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UNIQUENESS OF RADIALLY SYMMETRIC LARGE SOLUTIONS

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Abstract. In this paper we discuss the uniqueness of the large solutions and metasolutions in a general class of radially symmetric singular boundary value problems.

1. **The main theorem.** Throughout this paper, we consider $x_0 \in \mathbb{R}^N$, $N \geq 1$, $R > 0$, $R_2 > R_1 > 0$,

$$B_R(x_0) := \{\, x \in \mathbb{R}^N \ : \ |x - x_0| < R \,\},$$
$$A_{R_1,R_2}(x_0) := \{\, x \in \mathbb{R}^N \ : \ R_1 < |x - x_0| < R_2 \,\},$$

a continuous function $f \in \mathcal{C}[0,\infty)$ such that

$$f(t) > 0 \qquad \text{for all} \qquad t > 0, \tag{1}$$

a function $N \in \mathcal{C}[0,\infty) \cap \mathcal{C}^1(0,\infty)$ satisfying

$$N(0) = 0, \qquad N'(u) > 0 \quad \text{for all} \quad u > 0, \qquad \lim_{u \uparrow \infty} N(u) = \infty, \tag{2}$$

and

$$\Omega \in \{B_R(x_0), A_{R_1,R_2}(x_0)\}. \tag{3}$$

Also, we set

$$d(x) := \operatorname{dist}(x, \partial\Omega), \qquad x \in \Omega.$$

Under these assumptions, for every $M > 0$, the boundary value problem

$$\begin{cases} -\Delta u = \lambda u - f(d(x))N(u)u & \text{in } \Omega, \\ u = M & \text{on } \partial\Omega, \end{cases} \tag{4}$$

possesses a unique positive solution, subsequently denoted by $\theta_{[\lambda,M,\Omega]}$, which is radially symmetric (see López-Gómez [20, 17] and García-Melián et al. [8]). Moreover, the map $M \to \theta_{[\lambda,M,\Omega]}$ is increasing. But, in general, the point-wise limit

$$\theta_{[\lambda,\infty,\Omega]} := \lim_{M \uparrow \infty} \theta_{[\lambda,M,\Omega]} \tag{5}$$

might become somewhere infinity, unless the a priori bounds of Keller [12] and Osserman [25] hold. By (2), for every $\mu > 0$ and $a > 0$, $u^* := N^{-1}(\mu/a)$ is the unique positive zero of the auxiliary function

$$h(u) := auN(u) - \mu u, \qquad u \geq 0.$$

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It is said that $N(u)$ satisfies the *Keller-Osserman condition* if, for every $\mu > 0$, $a > 0$, and $u > u^*$, the following hold

$$I(u) := \int_u^\infty \left(\int_u^s h \right)^{-\frac{1}{2}} ds < \infty, \qquad \lim_{u \uparrow \infty} I(u) = \infty. \tag{6}$$

It turns out that (5) solves the singular problem

$$\begin{cases} -\Delta u = \lambda u - f(d(x))N(u)u & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{7}$$

if, and only if, N satisfies the Keller-Osserman condition (6) (cf. López-Gómez [20] and Delgado et al. [5]). In such case, $\theta_{[\lambda,\infty,\Omega]}$ must be radially symmetric, because it is a limit of radially symmetric functions. By a solution of (7) it is meant a positive strong solution L (as discussed by Gilberg and Trudinger [10]) such that

$$\lim_{d(x) \downarrow 0} L(x) = \infty.$$

These solutions are refereed to as the *large* (or *explosive*) solutions of

$$-\Delta u = \lambda u - f(d(x))N(u)u \qquad \text{in} \quad \Omega.$$

According to (1), it follows, e.g., from [20, Theorem 4.11], that, for every $\lambda \in \mathbb{R}$, (7) has a minimal and a maximal positive solution, denoted by $L^{\min}_{[\lambda,\Omega]}$ and $L^{\max}_{[\lambda,\Omega]}$, respectively; in the sense that any other solution L of (7) satisfies

$$L^{\min}_{[\lambda,\Omega]} \le L \le L^{\max}_{[\lambda,\Omega]}.$$

Moreover,

$$\theta_{[\lambda,\infty,\Omega]} = L^{\min}_{[\lambda,\Omega]}, \qquad L^{\max}_{[\lambda,\Omega]} = \lim_{\varepsilon \downarrow 0} L^{\min}_{[\lambda,\Omega_\varepsilon]}, \tag{8}$$

where, for sufficiently small $\varepsilon > 0$,

$$\Omega_\varepsilon := \{ x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) > \varepsilon \}$$

and $L^{\min}_{[\lambda,\Omega_\varepsilon]}$ stands for the minimal large solution in the shortened domain Ω_ε, $\varepsilon \sim 0$. But, in general, it is far from known whether or not

$$L^{\min}_{[\lambda,\Omega]} = L^{\max}_{[\lambda,\Omega]}, \tag{9}$$

which entails the uniqueness of the large solution of (7). The following result, which is the main theorem of this paper, is optimal.

Theorem 1. *Suppose* (3), $\lambda \ge 0$, $f \in \mathcal{C}[0,\infty)$ *satisfies*

$$0 < f(t) \le f(s) \qquad \text{if} \quad 0 < t \le s, \tag{10}$$

and N satisfies (2) *and the Keller-Osserman condition* (6). *Suppose, in addition, that there exists $\alpha = \alpha(N) > 0$ such that*

$$\varrho^2 N(\varrho^{-\alpha} u) \le N(u) \qquad \text{for all} \quad (\varrho, u) \in (1,\infty) \times [0,\infty). \tag{11}$$

Then, (9) *holds, and, therefore,* (7) *has a unique positive solution,*

$$L_{[\lambda,\Omega]} := L^{\min}_{[\lambda,\Omega]} = L^{\max}_{[\lambda,\Omega]}.$$

Moreover, $L_{[\lambda,\Omega]}$ is radially symmetric.

Conditions (2) and (6) guarantee the existence and the uniqueness of $\theta_{[\lambda,M,\Omega]}$, $M > 0$, as well as their stabilization to $L^{\min}_{[\lambda,\Omega]}$ as $M \uparrow \infty$. Condition (10) enables us to deal with the degenerate case when $f(0) = 0$. The uniqueness of the large solution here is based upon it, though it is not necessary for it. Condition (11) holds for the most common choice of N in the available literature. Namely,

$$N(u) = u^{p-1}, \qquad u \geq 0, \tag{12}$$

for some $p > 1$. Indeed, (11) holds if and only if

$$\varrho^2 \left(\varrho^{-\alpha} u\right)^{p-1} \leq u^{p-1} \qquad \text{for all} \quad (\varrho, u) \in (1,\infty) \times [0,\infty),$$

or, equivalently, $\varrho^{2-\alpha(p-1)} \leq 1$ if $\varrho > 1$, which holds true for the special choice $\alpha = 2/(p-1)$. Although apparently restrictive, (11) holds for larger classes of N's satisfying (2) and (6), as, e.g.,

$$N(u) = a_p u^{p-1} + a_q u^{q-1}, \qquad u \geq 0, \tag{13}$$

where $a_p > 0$, $a_q > 0$, $1 < q < p$. Indeed, in case (13), condition (11) becomes into

$$a_p \varrho^{2-\alpha(p-1)} u^{p-1} + a_q \varrho^{2-\alpha(q-1)} u^{q-1} \leq a_p u^{p-1} + a_q u^{q-1}, \qquad \varrho > 1, \;\; u \geq 0,$$

which is satisfied for $\alpha = 2/(q-1)$, because in such case, (11) holds if and only if

$$\varrho^{2\left(1-\frac{p-1}{q-1}\right)} \leq 1, \qquad \varrho > 1,$$

which is true, since $1 - (p-1)/(q-1) < 0$.

Theorem 1 extends López-Gómez [22, Theorem 1.4], [21, Theorem 1.1], originally established for the special case (12), to cover the general case when (11) occurs.

2. **Other uniqueness results.** In this section, we shortly discuss some of the main uniqueness results available in the literature. Most of them have been obtained for the special choice (12).

When $f(0) > 0$, condition (10) is not necessary for the validity of Theorem 1, by some classical results of Loewner and Nirenberg [16], Kondratiev and Nikishin [13], Bandle and Marcus [1], Lazer and Mckenna [14], [15], Marcus and Véron [24], and Véron [27]. These results were sharpened by Du and Huang [6] and, independently, by García-Melián et al. [9] to cover the more general case when

$$\ell := \lim_{t \downarrow 0} \frac{f(t)}{t^\gamma} > 0, \tag{14}$$

for some $\gamma \geq 0$, and by Cirstea and Radulescu [2], [3], [4], to cover the general case when $f \in \mathcal{C}^1[0,\infty)$ satisfies $f(0) = 0$ and

$$\lim_{t \downarrow 0} \frac{\int_0^t \sqrt{f}}{\sqrt{f(t)}} = 0, \qquad \ell_1 := \lim_{t \downarrow 0} \frac{d}{dt} \frac{\int_0^t \sqrt{f}}{\sqrt{f(t)}} \in [0,1]. \tag{15}$$

Note that (14) implies (15), and that $f' \geq 0$ and

$$Q \in \mathcal{C}^1[0,R] \quad \text{and} \quad Q(0) = 0, \quad \text{where} \quad Q(t) := \frac{\int_0^t \sqrt{f}}{\sqrt{f(t)}}, \quad t \geq 0, \tag{16}$$

imply (15). Indeed, $Q(0) = 0$ implies the first relation of (15), whereas the second one follows from

$$0 \leq \frac{f'(t)}{f(t)} \frac{\int_0^t \sqrt{f}}{\sqrt{f(t)}} = 2 \left(1 - \frac{d}{dt} \frac{\int_0^t \sqrt{f}}{\sqrt{f(t)}}\right),$$

which implies

$$0 \leq \ell_1 = \lim_{t\downarrow 0} \frac{d}{dt} \frac{\int_0^t \sqrt{f}}{\sqrt{f(t)}} \leq 1.$$

More recently, Ouyang and Xie [26] got uniqueness by imposing $\Omega = B_R(x_0)$ and

$$\tilde{Q} \in \mathcal{C}^1[0,R] \quad \text{and} \quad \tilde{Q}(0) = 0, \quad \text{where} \quad \tilde{Q}(t) := \frac{\int_0^t f}{f(t)}, \quad t \geq 0. \tag{17}$$

Condition (17) is reminiscent from (16), and it seems slightly stronger, since

$$\frac{\int_0^t \sqrt{f}}{\sqrt{f(t)}} \leq \frac{\left(\int_0^t f\right)^{\frac{1}{2}} t^{\frac{1}{2}}}{\sqrt{f(t)}} = \left(\frac{\int_0^t f}{f(t)}\right)^{\frac{1}{2}} t^{\frac{1}{2}},$$

and, hence, $Q(0) = 0$ if $\tilde{Q}(0) = 0$. Incidentally, to prove their main theorem, Ouyang and Xie [26] adapted *mutatis mutandis* a device coming from López-Gómez [19], which consists in constructing an appropriate sub and supersolution pair from the associated one-dimensional problem to capture the blow-up rate of the large solutions on $\partial\Omega$. But [19] was not incorporated to the list of references of [26].

Except in [21] and [22], in order to prove the uniqueness, the strategy adopted in all available references consists in showing that all large solutions have the same blow-up rate at the boundary to conclude that this actually entails uniqueness. The following result, which goes back to [21] and [22], provides us with most of previous uniqueness results and corresponding blow-up rates.

Theorem 2. *Suppose* (3) *and* (12)*, with* $p > 1$*, and* $\lambda \geq 0$*. Let* $f \in \mathcal{C}[0,\infty)$ *be a bounded positive function such that* $f(0) = 0$*,* $f(t) \geq f(s) > 0$ *whenever* $t \geq s > 0$ *are sufficiently small, and*

$$\lim_{t\downarrow 0} \frac{F(t)F''(t)}{[F'(t)]^2} = I_0 \in (0,\infty), \tag{18}$$

where $F(t)$ *stands for the function defined through*

$$F(t) := \int_t^\infty \left(\int_0^s f^{\frac{1}{p+1}}\right)^{-\frac{p+1}{p-1}} ds, \qquad t > 0. \tag{19}$$

Then, any positive solution $L(x)$ *of* (7) *satisfies*

$$\lim_{d(x)\downarrow 0} \frac{L(x)}{F(d(x))} = I_0^{-\frac{p}{p-1}} \left(\frac{p+1}{p-1}\right)^{\frac{p+1}{p-1}} \tag{20}$$

and, therefore, (7) *has a unique solution.*

Theorem 1 does not impose any special requirement on f, like (14), (15), (16), (17), or (18), but, exclusively, its monotonicity. Condition (10) provides us with a uniqueness theorem for which the knowledge of the exact blow-up rates of the large solutions on $\partial\Omega$ is not necessary. Theorem 2 basically establishes that, in the special case (12), the large solution of (7) is unique if $f \in L^{1/(p+1)}$ and the function $F(t)$ defined through (19) satisfies the non-oscillation condition (18).

We conjecture that, for any N satisfying (2), (6) and (11), and any $f \in \mathcal{C}[0,\infty)$ such that $f(0) = 0$ and $f(t) \geq f(s) > 0$ if $t \geq s > 0$ are sufficiently small, the problem (7) has a unique solution. Our conjecture relies upon Theorem 1 and the fact that there always exist f_1, $f_2 \in \mathcal{C}[0,\infty)$, non-decreasing and such that

$$f_1(0) = f_2(0) = 0, \qquad f_1 \leq f \leq f_2, \qquad f_1(t) = f(t) = f_2(t) \quad \text{if} \quad t \sim 0.$$

According to Theorem 1, for each $i \in \{1,2\}$, the problem

$$\begin{cases} -\Delta u = \lambda u - f_i(d(x))N(u)u & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases}$$

has a unique positive solution. Let denote it by L_i. By comparison, $L_2 \leq L \leq L_1$ for every solution L of (7). Moreover, as the blow-up rates of the large solutions should only depend upon the values of f around $\partial\Omega$ and $f = f_1 = f_2$ there in, we find that L_1 and L_2, and, hence, L, should have the same blow-up rates on $\partial\Omega$.

3. **Proof of Theorem 1.** This section consists of the proof of Theorem 1.

3.1. **Case $\Omega = B_R(x_0)$.** Subsequently, for each $\varepsilon \in (0,R)$, we set

$$\varrho_\varepsilon := \frac{R}{R-\varepsilon}$$

and consider the auxiliary function

$$\bar{L}_\varepsilon(x) := L^{\min}_{[\lambda, B_R(x_0)]}(x_0 + \varrho_\varepsilon(x - x_0)), \qquad 0 \leq |x - x_0| \leq R - \varepsilon.$$

The function $\bar{L}_\varepsilon$ satisfies $\bar{L}_\varepsilon = \infty$ on $\partial B_{R-\varepsilon}(x_0)$. Moreover, for every $x \in B_{R-\varepsilon}(x_0)$,

$$\begin{aligned} -\Delta \bar{L}_\varepsilon(x) &= -\varrho_\varepsilon^2 \Delta L^{\min}_{[\lambda, B_R(x_0)]}(x_0 + \varrho_\varepsilon(x - x_0)) \\ &= \varrho_\varepsilon^2 \lambda \bar{L}_\varepsilon(x) - \varrho_\varepsilon^2 f(R - \varrho_\varepsilon |x - x_0|) N\left(\bar{L}_\varepsilon(x)\right) \bar{L}_\varepsilon(x) \\ &\geq \lambda \bar{L}_\varepsilon(x) - \varrho_\varepsilon^2 f(R - |x - x_0|) N\left(\bar{L}_\varepsilon(x)\right) \bar{L}_\varepsilon(x) \\ &= \lambda \bar{L}(x) - \varrho_\varepsilon^2 f(d(x)) N\left(\bar{L}_\varepsilon(x)\right) \bar{L}_\varepsilon(x), \end{aligned}$$

because $\varrho_\varepsilon > 1$, $\lambda \geq 0$, and, due to (10),

$$f(R - \varrho_\varepsilon |x - x_0|) \leq f(R - |x - x_0|).$$

Let $\alpha > 0$ be satisfying condition (11) and consider the auxiliary function

$$\hat{L}_\varepsilon := \varrho_\varepsilon^\alpha \bar{L}_\varepsilon \qquad \text{in } B_{R-\varepsilon}(x_0).$$

Then, $\hat{L}_\varepsilon = \infty$ on $\partial B_{R-\varepsilon}(x_0)$, and

$$-\Delta \hat{L}_\varepsilon(x) \geq \lambda \hat{L}_\varepsilon(x) - \varrho_\varepsilon^2 f(d(x)) N\left(\varrho_\varepsilon^{-\alpha} \hat{L}_\varepsilon(x)\right) \hat{L}_\varepsilon(x).$$

Thus, we find from (11) that

$$-\Delta \hat{L}_\varepsilon(x) \geq \lambda \hat{L}_\varepsilon(x) - f(d(x)) N\left(\hat{L}_\varepsilon(x)\right) \hat{L}_\varepsilon(x),$$

and, therefore, $\hat{L}_\varepsilon$ provides us with a supersolution of the singular problem

$$\begin{cases} -\Delta L = \lambda L - f(d(x))N(L)L & \text{in } B_{R-\varepsilon}(x_0), \\ L = \infty & \text{on } \partial B_{R-\varepsilon}(x_0). \end{cases}$$

By the construction of $L^{\max}_{[\lambda, B_R(x_0)]}$, it follows from the maximum principle that

$$L^{\max}_{[\lambda, B_R(x_0)]}(x) \leq \hat{L}_\varepsilon(x) = \varrho_\varepsilon^\alpha L^{\min}_{[\lambda, B_R(x_0)]}(x_0 + \varrho_\varepsilon(x - x_0))$$

for every $\varepsilon \in (0,R)$ and $x \in B_{R-\epsilon}(x_0)$. Consequently, passing to the limit as $\varepsilon \downarrow 0$ shows that

$$L^{\max}_{[\lambda, B_R(x_0)]}(x) \leq L^{\min}_{[\lambda, B_R(x_0)]}(x)$$

for each $x \in B_R(x_0)$, which concludes the proof of the theorem in this case.

3.2. **Case** $\Omega = A_{R_1,R_2}(x_0)$**.** Then, setting

$$R_m := \frac{R_1+R_2}{2}, \qquad r := |x-x_0|,$$

we have that

$$d(x) := \mathrm{dist}(x,\partial\Omega) = \begin{cases} R_2 - r & \text{if } R_m \le r \le R_2, \\ r - R_1 & \text{if } R_1 \le r \le R_m. \end{cases}$$

Moreover, since $L^{\min}_{[\lambda,\Omega]}$ and $L^{\max}_{[\lambda,\Omega]}$ are radially symmetric, we have that

$$L^{\min}_{[\lambda,\Omega]}(x) = \psi^{\min}(r), \qquad L^{\max}_{[\lambda,\Omega]}(x) = \psi^{\max}(r), \qquad x \in \Omega = A_{R_1,R_2}(x_0),$$

where $\psi^{\min}(r)$ and $\psi^{\max}(r)$ are the reflections around $r = R_m$ of the minimal and the maximal solutions of the singular one-dimensional problem

$$\begin{cases} -\psi'' - \frac{N-1}{r}\psi' = \lambda\psi - f(R_2-r)N(\psi)\psi, & R_m < r < R_2, \\ \psi'(R_m) = 0, \quad \psi(R_2) = \infty. \end{cases} \tag{21}$$

Next, we will show that any positive solution ψ of (21) satisfies

$$\psi'(r) \ge 0 \qquad \text{for all} \quad r \in [R_m, R_2). \tag{22}$$

This is clear if $\lambda \le 0$, because, in such case, multiplying the differential equation by r^{N-1} and integrating in (R_m, r) shows that, for every $r \in (R_m, R_2)$,

$$r^{N-1}\psi'(r) = \int_{R_m}^r s^{N-1}[f(R_2-s)N(\psi(s)) - \lambda]\psi(s)\,ds > 0.$$

When $\lambda > 0$, to prove (22) we will argue by contradiction. Suppose $\lambda > 0$ and (21) possesses a positive solution ψ for which there exists $\tilde r \in (R_m, R_2)$ such that

$$\psi'(\tilde r) < 0.$$

Then, since

$$\psi'(R_m) = 0, \qquad \lim_{r\uparrow R_2} \psi(r) = \infty,$$

there exist $R_m \le r_0 < \tilde r < r_1 < R_2$ such that

$$\begin{cases} \psi'(r_0) = \psi'(r_1) = 0, \quad \psi'(r) \le 0 \quad \text{if} \quad r \in (r_0, r_1), \\ \psi''(r_0) \le 0, \quad \psi''(r_1) \ge 0. \end{cases} \tag{23}$$

Subsequently, we consider the function $H(\xi)$ defined by

$$H(\xi) := \lambda\xi - f(R_2-r_0)N(\xi)\xi, \qquad \xi > 0.$$

By (2), the value $\xi_0 := N^{-1}(\lambda/f(R_2-r_0))$ provides us with the unique positive zero of $H(\xi)$. Actually, we have that $H(\xi) > 0$ if $\xi \in (0,\xi_0)$, $H(\xi_0) = 0$, and $H(\xi) < 0$ if $\xi > \xi_0$. Suppose $\psi(r_0) > \xi_0$. Then, due to (21) and (23), we find that

$$0 \le -\psi''(r_0) = -\psi''(r_0) - \frac{N-1}{r_0}\psi'(r_0) = H(\psi(r_0)) < 0,$$

which is impossible. Thus,

$$\psi(r_0) \le \xi_0, \tag{24}$$

and, hence, (21) and (23) imply that

$$\begin{aligned} 0 \ge -\psi''(r_1) &= -\psi''(r_1) - \frac{N-1}{r_1}\psi'(r_1) = \lambda\psi(r_1) - f(R_2-r_1)N(\psi(r_1))\psi(r_1) \\ &= \lambda\psi(r_1) - f(R_2-r_0)N(\psi(r_1))\psi(r_1) + [f(R_2-r_0) - f(R_2-r_1)]\,N(\psi(r_1))\psi(r_1) \\ &= H(\psi(r_1)) + [f(R_2-r_0) - f(R_2-r_1)]\,N(\psi(r_1))\psi(r_1). \end{aligned}$$

As $\psi' < 0$ in (r_0, r_1), we have that $\psi(r_1) < \psi(r_0)$ and, so, by (24), $\psi(r_1) < \xi_0$, which implies

$$H(\psi(r_1)) > 0.$$

Moreover, $r_0 < r_1$ implies $R_2 - r_0 > R_2 - r_1$ and, hence, due to (10),

$$f(R_2 - r_0) \geq f(R_2 - r_1).$$

Therefore,

$$0 \geq H(\psi(r_1)) + [f(R_2 - r_0) - f(R_2 - r_1)]\, N(\psi(r_1))\psi(r_1) > 0,$$

which is impossible too. Consequently, condition (22) must be satisfied.

Subsequently, we set

$$a_\varepsilon := \frac{R_2 - R_m}{R_2 - R_m - \varepsilon} > 1, \qquad \varepsilon \in (0, R_2 - R_m),$$

and consider the function $\bar{\psi}_\varepsilon$ defined by

$$\bar{\psi}_\varepsilon(r) := \psi^{\min}\left(a_\varepsilon(r - R_m) + R_m\right), \qquad R_m \leq r < R_2 - \varepsilon.$$

By definition,

$$\bar{\psi}'_\varepsilon(R_m) = a_\varepsilon \left(\psi^{\min}\right)'(R_m) = 0$$

and

$$\lim_{r \uparrow R_2 - \varepsilon} \bar{\psi}_\varepsilon(r) = \lim_{\varrho \uparrow R_2} \psi^{\min}(\varrho) = \infty.$$

Moreover, setting

$$\varrho := R_m + a_\varepsilon(r - R_m), \qquad R_m \leq \varrho \leq R_2 - \varepsilon,$$

we find from (21) that, for every, $r \in (R_m, R_2 - \varepsilon)$,

$$\begin{aligned} -\bar{\psi}''_\varepsilon(r) - \frac{N-1}{r}\bar{\psi}'_\varepsilon(r) &= -a_\varepsilon^2 \left(\psi^{\min}\right)''(\varrho) - \frac{N-1}{r} a_\varepsilon \left(\psi^{\min}\right)'(\varrho) \\ &= -a_\varepsilon^2 \left(\psi^{\min}\right)''(\varrho) - \frac{N-1}{\varrho}\frac{\varrho}{r} a_\varepsilon \left(\psi^{\min}\right)'(\varrho). \end{aligned}$$

Thus, the following estimates

$$\lambda > 0, \quad \left(\psi^{\min}\right)' \geq 0, \quad a_\varepsilon > 1, \quad \frac{\varrho}{r} = \frac{R_m(1 - a_\varepsilon) + a_\varepsilon r}{r} \leq a_\varepsilon,$$

imply

$$\begin{aligned} -\bar{\psi}''_\varepsilon(r) - \frac{N-1}{r}\bar{\psi}'_\varepsilon(r) &= -a_\varepsilon^2 \left(\psi^{\min}\right)''(\varrho) - \frac{N-1}{\varrho}\frac{\varrho}{r} a_\varepsilon \left(\psi^{\min}\right)'(\varrho) \\ &\geq a_\varepsilon^2 \left[-\left(\psi^{\min}\right)''(\varrho) - \frac{N-1}{\varrho}\left(\psi^{\min}\right)'(\varrho)\right] \\ &= a_\varepsilon^2 \left[\lambda \psi^{\min}(\varrho) - f(R_2 - \varrho) N(\psi^{\min}(\varrho))\psi^{\min}(\varrho)\right] \\ &\geq \lambda \psi^{\min}(\varrho) - a_\varepsilon^2 f(R_2 - \varrho) N(\psi^{\min}(\varrho))\psi^{\min}(\varrho). \end{aligned}$$

On the other hand, we have that

$$\bar{\psi}_\varepsilon(r) := \psi^{\min}(\varrho), \qquad f(R_2 - r) \geq f(R_2 - \varrho),$$

because $r \leq \varrho$, and, consequently, for every $r \in (R_m, R_2 - \varepsilon)$,

$$-\bar{\psi}''_\varepsilon(r) - \frac{N-1}{r}\bar{\psi}'_\varepsilon(r) \geq \lambda \bar{\psi}_\varepsilon(r) - a_\varepsilon^2 f(R_2 - r) N(\bar{\psi}_\varepsilon(r))\bar{\psi}_\varepsilon(r).$$

Let $\alpha > 0$ be satisfying condition (11) and consider the auxiliary function

$$\hat{\psi}_\varepsilon(r) := a_\varepsilon^\alpha \bar{\psi}_\varepsilon(r), \qquad R_m < r < R_2 - \varepsilon.$$

Then, $\hat{\psi}'_\varepsilon(R_m) = 0$, $\lim_{r\uparrow R_2-\varepsilon} \hat{\psi}_\varepsilon(r) = \infty$, and, for any $r \in (R_m, R_2 - \varepsilon)$,

$$-\hat{\psi}''_\varepsilon(r) - \frac{N-1}{r}\hat{\psi}'_\varepsilon(r) \geq \lambda\hat{\psi}_\varepsilon(r) - a_\varepsilon^2 f(R_2 - r)N(a_\varepsilon^{-\alpha}\hat{\psi}_\varepsilon(r))\hat{\psi}_\varepsilon(r).$$

Consequently, thanks to (11),

$$-\hat{\psi}''_\varepsilon(r) - \frac{N-1}{r}\hat{\psi}'_\varepsilon(r) \geq \lambda\hat{\psi}_\varepsilon(r) - f(R_2 - r)N(\hat{\psi}_\varepsilon(r))\hat{\psi}_\varepsilon(r), \quad R_m < r < R_2 - \varepsilon,$$

and, therefore, $\hat{\psi}_\varepsilon$ provides us with a supersolution of the singular problem

$$\begin{cases} -\psi'' - \frac{N-1}{r}\psi' = \lambda\psi - f(R_2 - r)N(\psi)\psi, & R_m < r < R_2 - \varepsilon, \\ \psi'(R_m) = 0, \quad \psi(R_2 - \varepsilon) = \infty. \end{cases} \tag{25}$$

By the maximum principle, this implies that, for all sufficiently small $\varepsilon > 0$,

$$a_\varepsilon^\alpha \bar{\psi}_\varepsilon(r) = \hat{\psi}_\varepsilon(r) \geq \psi^{\max}(r) \qquad R_m < r < R_2 - \varepsilon.$$

Finally, passing to the limit as $\varepsilon \downarrow 0$ in the previous inequality show that $\psi^{\min} \geq \psi^{\max}$ in (R_m, R_2), which concludes the proof of the theorem.

4. **Applications to Population Dynamics.** Let $\Omega \subset \mathbb{R}^N$ be an arbitrary domain with smooth boundary, $\partial\Omega$, such that $\bar{B}_R(x_0) \subset \Omega$ and consider the function

$$a(x) := \begin{cases} f(d(x)), & x \in \bar{B}_R(x_0), \\ 0, & x \in \bar{\Omega} \setminus \bar{B}_R(x_0), \end{cases}$$

and the associated parabolic model

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - a(x)N(u)u, & (x,t) \in \Omega \times (0,\infty), \\ u = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{26}$$

where $u_0 \in \mathcal{C}(\bar{\Omega})$ satisfies $u_0 > 0$ ($u_0 \geq 0$ and $u_0 \neq 0$). In Population Dynamics, (26) models the evolution of a single species u dispersing in Ω, which consists of two regions. In $B_R(x_0)$, where $a > 0$, u grows according to the Verhulst law, whereas in $\Omega_0 := \Omega \setminus \bar{B}_R(x_0)$, where $a = 0$, the species has a genuine exponential, or Malthusian, growth. In (26), λ is the intrinsic growth rate of u and u_0 the initial population. In this paper, the inhabiting area Ω is assumed to be entirely surrounded by completely hostile regions, for we are dealing with homogeneous Dirichlet boundary conditions.

The problem (26) has a unique solution $u(x, t; u_0)$ globally defined in time, $t > 0$, because of the structure of the nonlinearity, and, from the point of view of the applications, it is imperative to ascertain the asymptotic behaviour

$$L_\lambda := \lim_{t\uparrow\infty} u(\cdot, t; u_0).$$

According to experience, it is natural to conjecture that L_λ must be a non-negative equilibrium of (26), i.e., a non-negative solution to the elliptic problem

$$\begin{cases} -\Delta u = \lambda u - a(x)N(u)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{27}$$

but, since Fraile et al. [7], García-Melián et al. [8], Gómez-Reñasco and López-Gómez [11], López-Gómez and Sabina [23], and López-Gómez [17] (see [20] for a rather self-contained monograph on the subject), these predictions fail to be true for sufficiently large λ, because (27) cannot admit a positive solution if $\lambda \geq \sigma[-\Delta; \Omega_0]$ and $u = 0$ is unstable for such λ's. Subsequently, for a given smooth domain D, $\sigma[-\Delta; D]$ stands for the principal eigenvalue of $-\Delta$ in D under homogeneous Dirichlet boundary conditions. Precisely, the following result holds.

Theorem 3. *The problem* (27) *admits a positive solution if and only if*

$$\sigma[-\Delta;\Omega] < \lambda < \sigma[-\Delta;\Omega_0]. \tag{28}$$

Moreover, it is unique if it exists, and if we denote it by $\theta_{[\lambda,\Omega]}$*, then the map* $\lambda \mapsto \theta_{[\lambda,\Omega]}$ *is point-wise increasing and, for every* $x \in \Omega$*, we have that*

$$\lim_{\lambda \uparrow \sigma[-\Delta;\Omega_0]} \theta_{[\lambda,\Omega]}(x) = \begin{cases} L^{\min}_{[\sigma[-\Delta;\Omega_0],B_R(x_0)]}(x), & x \in \bar{B}_R(x_0), \\ \infty, & x \in \Omega_0. \end{cases}$$

Furthermore, for every $u_0 > 0$*, we have that*

$$\lim_{t\uparrow\infty} u(\cdot,t;u_0) = \begin{cases} 0 & \text{if} \quad \lambda \le \sigma[-\Delta;\Omega], \\ \theta_{[\lambda,\Omega]} & \text{if} \quad \sigma[-\Delta;\Omega] < \lambda < \sigma[-\Delta;\Omega_0]. \end{cases}$$

Theorem 3 provides us with the dynamics of (26) when $\lambda < \sigma[-\Delta;\Omega_0]$ and, at $\lambda = \sigma[-\Delta;\Omega_0]$, it suggests that

$$\lim_{t\uparrow\infty} u(x,t;u_0) = \begin{cases} L^{\min}_{[\sigma[-\Delta;\Omega_0],B_R(x_0)]}(x), & x \in \bar{B}_R(x_0), \\ \infty, & x \in \Omega_0. \end{cases}$$

This naturally leads to the concept of *metasolution*, which goes back to Gómez-Reñasco and López-Gómez [11], and [17], and can be made precise as follows.

Definition 1. A function $\mathfrak{M} : \Omega \to [0,\infty]$ is said to be a metasolution of (26) supported in $B_R(x_0)$ if there exists a solution L of (7) such that

$$\mathfrak{M} = \begin{cases} L & \text{in} \quad B_R(x_0), \\ \infty, & \text{in} \quad \Omega \setminus B_R(x_0). \end{cases}$$

By Theorem 1, for any $\lambda \ge 0$, (7) has a unique solution, denoted by $L_{[\lambda,B_R(x_0)]}$, and, hence, (26) has a unique metasolution supported in $B_R(x_0)$. Namely,

$$\mathfrak{M}_\lambda = \begin{cases} L_{[\lambda,B_R(x_0)]} & \text{in} \quad B_R(x_0), \\ \infty, & \text{in} \quad \Omega \setminus B_R(x_0). \end{cases}$$

The following result establishes that the metasolutions provide us with the asymptotic behaviour of all solutions of (26) whenever $\lambda \ge \sigma[-\Delta;\Omega_0]$. The proof can be easily completed from Du and Huang [6], and López-Gómez [17], [18].

Theorem 4. *Suppose the assumptions of Theorem 1 are satisfied, and* $\lambda \ge \sigma[-\Delta;\Omega_0]$*. Then, for every* $u_0 > 0$*,*

$$\lim_{t\uparrow\infty} u(\cdot,t;u_0) = \mathfrak{M}_\lambda \qquad \text{in} \quad \Omega.$$

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