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SPECTRUM OF SOME TRIANGULATED CATEGORIES

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Abstract. In this note we announce the computation of the triangular spectrum (as defined by P. Balmer) of two classes of tensor triangulated categories which are quite common in algebraic geometry. One of them is the derived category of G-equivariant sheaves on a smooth quasi projective scheme X for a finite group G which acts on X. The other class is the derived category of split super-schemes.

This is an announcement of results for which the details will appear elsewhere.

Triangulated categories have been one of the most influential objects in mathematics. Introduced by Grothendeick and Verdier to study Serre duality in a relative setting, this idea was soon developed by Verdier and Illusie, who studied the triangulated derived category of the abelian category of coherent sheaves, and the triangulated category of perfect complexes, respectively. Slowly, the abstract homological construction of triangulated categories permeated into other subjects such as topology, modular representation theory and even Kasparov's KK theory. Balmer's paper [2] gives a nice summary of the elegant history.

In algebraic geometry, triangulated categories mostly appear as the derived category of the abelian category of coherent sheaves on a variety and as the category of perfect complexes on a variety. The later category, as was observed by Neeman [10], is just the compact objects of the derived category of the abelian category of quasi-coherent sheaves (in case the scheme is quasi compact and separated). Henceforth, we shall refer to the derived category of the category of coherent sheaves simply as the derived category of the variety . Gabriel [5] and Rosenberg [11] proved that the category of quasi coherent sheaves completely determine the underlying variety. Bondal and Orlov [3] proved that a smooth projective variety can be reconstructed from the derived category of coherent sheaves provided that either the canonical bundle or the anti-canonical bundle is ample. The ampleness assumption is crucial as Mukai [8] gave an example of two non-isomorphic varieties whose derived categories are equivalent. This example of Mukai says that the reconstruction of variety from its derived category is not possible.

Balmer [2][1] proved that in addition to the triangulated structure on a derived category, if we also consider the tensor structure induced by the tensor structure in the category of coherent sheaves, we have enough information to reconstruct the

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variety. He provided a method to reconstruct the variety by means of constructing the "Spec" of the tensor triangulated category. This Spec construction is quite general and can be done on any tensor triangulated category. Balmer used topologically Noetherian schemes for reconstruction but Krause[4] extended it to the non-Noetherian case. One question that naturally arises is how good is Spec as an invariant of the tensor triangulated category? It turns out that there exist pairs of tensor triangulated categories which have isomorphic Specs (isomorphic as ringed spaces). We give two such examples. This raises the question of whether one can define a finer geometric invariant. Some possible answers are discussed in the first author's thesis with HBNI.

The quasi projective scheme X with an action of a finite group G is called the G-scheme for simplicity. We first compute the Spec of the derived category of the abelian category of coherent G-equivariant sheaves on some smooth quasi-projective scheme X. Since the scheme is quasi projective, there exists an orbit space, see [9], which we denote as X/G. Recall that a G equivariant or G linearized sheaf is defined as follows:

Definition 1. A G-sheaf (or G-equivariant sheaf or an equivariant sheaf with respect to the group G) on X is a sheaf $\mathcal{F}$ together with isomorphisms $\rho_g : \mathcal{F} \to g^*\mathcal{F}$ for all $g \in G$ such that the following diagram

$$\begin{array}{ccccc} \mathcal{F} & \xrightarrow{\rho_h} & h^*\mathcal{F} & \xrightarrow{h^*\rho_g} & h^*g^*\mathcal{F} \\ & & \searrow^{\rho_{gh}} & & \| \\ & & & & (gh)^*\mathcal{F} \end{array}$$

is commutative for any pair $g, h \in G$. A G-sheaf is a pair $(\mathcal{F}, \rho)$.

The category of coherent G-sheaves is denoted $\mathfrak{Coh}^G(X)$, and for simplicity, we denote by $\mathcal{D}^G(X)$ the bounded derived category of coherent G-sheaves. Consider the affine map $\pi : X \to X/G$. Then $\mathcal{D}^G(X)$ admits a functor from $\mathcal{D}^{per}(X/G)$,

$$\pi^* : \mathcal{D}^{per}(X/G) \to \mathcal{D}^G(X).$$

Since we only consider quasi projective varieties, the perfect complexes are nothing but bounded complexes of vector bundles.

We prove the following theorem.

Theorem 2. *Assume that the scheme X is smooth quasi projective with an action of a finite group G. The induced map*

$$\mathrm{Spec}(\pi^*) : \mathrm{Spec}(\mathcal{D}^G(X)) \to \mathrm{Spec}(\mathcal{D}^{per}(X/G))$$

is an isomorphism of locally ringed spaces.

Here, Spec denotes the construction due to Balmer[2]. The proof involves some computation using results from representation theory.

Super-schemes, studied by Manin and Deligne (see for example [7]), are also an important object of study in modern algebraic geometry, especially due to applications in physics. The following definition of split super-scheme is given in [6, pp. 84-85].

Definition 3.

(1) A ringed space $(X, \mathcal{O}_X)$ is called a if the ring $\mathcal{O}_X(U)$ associated to any open subset U is super-commutative and each stalk is local ring. A *super-space* is called a if, in addition, the ringed space $(X, \mathcal{O}_{X,0})$ is a scheme and $\mathcal{O}_{X,1}$ is a coherent sheaf over $\mathcal{O}_{X,0}$ (where the subscript 0 denotes the even part and the subscript 1 denotes the odd part). We shall denote by J_X the ideal sheaf generated by $\mathcal{O}_{X,1}$ inside $\mathcal{O}_X$.

(2) A super-scheme $(X, \mathcal{O}_X)$ is called *split* if the graded sheaf $Gr\mathcal{O}_X$ with mod 2 grading is isomorphic as a locally super-ringed sheaf to $\mathcal{O}_X$. Here the graded sheaf is

$$Gr\mathcal{O}_X := \oplus_{i \geq 0} J_X^i / J_X^{i+1} \text{ where } J_X^0 := \mathcal{O}_X.$$

Manin has also given example of super-schemes which are not split super-schemes. An important example of a split super-scheme is super projective space $\mathbb{P}^{n|m}$. We consider the triangulated category $\mathcal{D}^{per}(X)$ of " perfect complexes " (the definition being modified appropriately in the super setting) on this super-scheme.

Theorem 4. *Let X be a split super-scheme. Let $X_0 = (X, \mathcal{O}_{X,0})$ be the 0-th part of this super-scheme. Here X_0 is by definition a scheme. Then we have an isomorphism of locally ringed spaces*

$$f : X_0 \to \operatorname{Spec}(\mathcal{D}^{per}(X)).$$

The proof of homeomorphism adapts the classification of thick tensor ideals, due to Thomason[12], as demonstrated by Balmer[2]. Following Balmer[2] once more, we use the generalized localization theorem of Neeman[10, Theorem 2.1] to finish the proof.

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