PROCEEDINGS OF THE FOURTH INTERNATIONAL CONFERENCE ON
DYNAMICAL SYSTEMS AND DIFFERENTIAL EQUATIONS
May 24 – 27, 2002, Wilmington, NC, USA pp. **694–700**

STABILITY OF A LINEAR FUNCTIONAL EQUATION IN BANACH MODULES

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Abstract. We prove the generalized Hyers-Ulam-Rassias stability of a linear functional equation in Banach modules over a unital C^*-algebra.

1. **Introduction.** The stability of functional equations has been investigated by several authors ([1], [2], [3]).

Recently, T. Trif [6, Theorem 2.1] proved that, for vector spaces V and W, a mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation

$$n\ {}_{n-2}C_{k-2} f(\frac{x_1 + \cdots + x_n}{n}) + {}_{n-2}C_{k-1} \sum_{i=1}^{n} f(x_i) = k \sum_{1 \le i_1 < \cdots < i_k \le n} f(\frac{x_{i_1} + \cdots + x_{i_k}}{k})$$

for all $x_1, \cdots, x_n \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$. He proved the stability of this equation (see [6, Theorem 3.1 and Theorem 3.2]).

We consider the following functional equation derived from the Trif's functional equation.

Lemma 1. *Let V and W be vector spaces and let n, p, k be positive integers. A mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation*

$$npf(\frac{x_1 + \cdots + x_{np}}{np}) + (pk - p) \sum_{i=1}^{n} f(\frac{x_{pi-p+1} + \cdots + x_{pi}}{p}) = k \sum_{i=1}^{np} f(\frac{x_i + \cdots + x_{i+k-1}}{k}) \tag{1}$$

for all $x_1 = x_{np+1}, \cdots, x_{k-1} = x_{np+k-1}, x_k, \cdots, x_{np} \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

Proof. Assume that a mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation (1).

2000 *Mathematics Subject Classification.* Primary 39B52, 47B48, 46L05.
Key words and phrases. linear functional equation, Banach module over C^*-algebra, stability.

Replace x and y by kx and ky, respectively, and let $x_3 = \cdots = x_{np} = 0$ in (1). Then we get

$$npf(\frac{kx+ky}{np}) + p(k-1)f(\frac{kx+ky}{p}) = k(k-1)f(x+y) + kf(x) + kf(y) \qquad (2)$$

for all $x, y \in V$. Let $y = 0$ and replace x by $x + y$ in (2). Then we get

$$npf(\frac{kx+ky}{np}) + p(k-1)f(\frac{kx+ky}{p}) = k^2 f(x+y) \qquad (3)$$

for all $x, y \in V$. It follows from (2) and (3) that

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in V$.

The converse is obvious. $\square$

Throughout this paper, let A be a unital C^*-algebra with norm $|\cdot|$ and $\mathcal{U}(A)$ the unitary group of A. Let ${}_A\mathcal{B}$ and ${}_A\mathcal{C}$ be left Banach A-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively. Let n, p and k be positive integers.

The main purpose of this paper is to prove the generalized Hyers-Ulam-Rassias stability of the functional equation (1) in Banach modules over a unital C^*-algebra.

2. Stability of a linear functional equation in Banach modules over a C^*-algebra.

We are going to prove the generalized Hyers-Ulam-Rassias stability of the functional equation (1) in Banach modules over a unital C^*-algebra for special cases.

Theorem 1. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_A\mathcal{B}^{p^n} \to [0, \infty)$ such that*

$$\begin{aligned}\widetilde{\varphi}(x_1, \cdots, x_{p^n}) := \sum_{j=0}^{\infty} p^j \varphi(&\frac{(p-1)k+1}{p^j}x_1, -\frac{k}{p^j}x_2, \cdots, -\frac{k}{p^j}x_p, \cdots, \\ &\frac{(p-1)k+1}{p^j}x_{p^n-p+1}, -\frac{k}{p^j}x_{p^n-p+2}, \cdots, -\frac{k}{p^j}x_{p^n}) < \infty, \qquad (4)\end{aligned}$$

$$\begin{aligned}\|D_u f(x_1, \cdots, x_{p^n})\| = \|p^n u f(\frac{x_1 + \cdots + x_{p^n}}{p^n}) + p^2 k \sum_{i=1}^{p^{n-1}} uf(\frac{x_{pi-p+1} + \cdots + x_{pi}}{p}) \\ - (pk+1)\sum_{i=1}^{p^n} f(\frac{ux_i + \cdots + ux_{i+pk}}{pk+1})\| \le \varphi(x_1, \cdots, x_{p^n}) \qquad (5)\end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x_1 = x_{p^n+1}, \cdots, x_{pk} = x_{p^n+pk}, x_{pk+1}, \cdots, x_{p^n} \in {}_A\mathcal{B}$. Then there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ such that

$$\|f(x) - T(x)\| \le \frac{1}{(pk+1)p^{n-1}} \widetilde{\varphi}(\underbrace{x, \cdots, x}_{p^n\text{-}times}) \qquad (6)$$

for all $x \in {}_A\mathcal{B}$.

Proof. Put $u = 1 \in \mathcal{U}(A)$. Let $x_{pi-p+1} = x$ and $x_{pi-p+2} = \cdots = x_{pi} = y$ in (5) for all $i = 1, \cdots, p^{n-1}$. Then we get

$$\begin{aligned}\|p^n f(\frac{x+(p-1)y}{p}) &+ pk\cdot p^n f(\frac{x+(p-1)y}{p}) \\ &- (pk+1)p^{n-1} f(\frac{(k+1)x+(p-1)ky}{pk+1}) \\ &- (p-1)(pk+1)p^{n-1} f(\frac{kx+((p-1)k+1)y}{pk+1})\| \\ &\le \varphi(x, \underbrace{y, \cdots, y}_{p-1\text{-times}}, \cdots, x, \underbrace{y, \cdots, y}_{p-1\text{-times}})\end{aligned} \tag{7}$$

for all $x, y \in {}_A\mathcal{B}$. Replacing x and y by $((p-1)k+1)x$ and $-kx$ in (7), respectively, we get

$$\begin{aligned}&\|(pk+1)p^n f(\frac{x}{p}) - (pk+1)p^{n-1} f(x)\| \\ &\le \varphi(((p-1)k+1)x, \underbrace{-kx, \cdots, -kx}_{p-1\text{-times}}, \cdots, ((p-1)k+1)x, \underbrace{-kx, \cdots, -kx}_{p-1\text{-times}})\end{aligned}$$

for all $x \in {}_A\mathcal{B}$. So one can obtain

$$\begin{aligned}\|f(x) - pf(\frac{x}{p})\| \le \frac{1}{(pk+1)p^{n-1}} \varphi(((p-1)k+1)x, &\underbrace{-kx, \cdots, -kx}_{p-1\text{-times}}, \cdots, \\ &((p-1)k+1)x, \underbrace{-kx, \cdots, -kx}_{p-1\text{-times}}),\end{aligned}$$

and hence

$$\begin{aligned}\|p^j f(\frac{1}{p^j}x) - p^{j+1} f(\frac{1}{p^{j+1}}x)\| \le \frac{p^j}{(pk+1)p^{n-1}} \varphi(\frac{(p-1)k+1}{p^j}x, \\ \underbrace{-\frac{k}{p^j}x, \cdots, -\frac{k}{p^j}x}_{p-1\text{-times}}, \cdots, \frac{(p-1)k+1}{p^j}x, \underbrace{-\frac{k}{p^j}x, \cdots, -\frac{k}{p^j}x}_{p-1\text{-times}})\end{aligned}$$

for all $x \in {}_A\mathcal{B}$. So we get

$$\begin{aligned}\|f(x) - p^j f(\frac{1}{p^j}x)\| \le \frac{1}{(pk+1)p^{n-1}} \sum_{m=0}^{j-1} p^m \varphi(\frac{(p-1)k+1}{p^m}x, \\ \underbrace{-\frac{k}{p^m}x, \cdots, -\frac{k}{p^m}x}_{p-1\text{-times}}, \cdots, \frac{(p-1)k+1}{p^m}x, \underbrace{-\frac{k}{p^m}x, \cdots, -\frac{k}{p^m}x}_{p-1\text{-times}})\end{aligned} \tag{8}$$

for all $x \in {}_A\mathcal{B}$.

Let x be an element in ${}_A\mathcal{B}$. For positive integers l and m with $l > m$,

$$\|p^l f(\frac{1}{p^l}x) - p^m f(\frac{1}{p^m}x)\| \le \frac{1}{(pk+1)p^{n-1}} \sum_{j=m}^{l-1} p^j \varphi(\frac{(p-1)k+1}{p^j}x, \underbrace{-\frac{k}{p^j}x, \cdots, -\frac{k}{p^j}x}_{p-1\text{-times}}, \cdots, \frac{(p-1)k+1}{p^j}x, \underbrace{-\frac{k}{p^j}x, \cdots, -\frac{k}{p^j}x}_{p-1\text{-times}}),$$

which tends to zero as $m \to \infty$ by (4). So $\{p^j f(\frac{1}{p^j}x)\}$ is a Cauchy sequence for all $x \in {}_A\mathcal{B}$. Since ${}_A\mathcal{C}$ is complete, the sequence $\{p^j f(\frac{1}{p^j}x)\}$ converges for all $x \in {}_A\mathcal{B}$. We can define a mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ by

$$T(x) = \lim_{j\to\infty} p^j f(\frac{1}{p^j}x) \tag{9}$$

for all $x \in {}_A\mathcal{B}$.

By (9), (4) and (5), we get

$$\begin{aligned}\|D_1 T(x_1, \cdots, x_{p^n})\| &= \lim_{j\to\infty} p^j \|D_1 f(\frac{1}{p^j}x_1, \cdots, \frac{1}{p^j}x_{p^n})\| \\ &\le \lim_{j\to\infty} p^j \varphi(\frac{1}{p^j}x_1, \cdots, \frac{1}{p^j}x_{p^n}) = 0\end{aligned}$$

for all $x_1, \cdots, x_{p^n} \in {}_A\mathcal{B}$. Hence $D_1 T(x_1, \cdots, x_{p^n}) = 0$ for all $x_1, \cdots, x_{p^n} \in {}_A\mathcal{B}$. By Lemma 1, T is additive. Moreover, by passing to the limit in (8) as $j \to \infty$, we get the inequality (6).

Now let $L : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be another additive mapping satisfying

$$\|f(x) - L(x)\| \le \frac{1}{(pk+1)p^{n-1}} \widetilde{\varphi}(\underbrace{x, \cdots, x}_{p^n\text{-times}})$$

for all $x \in {}_A\mathcal{B}$.

$$\begin{aligned}\|T(x) - L(x)\| &= p^j \|T(\frac{1}{p^j}x) - L(\frac{1}{p^j}x)\| \\ &\le p^j \|T(\frac{1}{p^j}x) - f(\frac{1}{p^j}x)\| + p^j \|f(\frac{1}{p^j}x) - L(\frac{1}{p^j}x)\| \\ &\le \frac{2}{(pk+1)p^{n-1}} p^j \widetilde{\varphi}(\underbrace{\frac{1}{p^j}x, \cdots, \frac{1}{p^j}x}_{p^n\text{-times}}),\end{aligned}$$

which tends to zero as $j \to \infty$ by (4). Thus $T(x) = L(x)$ for all $x \in {}_A\mathcal{B}$. This proves the uniqueness of T.

By the assumption, for each $u \in \mathcal{U}(A)$,

$$p^j \|D_u f(\underbrace{\frac{1}{p^j}x, \cdots, \frac{1}{p^j}x}_{p^n\text{-times}})\| \le p^j \varphi(\underbrace{\frac{1}{p^j}x, \cdots, \frac{1}{p^j}x}_{p^n\text{-times}})$$

for all $x \in {}_A\mathcal{B}$, and

$$p^j \|D_u f(\underbrace{\frac{1}{p^j}x, \cdots, \frac{1}{p^j}x}_{p^n\text{-times}})\| \to 0$$

as $j \to \infty$ for all $x \in {}_A\mathcal{B}$. So

$$D_uT(\underbrace{x, \cdots, x}_{p^n\text{-times}}) = \lim_{j\to\infty} p^j D_u f(\underbrace{\frac{1}{p^j}x, \cdots, \frac{1}{p^j}x}_{p^n\text{-times}}) = 0$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Hence

$$D_uT(\underbrace{x, \cdots, x}_{p^n\text{-times}}) = p^n uT(x) + pk \cdot p^n uT(x) - (pk+1)p^n T(ux) = 0$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. So

$$uT(x) = T(ux)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$.

Now let $a \in A$ ($a \neq 0$) and M an integer greater than $4|a|$. Then

$$|\frac{a}{M}| = \frac{1}{M}|a| < \frac{|a|}{4|a|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By [4, Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(A)$ such that $3\frac{a}{M} = u_1 + u_2 + u_3$. So

$$\begin{aligned} T(ax) &= T(\frac{M}{3} \cdot 3\frac{a}{M}x) = M \cdot T(\frac{1}{3} \cdot 3\frac{a}{M}x) = \frac{M}{3}T(3\frac{a}{M}x) \\ &= \frac{M}{3}T(u_1x + u_2x + u_3x) = \frac{M}{3}(T(u_1x) + T(u_2x) + T(u_3x)) \\ &= \frac{M}{3}(u_1 + u_2 + u_3)T(x) = \frac{M}{3} \cdot 3\frac{a}{M}T(x) = aT(x) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$. Obviously, $T(0x) = 0T(x)$ for all $x \in {}_A\mathcal{B}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{B}$. So the unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is an A-linear mapping, as desired. □

Theorem 2. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_A\mathcal{B}^{p^n} \to [0, \infty)$ such that*

$$\begin{aligned} \widetilde{\varphi}(x_1, \cdots, x_{p^n}) := \sum_{j=0}^{\infty} p^j \varphi(&-\frac{(p-1)k+p-2}{p^j}x_1, \frac{k+1}{p^j}x_2, \cdots, \frac{k+1}{p^j}x_p, \cdots, \\ &-\frac{(p-1)k+p-2}{p^j}x_{p^n-p+1}, \frac{k+1}{p^j}x_{p^n-p+2}, \cdots, \frac{k+1}{p^j}x_{p^n}) < \infty, \end{aligned} \tag{10}$$

$$\begin{aligned} \|D_uf(x_1, \cdots, x_{p^n})\| = \|p^n uf(&\frac{x_1 + \cdots + x_{p^n}}{p^n}) \\ &+ p(pk+p-2)\sum_{i=1}^{p^{n-1}} uf(\frac{x_{pi-p+1} + \cdots + x_{pi}}{p}) \\ &-(pk+p-1)\sum_{i=1}^{p^n} f(\frac{ux_i + \cdots + ux_{i+pk+p-2}}{pk+p-1})\| \le \varphi(x_1, \cdots, x_{p^n}) \end{aligned} \tag{11}$$

for all $x_1 = x_{p^n+1}, \cdots, x_{pk+p-2} = x_{p^n+pk+p-2}, x_{pk+p-1}, \cdots, x_{p^n} \in {}_A\mathcal{B}$ *and all* $u \in \mathcal{U}(A)$. *Then there exists a unique* A*-linear mapping* $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ *such that*

$$\|f(x) - T(x)\| \le \frac{1}{(pk+p-1)p^{n-1}} \widetilde{\varphi}(\underbrace{x, \cdots, x}_{p^n\text{-times}}) \tag{12}$$

for all $x \in {}_A\mathcal{B}$.

Proof. Put $u = 1 \in \mathcal{U}(A)$. Let $x_{pi-p+1} = x$ and $x_{pi-p+2} = \cdots = x_{pi} = y$ in (11) for all $i = 1, \cdots, p^{n-1}$. Then we get

$$\begin{aligned} \|p^n f(\frac{x+(p-1)y}{p}) &+ (pk+p-2)p^n f(\frac{x+(p-1)y}{p}) \\ &-(pk+p-1)p^{n-1} f(\frac{kx+((p-1)k+p-1)y}{pk+p-1}) \\ &-(p-1)(pk+p-1)p^{n-1} f(\frac{(k+1)x+((p-1)k+p-2)y}{pk+p-1})\| \\ &\le \varphi(x, \underbrace{y, \cdots, y}_{p-1\text{-times}}, \cdots, x, \underbrace{y, \cdots, y}_{p-1\text{-times}}) \end{aligned} \tag{13}$$

for all $x, y \in {}_A\mathcal{B}$. Replacing x and y by $-((p-1)k+p-2)x$ and $(k+1)x$ in (13), respectively, we get

$$\begin{aligned} \|(pk+p-1)p^n f(\frac{x}{p}) &- (pk+p-1)p^{n-1} f(x)\| \\ &\le \varphi(-((p-1)k+p-2)x, \underbrace{(k+1)x, \cdots, (k+1)x}_{p-1\text{-times}}, \cdots, \\ &\quad -((p-1)k+p-2)x, \underbrace{(k+1)x, \cdots, (k+1)x}_{p-1\text{-times}}) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$. So one can obtain

$$\begin{aligned} \|f(x) - pf(\frac{x}{p})\| &\le \frac{1}{(pk+p-1)p^{n-1}} \varphi(-((p-1)k+p-2)x, \\ &\underbrace{(k+1)x, \cdots, (k+1)x}_{p-1\text{-times}}, \cdots, -((p-1)k+p-2)x, \underbrace{(k+1)x, \cdots, (k+1)x}_{p-1\text{-times}}), \end{aligned}$$

and hence

$$\begin{aligned} \|p^j f(\frac{1}{p^j}x) - p^{j+1} f(\frac{1}{p^{j+1}}x)\| &\le \frac{p^j}{(pk+p-1)p^{n-1}} \varphi(-\frac{(p-1)k+p-2}{p^j}x, \\ &\underbrace{\frac{k+1}{p^j}x, \cdots, \frac{k+1}{p^j}x}_{p-1\text{-times}}, \cdots, -\frac{(p-1)k+p-2}{p^j}x, \underbrace{\frac{k+1}{p^j}x, \cdots, \frac{k+1}{p^j}x}_{p-1\text{-times}}) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$. So we get

$$\begin{aligned} \|f(x) - p^j f(\frac{1}{p^j}x)\| &\le \frac{1}{(pk+p-1)p^{n-1}} \sum_{m=0}^{j-1} p^m \varphi(-\frac{(p-1)k+p-2}{p^m}x, \\ &\underbrace{\frac{k+1}{p^m}x, \cdots, \frac{k+1}{p^m}x}_{p-1\text{-times}}, \cdots, -\frac{(p-1)k+p-2}{p^m}x, \underbrace{\frac{k+1}{p^m}x, \cdots, \frac{k+1}{p^m}x}_{p-1\text{-times}}) \end{aligned} \tag{14}$$

for all $x \in {}_A\mathcal{B}$.

Let x be an element in ${}_A\mathcal{B}$. For positive integers l and m with $l > m$,

$$\|p^l f(\frac{1}{p^l}x) - p^m f(\frac{1}{p^m}x)\| \le \frac{1}{(pk+p-1)p^{n-1}} \sum_{j=m}^{l-1} p^j \varphi(-\frac{(p-1)k+p-2}{p^j}x, \underbrace{\frac{k+1}{p^j}x, \cdots, \frac{k+1}{p^j}x}_{p-1\text{-times}}, \cdots, -\frac{(p-1)k+p-2}{p^j}x, \underbrace{\frac{k+1}{p^j}x, \cdots, \frac{k+1}{p^j}x}_{p-1\text{-times}}),$$

which tends to zero as $m \to \infty$ by (10). So $\{p^j f(\frac{1}{p^j}x)\}$ is a Cauchy sequence for all $x \in {}_A\mathcal{B}$. Since ${}_A\mathcal{C}$ is complete, the sequence $\{p^j f(\frac{1}{p^j}x)\}$ converges for all $x \in {}_A\mathcal{B}$. We can define a mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ by

$$T(x) = \lim_{j\to\infty} p^j f(\frac{1}{p^j}x) \tag{15}$$

for all $x \in {}_A\mathcal{B}$.

By (15), (10) and (11), we get

$$\begin{aligned}\|D_1T(x_1, \cdots, x_{p^n})\| &= \lim_{j\to\infty} p^j \|D_1 f(\frac{1}{p^j}x_1, \cdots, \frac{1}{p^j}x_{p^n})\| \\ &\le \lim_{j\to\infty} p^j \varphi(\frac{1}{p^j}x_1, \cdots, \frac{1}{p^j}x_{p^n}) = 0\end{aligned}$$

for all $x_1, \cdots, x_{p^n} \in {}_A\mathcal{B}$. Hence $D_1T(x_1, \cdots, x_{p^n}) = 0$ for all $x_1, \cdots, x_{p^n} \in {}_A\mathcal{B}$. By Lemma 1, T is additive. Moreover, by passing to the limit in (14) as $j \to \infty$, we get the inequality (12).

The rest of the proof is similar to the proof of Theorem 1. □

Acknowledgements. This work was supported by Korea Research Foundation Grant KRF-2002-041-C00014. The author would like to thank the referees for a number of valuable suggestions regarding a previous version of this paper.

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Received September 2002; in revised April 2003.

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