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TRAVELING WAVES TO A REACTION–DIFFUSION EQUATION

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Abstract. In this paper, we study a nonlinear reaction–diffusion equation for its traveling waves. This equation can be regarded as a generalization of the Fisher equation and is used as a nonlinear model, in the one-dimensional situation, for studying insect and animal dispersal with growth dynamics. Applying the Lie symmetry method, we obtain two traveling wave solutions under certain parametric conditions and express them in terms of elliptic functions.

1. **Introduction.** The problems of the propagation of nonlinear waves have fascinated scientists for over two hundred years. The modern theory of nonlinear waves, like many areas of mathematics, had its beginnings in attempts to solve specific problems, the hardest among them being the propagation of waves in water. There was significant activity on this problem in the 19th century and the beginning of the 20th century, including the classic work of Stokes, Lord Rayleigh, Korteweg and de Vries, Boussinesque, Benard and Fisher to name some of the better remembered examples [1, 2]. One particularly noteworthy contribution was the explosion of activity unleashed by the numerical discovery of the soliton by Zabusky and Kruskal in the early sixties, and the earliest theoretical explanation by Gardner, Greene, Kruskal, and Miura in the latter part of that decade, which subsequently led to the present-day theory of integrable partial differential equations. Nonlinear waves and coherent structures is an interdisciplinary area that has many important applications, including nonlinear optics, hydrodynamics, plasmas and solid-state physics. In fact, for any physical system where the dynamics is driven by, and mainly determined by, phase coherence of the individual waves, it has applications and consequences.

Modern theories describe nonlinear waves and coherent structures in a diverse variety of fields, including general relativity, high energy particle physics, plasmas, atmosphere and oceans, animal dispersal, random media, chemical reactions, biology, nonlinear electrical circuits, and nonlinear optics. For example, in the latter, the mathematics developed for describing the propagation of information via optical solitons is most striking, attaining an incredible accuracy. It has been experimentally verified and spans twelve orders of magnitude: from the wavelength of light to transoceanic distances. It also guides the practical applications in modern telecommunications. Many other nonlinear wave theories mentioned above can claim similar success.

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Nowadays it has been universally acknowledged in the physical, chemical and biological communities that the reaction-diffusion equation plays an important role in dissipative dynamical systems. Typical examples are provided by the fact that there are many phenomena in biology where a key element or precursor of a developmental process seems to be the appearance of a traveling wave of chemical concentration (or mechanical deformation). When reaction kinetics and diffusion are coupled, traveling waves of chemical concentration can effect a biochemical change much faster than straight diffusional processes. This usually gives rise to reaction-diffusion equations which in one dimensional space can look like

$$\frac{\partial u}{\partial t} = k_0 \frac{\partial^2 u}{\partial x^2} + f(u), \tag{1}$$

for a chemical concentration u, where k_0 is the diffusion coefficient, and $f(u)$ represents the kinetics.

When $f(u)$ is linear, i.e., $f(u) = k_2 u + k_1$, where both k_1 and k_2 are constants, then in many instances equation (1) can be solved by the separation of variables methods. However if, as in many of the applications considered in [3], $f(u)$ is nonlinear, then the problem is much more intractable. Indeed, it is not usually possible to obtain general analytical traveling wave solutions and one must analyze such problems numerically [4]. Despite this, however, under some particular circumstances, many nonlinear evolutionary equations have traveling wave solutions of special types, which are of fundamental importance to our understanding of biological phenomena modeled evolutionary equations. The classic and simplest case of the nonlinear reaction-diffusion equation is when $f(u) = k_3 u(1 - u)$, which is the so-called Fisher equation. It was suggested by Fisher as a deterministic version of a stochastic model for the spatial spread of a favored gene in a population [5]. (Although this equation is now referred to as the Fisher equation, the discovery, investigation and analysis of traveling waves in chemical reactions was first presented by Luther at a conference [6]. There, he stated that the wave speed is a simple consequence of the differential equations. This recently re-discovered paper has been translated by Arnold et al. [7] and Luther's remarkable discovery and analysis of chemical waves has been put in a modern context by Showalter and Tyson [8]). In the 20th century, the Fisher equation has became the basis for a variety of models for spatial spread. The typical examples are that Aoki discussed gene-culture waves of advance [9] and Ammerman and Cavali-Sforza, in an interesting direct application of the model, applied it to the spread of early farming in Europe [10, 11]. Meanwhile, the qualitative analysis in the phase plane and traveling wave solutions of the Fisher equation have been widely investigated. The seminal and now classical references are that by Kolmogorov, Petrovsky and Piscunov [12], Albowitz and Zeppetella [13], Fife [14] and Britten [15]. In [12], Kolmogorov et al. showed that any initial concentration which is one for large negative spatial variable x and vanishes for large x, evolves to a traveling wavefront with minimal velocity $v = 2\sqrt{k_0}$. Different initial values propagate with different traveling waves, depending on the behavior at $x \to \pm\infty$. The first explicit analytic form of a cline solution for the Fisher equation was obtained by Albowitz and Zeppetella making use of the Painlevé analysis [13]. A full discussion of this equation and an extensive bibliography can be seen in [14, 15]. The singular property, auto-Bäcklund transformation and analytic solutions including some heterclinic and homoclinic solutions of the Fisher equation were obtained by Guo and Chen via the expanded Painlevé for carrier flow equation in semiconductor devices [16, 17]. A discrete singular convolution algorithm was introduced to solve Fisher's equation and predicted long-time traveling wave behavior by Zhao and Wei [18]. For more results of traveling wave solutions for equation (1) and the generalized versions, we refer the reader to [19] and references therein.

In the present work, we consider equation (1) with $f(u) = u(\mu + \beta u - \gamma u^2)$, namely

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + u(\mu + \beta u - \gamma u^2), \tag{2}$$

where α, β, μ and γ are real constants. This equation can be regarded as a generalization of the Fisher equation, which is used as a density-dependent diffusion model, in the one-dimensional situation, for studying insect and animal dispersal with growth dynamics [3], and as a genetic model arising from the classical theory of population genetics and combustion [20, 21]. During the past decade, considerable attention has been received to exact solutions and traveling wave solutions of equation (2). When $\mu = 0$ or $\beta = 0$, exact solutions of equation (2) have been found by Clarkson and Mansfield [22, 23] using the nonclassical method, by Chen and Guo using a truncated Painlevé expansion [24], and by Chowdhury, Estévez and Gordoa [25, 26] using a complete Painlevé test. When $\mu = 0$, Herrera, Minzoni and Ondarza [27] obtained explicit traveling wave solutions of (2) if choosing $p = 2$ in their equations. Wu and Hu derived an exact solitary wave solution to equation (2) by utilizing an analytic method [28]. Some special cases with $\mu + \beta - \gamma = 0$ of this equation have been investigated by Cohen et al. [29–32]. Ma and Fuchssteiner dealt with equation (2) by using a Cole-Hopf transformation and Bäcklund transformations [33]. The study of the properties of the traveling waves and their applications were described in [34, 35]. Note that since two nonlinearities occur in equation (2), some classical methods such as the Fourier transform and the approaches for integrable systems become invalid. Therefore, to seek traveling wave solutions of equation (2), qualitative analysis together with ingenious mathematical techniques for treating such this nonlinear system appears to be more powerful and important. Recently, qualitative results for some physical and biological systems have been studied extensively [36–38 et al.], and some innovative mathematical methods, such as the Lie group analysis and symmetry method [39–46 et al.] have been developed and widely applied to many nonlinear systems. Our goal in this paper is to find traveling wave solutions of equation (2) under certain parametric conditions which can only be expressed in elliptic functions.

The rest of this paper is organized as follows. In Section 2, we obtain a class of traveling wave solutions for equation (2) in terms of elliptic functions by means of the Lie symmetry method. Section 3 is a brief conclusion.

2. **Traveling Waves in Terms of Elliptic Functions.** Assume that equation (2) has a traveling wave solution of the form

$$u(x,t) = u(\xi), \qquad \xi = x - vt, \tag{3}$$

where v is the wave velocity and $v \in \mathbb{R}$. Substituting (3) into equation (2) gives

$$u''(\xi) - ru'(\xi) - au^3 - bu^2 - du = 0, \tag{4}$$

where $r = -\frac{v}{\alpha}$, $a = \frac{\gamma}{\alpha}$, $b = -\frac{\beta}{\alpha}$ and $d = -\frac{\mu}{\alpha}$. We know that when $abd \neq 0$, in the general case, equation (4) does not pass Painlevé test, and is not integrable either.

In order to avoid complicated calculations, we make the natural logarithmic transformation

$$\xi = \frac{1}{r} \ln \tau, \tag{5}$$

then equation (4) becomes

$$r^2\tau^2\frac{d^2u}{d\tau^2} - au^3 - bu^2 - du = 0. \tag{6}$$

Take the coordinate transformation as

$$q = \tau^k, \quad u = \tau^{-\frac{1}{2}(k-1)} \cdot \rho(q), \tag{7}$$

then equation (6) reduces to a simple form:

$$\frac{d^2\rho}{dq^2} = \frac{a}{r^2k^2}q^{\frac{1-3k}{k}}\rho^3 + \frac{b}{r^2k^2}q^{\frac{1-5k}{2k}}\rho^2 = F(q,\ \rho), \tag{8}$$

where k is given by

$$k^2 = 1 + \frac{4d}{r^2}. \tag{9}$$

To present our result in a straightforward manner, here we summarize our main result as follows:

Theorem 1. *Suppose that the velocity v satisfies $v^2 = -\frac{9}{4}\alpha\mu$. Then equation (2) has two traveling wave solutions. One is*

$$u_1(x,\ t) = e^{\pm\frac{\sqrt{-\alpha\mu}}{2\alpha}\left(x\mp\frac{3}{2}\sqrt{-\alpha\mu}t+\xi_0\right)} \cdot \rho(q), \tag{10}$$

where ξ_0 is an arbitrary constant, ρ and q are given by the parametric form

$$q = a_2C_1^5\varphi^5, \quad \rho = b_2C_1\varphi(\omega\varphi - k_1),$$

where C_1 is an arbitrary constant, $\frac{\gamma}{\mu} = \mp\frac{1}{2}a_2^{2/5}b_2^{-2}$, $\frac{\beta}{\mu} = \pm\frac{3}{2}a_2^{1/5}b_2^{-1}k_1$, and φ is the elliptic function defined by

$$\varphi \quad = \quad \int(1 \pm \omega^4)^{-1/2}d\omega + C_3, \tag{11}$$

where the modulus $k_1 = \pm 1$ and C_3 is an arbitrary constant.

The other solution is

$$u_2(x,\ t) = e^{\mp\frac{2\sqrt{-\alpha\mu}}{\alpha}\left(x\mp\frac{3}{2}\sqrt{-\alpha\mu}t+\xi_0\right)} \cdot \rho(q), \tag{12}$$

where ρ and q are given by the parametric form

$$q = a_2C_1^5\varphi^{-5}, \quad \rho = b_2C_1^4\varphi^{-4}(\omega\varphi - k_1),$$

where $\frac{\gamma}{\mu} = \mp\frac{1}{2}a_2^{8/5}b_2^{-2}$, $\frac{\beta}{\mu} = \pm\frac{3}{2}a_2^{4/5}b_2^{-1}k_1$, and ω is the same as (11).

Proof of Theorem 1. Consider the set of transformations

$$L_\varepsilon = \begin{cases} q^* = \phi_0(q,\ \rho,\ \varepsilon), & q^*|_{\varepsilon=0} = q \\ \rho^* = \psi_0(q,\ \rho,\ \varepsilon), & \rho^*|_{\varepsilon=0} = \rho, \end{cases} \tag{13}$$

where ϕ_0 and ψ_0 are smooth functions of their arguments and ε is a real parameter. The set L_ε is called a continuous one-parameter Lie group of point transformations if the relation $L_{\varepsilon_1+\varepsilon_2} = L_{\varepsilon_1} \circ L_{\varepsilon_2}$ holds for any real ε_1 and ε_2. Taylor's expansion of q^* and ρ^* in (13) with respect to the parameter ε at $\varepsilon = 0$ yields:

$$\begin{aligned} q^* &= q + \varepsilon\xi(q,\ \rho) + O(\varepsilon^2), \\ \rho^* &= \rho + \varepsilon\eta(q,\ \rho) + O(\varepsilon^2), \end{aligned} \tag{14}$$

here

$$\xi(q,\ \rho) = \left.\frac{\partial\phi_0(q,\ \rho,\ \varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}, \quad \eta(q,\ \rho) = \left.\frac{\partial\psi_0(q,\ \rho,\ \varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}.$$

For the infinitesimal transformation (14), the associated infinitesimal generator and prolonged operator are given as

$$X = \xi(q,\ \rho)\frac{\partial}{\partial q} + \eta(q,\ \rho)\frac{\partial}{\partial \rho},$$

and

$$\Delta X = \xi(q,\ \rho)\frac{\partial}{\partial q} + \eta(q,\ \rho)\frac{\partial}{\partial \rho} + \sum_{i=1}^{2}\zeta_i\Big(q,\ \rho,\ \rho_q^{(i)}\Big)\frac{\partial}{\partial \rho_q^{(i)}},$$

respectively, where

$$\begin{aligned} \rho_q^{(k+1)} &= \frac{\partial \rho_q^{(k)}}{\partial q}, \\ \zeta_1 &= \eta_q + (\eta_\rho - \xi_q)\rho_q' - \xi_\rho(\rho_q')^2 = D_q(\eta) - \rho_q' D_q(\xi), \\ \zeta_{k+1} &= D_q(\zeta_k) - \rho_q^{(k+1)} D_q(\xi), \end{aligned}$$

here $D_q = \partial_q + \rho_q'\partial_\rho$ is the operator of total derivative with respect to q.

If we assume that equation (8) admits the Lie group (13), from the invariance condition

$$\Delta X\Big(\rho'' - F(q,\ \rho)\Big)\big|_{\rho''=F(q,\ \rho)} = 0,$$

we have

$$\begin{aligned} &\eta_{qq} + (2\eta_{q\rho} - \xi_{qq})\rho_q' + (\eta_{\rho\rho} - 2\xi_{q\rho})(\rho_q')^2 - \xi_{\rho\rho}(\rho_q')^3 \\ = \ &(2\xi_q - \eta_\rho + 3\xi_\rho\rho_q')F(q,\ \rho) + \xi F_q(q,\ \rho) + \eta F_\rho(q,\ \rho), \end{aligned} \tag{15}$$

This is a second-order partial differential equation for two unknown functions $\xi(q,\ \rho)$ and $\eta(q,\ \rho)$. Since the unknown functions do not depend on the derivative ρ_q', after setting the coefficients of the powers $(\rho_q')^i$ $(i = 0,\ 1,\ 2,\ 3)$ in (15) to zero, one can get the determining equations which can be split and represented as the system

$$\xi_{\rho\rho} = 0, \tag{16}$$

$$2\xi_{q\rho} - \eta_{\rho\rho} = 0, \tag{17}$$

$$2\eta_{q\rho} - \xi_{qq} - 3\xi_\rho F(q,\ \rho) = 0, \tag{18}$$

$$\eta_{qq} + (\eta_\rho - 2\xi_q)F(q,\ \rho) - \xi F_q(q,\ \rho) - \eta F_\rho(q,\ \rho) = 0. \tag{19}$$

From equations (16) and (17), we get

$$\xi(q,\ \rho) = g(q)\rho + f(q), \quad \eta(q,\ \rho) = g'(q)\rho^2 + \phi(q)\rho + \psi(q), \tag{20}$$

where $g(q),\ f(q),\ \phi(q)$ and $\psi(q)$ are functions to be determined later. Substituting (20) into equations (18) and (19) respectively, we have

$$3g''(q)\rho + 2\phi'(q) - f''(q) - 3g(q)F(q,\ \rho) = 0, \tag{21}$$

$$\begin{aligned} &g'''(q)\rho^2 + \phi''(q)\rho + \psi''(q) + [\phi(q) - 2f'(q)]F(q,\ \rho) \\ &-[g(q)\rho + f(q)]F_q(q,\ \rho) - [g'(q)\rho^2 + \phi(q)\rho + \psi(q)]F_\rho(q,\ \rho) = 0. \end{aligned} \tag{22}$$

Solving equation (21) gives $g(q) = 0$ and $\phi(q) = \frac{1}{2}f'(q) + c_0$, where c_0 is an arbitrary constant. Substitution of $g(q)$ and $\phi(q)$ into equation (22), we have

$$\begin{aligned} &\psi'' + \frac{1}{2}f'''(q)\rho - f(q)F_q(q,\ \rho) + \Big(c_0 - \frac{3}{2}f'(q)\Big)F(q,\ \rho) \\ &-\Big[\Big(c_0 + \frac{1}{2}f'(q)\Big)\rho + \psi(q)\Big]F_\rho(q,\ \rho) = 0. \end{aligned} \tag{23}$$

When $k \neq \pm 1$ and $\pm 1/3$, setting ψ'' and the coefficients of ρ and ρ^2 in (23) to zero, we have

$$\begin{aligned}
&\psi(q) = a_1 q + b_1,\\
&f(q) = \frac{32a_1bk}{r^2(1-k)(1+k)(1+3k)} q^{\frac{1}{2k}+\frac{3}{2}} + \frac{32b_1bk}{r^2(1-3k)(1-k)(1+k)} q^{\frac{1}{2k}+\frac{1}{2}}\\
&\qquad + \frac{c_1}{2}q^2 + c_2 q + c_3,\\
&\left[\frac{5}{2}f'(q) + c_0\right] b q^{\frac{1}{2k}-\frac{5}{2}} + b\left[\frac{1}{2k} - \frac{5}{2}\right] f(q) q^{\frac{1}{2k}-\frac{7}{2}} + 3a(a_1q+b_1)q^{\frac{1}{k}-3} = 0,
\end{aligned}$$

where $a_1,\ b_1$ and $c_j\ (j = 0,\ 1,\ 2,\ 4)$ are constants.

Substituting $\psi(q)$ and $f(q)$ into equation (23), using the third equation and equating corresponding coefficients of powers of q while setting the coefficient of ρ^3 to zero, we find that there are only two possibilities if (23) holds:

$$\begin{aligned}
&(i)\ k = \frac{5}{3},\quad \text{we choose}\quad b_1 = a,\ a_1 = c_1 = c_2 = c_3 = c_0 = 0;\\
&(ii)\ k = -\frac{5}{3},\quad \text{we choose}\quad a_1 = -a,\ b_1 = c_1 = c_2 = c_3 = c_0 = 0.
\end{aligned}$$

These two possibilities yield two cases of prolongation of the admissible group:

Case 1: when $k = \frac{5}{3}$, we obtain that equation (8) admits the infinitesimal generator

$$\begin{aligned}
X_1 &= \xi(q,\ \rho)\frac{\partial}{\partial q} + \eta(q,\ \rho)\frac{\partial}{\partial \rho},\\
\xi(q,\ \rho) &= \frac{15ab}{2r^2}q^{\frac{4}{5}},\quad \eta(q,\ \rho) = \frac{3ab}{r^2}q^{-\frac{1}{5}}\rho + a.
\end{aligned}$$

Case 2: when $k = -\frac{5}{3}$, equation (8) admits the infinitesimal generator

$$\begin{aligned}
X_2 &= \xi(q,\ \rho)\frac{\partial}{\partial q} + \eta(q,\ \rho)\frac{\partial}{\partial \rho},\\
\xi(q,\ \rho) &= \frac{15ab}{2r^2}q^{\frac{6}{5}},\quad \eta(q,\ \rho) = \frac{9ab}{2r^2}q^{\frac{1}{5}}\rho - aq.
\end{aligned}$$

We know that if an ordinary differential equation admits an infinitesimal generator, then the order of the equation can be reduced by one [44, 45]. Now we consider Cases 1 and 2. Assume that

$$\Psi = g(q,\ \rho) \quad \text{and} \quad \nu = f(q,\ \rho)$$

are nontrivial solutions of the first-order linear differential equations

$$\xi(q,\ \rho)\frac{\partial f}{\partial q} + \eta(q,\ \rho)\frac{\partial f}{\partial \rho} = \chi, \tag{24}$$

$$\xi(q,\ \rho)\frac{\partial g}{\partial q} + \eta(q,\ \rho)\frac{\partial g}{\partial \rho} = 0, \tag{25}$$

where χ is a nonzero constant and can be chosen arbitrarily. Suppose that the general solution of the characteristic equation

$$\frac{dq}{\xi(q,\ \rho)} = \frac{d\rho}{\eta(q,\ \rho)}$$

has the form $U(q,\ \rho) = C$, where C is arbitrary, then the general solutions of equations (24) and (25) can be expressed by

$$f(q,\ \rho) = k \int \frac{dq}{\xi^*(q,\ U)} + \Phi_1(U), \tag{26}$$

$$g(q,\ \rho) = \Phi_2(U), \quad U = U(q,\ \rho), \tag{27}$$

where $\Phi_1(U)$ and $\Phi_2(U)$ are arbitrary functions, $\xi^*\big(q,\ U(q,\ \rho)\big) \equiv \xi(q,\ \rho)$, and U in the integral is regarded as a parameter later. Using formulas (26) and (27), and choosing an appropriate value $\chi = 5$ in (24), we obtain that solutions for equation (8) in the parametric form:

(I). When $k = \frac{5}{3}$,

$$q = a_2 C_1^5 \varphi^5, \quad \rho = b_2 C_1 \varphi(\omega\varphi - k_1), \tag{28}$$

where C_1 is an arbitrary constant, $\frac{\gamma}{\mu} = \mp\frac{1}{2} a_2^{2/5} b_2^{-2}$, $\frac{\beta}{\mu} = \pm\frac{3}{2} a_2^{1/5} b_2^{-1} k_1$, and φ is given as (11).

(II). When $k = -\frac{5}{3}$,

$$q = a_2 C_1^5 \varphi^{-5}, \quad \rho = b_2 C_1^4 \varphi^{-4}(\omega\varphi - k_1), \tag{29}$$

where C_1 is an arbitrary constant, $\frac{\gamma}{\mu} = \mp\frac{1}{2} a_2^{8/5} b_2^{-2}$, $\frac{\beta}{\mu} = \pm\frac{3}{2} a_2^{4/5} b_2^{-1} k_1$, and φ is the same as (11).

Note that when $k = \pm\frac{5}{3}$, formula (9) gives $v^2 = -\frac{9}{4}\alpha\mu$. Using (28)–(29) as well as the inverse transformations of (5) and (7), we obtain solutions (10) and (12), respectively. Therefore, the proof of Theorem 1 is completed. □

3. **Conclusion.**

Many physical, chemical and biological phenomena can be described by nonlinear reaction-diffusion models. Typical examples are given by the Fisher equation after Fisher who proposed the one-dimensional version as a model for the spread of an advantageous gene in a population. The last few decades have seen an enormous growth of the applicability of nonlinear models and of the development of related nonlinear concepts. This has been driven by modern computer power as well as by the discovery of new mathematical techniques, which include two contrasting themes: (i) the theory of dynamical systems, most popularly associated with the study of chaos, and (ii) the theory of integrable systems associated, among other things, with the study of traveling solitary waves. However, not all systems arising from realistic phenomena are integrable. Therefore, qualitative analysis as well as other innovative mathematical methods for tackling such nonlinear systems seems to be more necessary and powerful. Applications of nonlinear systems range from atmospheric science to condensed matter physics and to biology, from the smallest scales of theoretical particle physics up to the largest scales of cosmic structure.

In this work, we are concerned with a nonlinear reaction-diffusion equation, which can be considered as a generalized Fisher equation. By applying the Lie symmetry method, we show that under the given parametric conditions traveling wave solutions of the generalized Fisher equation (2) can be expressed by a product of an exponential function and an elliptic function. At the present stage, the chemical and biological explanation is being sought for traveling waves of this special type. It is worthwhile to mention that we can apply the above technique to a more general reaction-diffusion equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + u(\mu + \beta u^q - \gamma u^p), \quad q,\ p > 0.$$

Some interesting results are discussed and will be presented in a forthcoming paper elsewhere.

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