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ON THE CENTRAL λ-STABILITY ZONE FOR LINEAR DISCRETE-TIME HAMILTONIAN SYSTEMS

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Abstract. In this paper we start from the discrete version of linear Hamiltonian systems with periodic coefficients

$$y_{k+1} - y_k = \lambda B_k y_k + \lambda D_k z_{k+1}$$
$$z_{k+1} - z_k = -\lambda A_k y_k - \lambda B_k^* z_{k+1}$$

where A_k and D_k are Hermitian matrices, A_k, B_k, D_k define N-periodic sequences, and λ is a complex parameter. For this system a Krein-type theory of the λ-zones of strong (robust) stability may be constructed. Within this theory the side λ-zones' width may be estimated using the multipliers' "traffic rules" of Krein while the central stability zone (centered around $\lambda = 0$) is estimated using the eigenvalues of a certain boundary value problem which is self-adjoint. In the discrete-time there occur some specific differences with respect to the continuous time case due to the fact that the transition matrix (hence the monodromy matrix also) is not entire with respect to λ but rational. During the paper we consider some specific cases (the matrix analogue of the discretized Hill equation, the J-unitary and symplectic systems, real scalar systems) for which the results on the eigenvalues are complete and obtain some simplified estimates of the central stability zones.

1. **Problem statement and state of the art.** The starting point of the paper is given by the paper of M.G. Krein [8] on the linear periodic Hamiltonian system with complex coefficients

$$\dot{x} = \lambda J H(t) x \tag{1}$$

where $H(t) = H^*(t) = H(t+T)$ the star denoting transposition and complex conjugation; for the definition of J, see (8) . The results of Krein may be summarized as follows: the total stability of (1) which is equivalent to the location of the multipliers on the unit circle is preserved for real λ belonging to a set of open intervals called λ-zones of stability. This property is called strong stability or λ-stability. In this theory Krein gives some estimates of the width of the stability zones: for the side zones they are based on multipliers classification and on the rules of their motion on the unit circle (the "traffic rules"); for the central zone (centered around $\lambda = 0$) the width is estimated using the properties of the eigenvalues of a boundary value problem for (1); this boundary value problem is self-adjoint. At this point some simplified estimates are introduced [8, 9] and generalization of the classical results of Liapunov [10] and Žukovskii [11] are obtained. The results of Krein stimulated

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further research; the main results are summarized in the book of Yakubovich and Staržinskii [12].

Let us consider a discretized version (by using Euler discretization method) of (1): it sends to the following discrete-time linear Hamiltonian system with periodic coefficients:

$$\begin{aligned} y_{k+1} - y_k &= \lambda B_k y_k + \lambda D_k z_{k+1} \\ z_{k+1} - z_k &= -\lambda A_k y_k - \lambda B_k^* z_{k+1} \end{aligned} \tag{2}$$

where A_k and D_k are Hermitian and all matrices define N-periodic sequences. Concerning discrete Hamiltonian systems a long list of references exists: we refer to the monographs by Ahlbrandt and Peterson [1], Kratz [7] and also to the papers of Bohner, Došlý and Kratz [2, 3, 4]. Our topics are somehow aside of their mainstream but some connections do exist.

According to the "dictionary" of the above references, system (2) may be considered as Hamiltonian provided $I + \lambda B_k^*$ is nonsingular: we have thus to eliminate from the admissible values of $\lambda \in \mathbb{C}$ the symmetric (with respect to the unit circle) of the eigenvalues of B_k. On the other hand system (2) may be written also as

$$x_{k+1} = C_k(\lambda) x_k \tag{3}$$

where

$$x = \begin{pmatrix} y \\ z \end{pmatrix}, \quad C_k(\lambda) = \begin{pmatrix} I & -\lambda D_k \\ 0 & I + \lambda B_k^* \end{pmatrix}^{-1} \begin{pmatrix} I + \lambda B_k & 0 \\ -\lambda A_k & I \end{pmatrix}.$$

It is easily shown that $C_k^*(\lambda) J C_k(\lambda) = J$ for real λ i.e. $C_k(\lambda)$ is in this case J-unitary. If besides λ all matrices are real we deduce that $C_k(\lambda)$ is symplectic. As pointed out in [2, 3, 4], in the discrete-time case Hamiltonian systems are a subset of the symplectic systems; if we refer to [12] where systems (2) with real coefficients are called canonical we may say that *in the discrete-time case canonical systems are a subset of the symplectic systems and Hamiltonian systems (with complex coefficients) are a subset of the J-unitary systems.*

In any case λ has to be real. On the other hand if use is made of the properties of analytic functions and of the eigenvalues of boundary value problems we have to consider $\lambda \in \mathbb{C}$ what takes us outside the class of J-unitary (or symplectic) systems.

The results on λ-stability in the continuous time case, more precisely the estimates of the central zone, strongly rely on the fact that only entire functions of λ are met (starting with the transition matrix and going on with the monodromy and the matrices in the boundary value problem). In the discrete-time case we may see from (2) that this is no longer true: in fact the assumption on invertibility of $I + \lambda B_k^*$ speaks for that. There are, nevertheless, notable exceptions. For instance, in [6] we considered the discretized version of

$$y'' + \lambda P(t) y = 0, \tag{4}$$

which lead to a system (2) with $B_k = 0,\ D_k = I,\ \ A_k = P_k$. Since $B_k = 0$ the above mentioned assumption is automatically fulfilled. Moreover $C_k(\lambda)$ is a polynomial matrix function hence it is of entire type.

Another case is suggested by [4]: starting from the Sturm Liouville equations the following symplectic system is considered

$$x_{k+1} = (S_k - \lambda \hat{S}_k) x_k, \tag{5}$$

where

$$S_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \tag{6}$$

is symplectic and

$$\hat{S}_k = \begin{pmatrix} 0 & 0 \\ W_k A_k & W_k B_k \end{pmatrix}, \quad W_k \geq 0.$$

The two cases cannot be reduced one to another because the structures of matrices are different. Nevertheless, if we want to obtain results on λ-stability for (5), the approach to be taken is exactly that of [6].

2. **The central stability zone and a boundary value problem.** We shall consider here system (2) written as

$$\begin{pmatrix} y_{k+1} - y_k \\ z_{k+1} - z_k \end{pmatrix} = \lambda J H_k \begin{pmatrix} y_k \\ z_{k+1} \end{pmatrix}, \tag{7}$$

where

$$H_k = \begin{pmatrix} A_k & B_k^* \\ B_k & D_k \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{8}$$

and the boundary value problem for (7) defined by

$$x_N = G x_0 \tag{9}$$

with $G = G^*$ (a Hermitian matrix). This boundary value problem is self-adjoint. Such boundary value problems are interesting for themselves (see the book of Gohberg and Krein [5]) but *an interest is due to a theorem which is an extension of a result of Krein from the continuous-time case* [8] and will be given in the following. We need first

Proposition 1. *Assume that* $H_k \in \mathcal{P}_n(N)$ *i.e.* $H_k \geq 0$, $\forall k \in \mathcal{I} = [0, N-1] \cap \mathbb{Z}$, $\sum_{0}^{N-1} H_k > 0$ *(the Hamiltonian system is of positive type). Then all characteristic numbers of the boundary value problem (7), (9) are real (if they exist).*

The proof goes along the lines of [8] and will be omitted. The characteristic numbers being the roots of the equation

$$\det(U_N(\lambda) - G) = 0 \tag{10}$$

are *at most a finite number*. Let

$$\cdots \leq \lambda_{-j} \leq \cdots \leq \lambda_{-2} \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \cdots \lambda_j \leq \cdots$$

be their ordered sequence: each characteristic number occurs in the sequence as many times as is its multiplicity as a root of (10). We denoted

$$U_k(\lambda) = C_{k-1}(\lambda) \cdots C_1(\lambda) C_0(\lambda) \tag{11}$$

hence $U_N(\lambda)$ is the monodromy matrix. Let us remark also that $\lambda_0 = 0$ is a characteristic number of the boundary value problem iff 1 is an eigenvalue of G. We are in position to state now the theorem mentioned previously as the extension of a result of Krein from the continuous-time case

Theorem 1. *Consider the above boundary value problem with $G = -I$. Let Λ_+ be the smallest (first) positive characteristic number of this boundary value problem and Λ_- be the largest (first) negative characteristic number of the boundary value problem. The open interval (Λ_-, Λ_+) belongs to the central λ-zone of stability for (7), provided (7) is of positive type in the sense of Krein i.e. $H_k \geq 0$, $\sum_0^{N-1} H_k > 0$. Moreover if H_k have real entries the central zone coincides with the interval (Λ_-, Λ_+).*

The proof goes along the lines given in [8] and we shall not reproduce it.

Theorem 1 gives an estimate of the central stability zone relying on the characteristic numbers of the boundary value problem defined by (7), (9) with $G = -I$ (the skew-periodic problem). In order to use this result in applications two other properties of the characteristic numbers are important: their dependence on H_k and the connection between their multiplicity as roots of (10) and the number of associated linearly independent solutions of (7), (9). We have

Theorem 2. *Let H_k^1, H_k^2, $k \in \mathcal{I}$ be two matrix sequences of the class $\mathcal{P}_n(N)$ such that $H_k^1 \leq H_k^2$, $k \in \mathcal{I}$. Then $\lambda_j(H^1) \geq \lambda_j(H^2)$, $\lambda_{-j}(H^1) \leq \lambda_{-j}(H^2)$. Here $\lambda_m(H^p)$ are the characteristic numbers of the problem defined by (7), (9) with $H_k = H_k^p$.*

Theorem 3. *The multiplicity k_j of any characteristic number λ_j of the problem (7), (9) - real root of (10) - coincides with the number d_j of non-trivial linearly independent solutions of (7), (9).*

The proof of these results, while not very straightforward, still goes along the proofs of [8] with some cautions among which we shall mention the use of the Smith-McMillan form of the rational matrices.

Let us remark now that the statement of Theorem 1 contains two assumptions that have to be proved : existence of characteristic numbers (since Proposition 1 states only that they are real provided they exist) and existence of at least one positive and one negative characteristic numbers for the boundary value problem. Here the results do not migrate *mutatis-mutandis* from continuous time to discrete time case. Indeed, in the continuous time case a result due to Krein [5, 8] reads as follows: *the skew-periodic boundary value problem defined by (7), (9) with $G = -I$ has at least one positive and one negative characteristic number.* **In the discrete-time case we do not know always that this result is true**.

The techniques of Krein are essentially based on the properties of the entire functions of complex variable. For this reason his results may be extended to these cases when $C_k(\lambda)$, $U_k(\lambda)$ and $U_N(\lambda)$ are polynomial matrices; this is true for the case defined by (5)-(6) and for the case defined by the discretization of (4) (see [6]). In the general case we may obtain, following Krein, the result given next

Proposition 2. *Consider the skew-periodic boundary value problem defined by (7), (9) with $G = -I$. If the matrices B_k, $k \in \mathcal{I}$ are such that none of their purely imaginary eigenvalues coincides with some eigenvalue of $\frac{1}{2}J \sum_0^{N-1} H_k$ (whose eigenvalues are all purely imaginary provided $\sum_0^{N-1} H_k > 0$) then the boundary value problem has characteristic numbers.*

This proposition is a quite straightforward adaptation of the results of Krein [8] mentioned above. On the other hand we may show rather simple cases when the assumption of Proposition 2 does not hold but this fact is not contradictory with other assumptions on H_k. Nevertheless there exist also cases which are important in applications when the assumption of Proposition 2 holds.

For instance, if y and z are scalars and the coefficients are real (in fact only b_k have to be real since a_k and d_k are such) then the "eigenvalues" of b_k are b_k and are real. Moreover, as it will be shown below, in this case there exist characteristic numbers of the boundary value problem. Indeed, following the way of Krein [8] the following sufficient condition may be proved

Proposition 3. *If at least one of the matrices B_k has at least one real eigenvalue then the skew-periodic boundary value problem defined by (7), (9) with $G = -I$ has characteristic numbers of opposite sign (provided they exist).*

Outline of proof. We shall assume that there exist characteristic numbers but only of one sign e.g. negative. In this case the central stability zone is semi-infinite in positive direction i.e. for all $\lambda > 0$ hence all systems multipliers $\rho_j(\lambda)$ are located on the unit circle:

$$\det(U_N(\lambda) - \rho I) = \prod_1^{2m} (\rho_j(\lambda) - \rho) \tag{12}$$

Therefore

$$|\det(U_N(\lambda) + I)| = \left| \prod_1^{2m} (\rho_j(\lambda) + 1) \right| \le 2^{2m}, \ \ 0 < \lambda < \infty \tag{13}$$

From there on the contradiction had been obtained by Krein [8] based on the fact that in the continuous time case $\det(U_N(\lambda) + I)$ was an entire function. In the cases of (5)-(6) and [6] this function is polynomial and the proof of Krein is valid. In the general case the function is rational its poles being $-\left(\overline{\lambda_j}^{-k}\right)^{-1}$ where λ_j^k are the eigenvalues of $B_k, k \in \mathcal{I}$. If at least one of them is real the rational function $\det(U_N(\lambda) + I)$ becomes unbounded in its neighborhood-a contradiction to (13). Therefore there exist also positive characteristic numbers; the proposition is proved.

Let us remark that if all poles are complex, then the contradiction is obtained only by analyzing the asymptotic behavior (for $\lambda \to \infty$) of the rational function $\det(U_N(\lambda) + I)$. Remark also that this analysis is required for obtaining the contradiction provided we take the way of Krein in proving the properties of the characteristic numbers. Krein himself was aware of the fact that this technique was perhaps too strong for the required task; he then suggested another approach based on Green's function and weighted integral equations where the standard theory of compact operators had to be applied. In the discrete-time case this should reduce to some special linear systems of algebraic equations.

The problem is still open and we shall not insist on this subject here. The fact is that *if we assume existence of positive and negative characteristic numbers for the skew-periodic boundary value problem then all other development of Krein concerning estimates of the central stability zone* [8] *may be extended to the discrete-time case; moreover, in the real scalar case the existence of characteristic numbers of opposite sign is ensured and the entire development of Krein may be extended.*

3. **Estimates of the central stability zone.** In the following we shall assume existence of the characteristic numbers Λ_+ (the smallest positive) and Λ_- (the largest negative) for the skew-periodic boundary problem (7)-(9) ($G = -I$). An important technical result is *the theorem of Perron on matrices with nonnegative elements: such matrices have at least one positive eigenvalue.*

For a given matrix A with nonnegative elements let $M(A)$ be the positive eigenvalue whose existence is ensured by the theorem of Perron.

Also, if B is some matrix we shall denote by B^a the matrix obtained from B by replacing its entries by their moduli. For instance

$$J^a = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

We may state now the following result

Theorem 4. *(Criterion I_n) If $|\lambda| < 2M(C)^{-1}$ where $C = J^a \sum_0^{N-1} H_k^a$ then $\lambda \in \mathbb{R}$ belongs to the central stability zone.*

The proof of this result goes along the lines from [8] with some caution due to the specific structure of the discrete Hamiltonian systems namely to the presence of the advanced argument in the second equation.

Let us consider the case n = 2 (m = 1) i.e. the scalar case and assume that all elements are real. In this case we have in (8)

$$H_k = \begin{bmatrix} a_k & b_k \\ b_k & d_k \end{bmatrix}.$$

Denote

$$\gamma_a = \sum_0^{N-1} a_k, \quad \gamma_b = \sum_0^{N-1} |b_k|, \quad \gamma_d = \sum_0^{N-1} d_k. \tag{14}$$

We shall have, by direct computation

$$C = J^a \sum_0^{N-1} H_k^a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \gamma_a & \gamma_b \\ \gamma_b & \gamma_d \end{bmatrix} = \begin{bmatrix} \gamma_b & \gamma_d \\ \gamma_a & \gamma_b \end{bmatrix}. \tag{15}$$

and therefore $M(C) = \gamma_b + \sqrt{\gamma_a \gamma_d}$.

Corollary 1. *(Criterion I_2) Consider the second order N-periodic system*

$$\begin{aligned} y_{k+1} - y_k &= \lambda b_k y_k + \lambda d_k z_{k+1} \\ z_{k+1} - z_k &= -\lambda a_k y_k - \lambda b_k z_{k+1} \end{aligned} \tag{16}$$

with real elements such that

$$a_k \geq 0, d_k \geq 0, a_k d_k - b_k^2 \geq 0, \left(\sum_0^{N-1} a_k\right)\left(\sum_0^{N-1} d_k\right) - \left(\sum_0^{N-1} b_k\right)^2 > 0.$$

If

$$|\lambda| < 2\left(\gamma_b + \sqrt{\gamma_a \gamma_d}\right)^{-1} \tag{17}$$

where $\gamma_a, \gamma_b, \gamma_d$ are those defined by (14) then all solutions of (16) are bounded.

Remark 1. In this case existence of Λ_+ and Λ_- is ensured since b_k are real.

Corollary 1 allows obtaining as a special case of I_2 the *discrete analogue of the well-known Liapunov criterion* [10]. Following [6] we consider the second order equation

$$y_{k+1} - 2y_k + y_{k-1} + \lambda^2 p_k y_k = 0 \tag{18}$$

with $p_k \geq 0, p_k = p_{k+N}, \sum_0^{N-1} p_k > 0$. This equation is written as a canonical system of second order

$$\begin{aligned} y_{k+1} - y_k &= \lambda z_{k+1} \\ z_{k+1} - z_k &= -\lambda p_k y_k \end{aligned} \tag{19}$$

where the new variable z_k has been introduced via the first equation of (19). We have here

$$H_k = \begin{bmatrix} p_k & 0 \\ 0 & 1 \end{bmatrix}$$

hence $b_k = 0$. In this case the functions occurring in the problem analysis are polynomial in λ and existence of Λ_+ and Λ_- is ensured [6]. Using the result of Corollary 1 (Criterion I_2) we have here

$$\gamma_b = 0, \ \ \gamma_a = \sum_0^{N-1} p_k, \ \gamma_d = N$$

Therefore $|\lambda| < 2\left(\sqrt{N \sum p_k}\right)^{-1}$ what reads

$$\lambda^2 < \frac{4}{N}\left(\sum_0^{N-1} p_k\right)^{-1} \tag{20}$$

which is exactly the discrete-time analogue of the Liapunov criterion.

There exist also other simplified criteria for λ to belong to the central zone [8]. Discrete-time analogues may be given for all of them.

4. **Concluding remarks.** The research programme of extending to discrete-time the stability results concerning continuous-time Hamiltonian systems seems fruitful. Not all results of [8] could be extended *mutatis-mutandis* but there exist hopes of overcoming the difficulties (e.g. concerning existence of characteristic numbers of both signs for the skew-periodic boundary value problem) using powerful tools of operator theory.

Other results could be extended in a quite straightforward manner and this is the case, for instance, with the famous Liapunov criterion. The research is still in progress for other problems of the λ-stability theory.

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