^i in satisfying:

(i) *There exists $\alpha_i > 1$ with $\alpha_1 \neq \alpha_2$ s.t. $\phi^i(\lambda z) = \lambda^{\alpha_i}\phi^i(z)$ $\forall \lambda > 0$ $\forall z \in D(\phi^i)$.*

(ii) $(\phi^2(z))^{1/\alpha_2} \leq C(\phi^1(z))^{1/\alpha_1}$ $\forall z \in D(\phi^1)$.

Suppose that there exists an element $u \in D(\partial\phi^2)$ such that

(iii) *u gives the best possible constant for* (ii), *i.e.,*

$$R(u) = \min\{R(z); z \in D(\phi^1), \ \phi^2(z) \neq 0\}, \quad R(z) = \frac{(\phi^1(z))^{1/\alpha_1}}{(\phi^2(z))^{1/\alpha_2}}.$$

(iv) $\alpha_1\phi^1(u) = \alpha_2\phi^2(u)$.

Then $u \in D(\partial\phi^1)$ and $\partial\phi^2(u) \subset \partial\phi^1(u)$. In particular, if $\partial\phi^1$ is single valued, then u becomes a nontrivial solution of $\partial\phi^1(u) = \partial\phi^2(u)$.

Remark 4. If ϕ is Fréchet differentiable, then $\partial\phi$, the subdifferential of ϕ is coincide with its Fréchet derivative.

Proof of Theorem 2. Let $X = H_0^2(\Omega)$, and define functionals as follows.

$$\phi^1(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx, \quad \phi^2(u) = \frac{1}{q}\int_\Omega |x|^\alpha |u|^q dx.$$

We verify the conditions of Proposition 6.
(i) This condition easily follows from the fact that the homogeneous degree of ϕ^1 is 2 and that of ϕ^2 is q.
(ii) This is a direct consequence of Propositions 4 and 5.

To verify the condition (iii), we need the following lemma.

Lemma 4. *Let (u_n) be a bounded sequence in $H_0^2(\Omega)$ and let $1 < q < \dfrac{2(\alpha+N)}{N-4}$. Then $\displaystyle\int_\Omega |x|^\alpha |u_n|^q dx \to \int_\Omega |x|^\alpha |u|^q dx$.*

Proof. By the Hölder inequality, for $\beta \in (0, 1)$,

$$\begin{aligned} \int_\Omega |x|^\alpha |u_n - u|^q dx &= \int_\Omega |x|^\alpha |u_n - u|^{q-\beta} |u_n - u|^\beta dx \\ &\leq \left(\int_\Omega |u_n - u| dx\right)^\beta \left(\int_\Omega |x|^{\alpha/(1-\beta)} |u_n - u|^{(q-\beta)/(1-\beta)} dx\right)^{1-\beta}. \end{aligned}$$

From Propositions 4 and 5, the latter integral is bounded if

$$\frac{\alpha}{1-\beta} > \frac{N-4}{2}\frac{q-\beta}{1-\beta} - N. \tag{13}$$

Since (13) is equivalent to

$$\alpha > \frac{N-4}{2}q - N + \beta\frac{N+4}{2},$$

If we take β sufficiently small, then the integral is bounded.

Finally, from the fact that $H_0^2(\Omega)$ is compactly embedded in $L^1(\Omega)$,

$$\int_\Omega |x|^\alpha |u_n - u|^q dx \to 0 \text{ as } n \to 0.$$

$\square$

Proof of Theorem 2 (continued) (iii) Let $c = \inf R(v)$, and let v_k is a minimizing sequence of R (i.e., $R(v_k) \to c$). Setting $u_k(x) := \dfrac{1}{\|\Delta v_k\|_2} v_k(x)$, $\|\Delta u_k\|_2 = 1$ and $R(v_k) = R(u_k)$. Then by Lemma 4, there exist an element $u_0 \in H_0^2(\Omega)$ and a subsequence (u_{k_j}) such that

$$\begin{aligned} \Delta u_{k_j} &\rightharpoonup \Delta u_0 \quad \text{weakly in } L^2(\Omega), \\ |x|^{\alpha/q} u_{k_j} &\to |x|^{\alpha/q} u_0 \quad \text{stlongly in } L^q(\Omega). \end{aligned}$$

From weak lower semi continuity of the norm,

$$c = \lim_{j\to\infty} R(u_{k_j}) \geq R(u_0).$$

The definition of c implies $R(u_0) = \min R(v)$.
(iv) Let w be an element satisfying (iii) and suppose that w does not satisfy (iv), then we can choose an appropriate $\lambda > 0$ so that $u = \lambda w$ satisfies both (iii) and (iv), since ϕ^1 and ϕ^2 are homogeneous functions of degree 2 and q respectively.

Then all assumptions of Proposition 6 are fulfilled. Thus it is proved that $(\mathrm{E})_\alpha$ is a nontrivial solution u. $\square$

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